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Some Results on the Solutions of Higher-Order Linear Differential Equations

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Abstract In this paper, we investigate the solutions of certain types of higher-order linear differential equations. By introducing a new concept of n-subnormal solution, we study the existence, growth, and numbers of solutions of this type, and we also estimate the growth of all other solutions.

Keywords Meromorphic functions · Differential equation · Hyper-order

Mathematics Subject Classification 30D35 · 34M10

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1 Introduction

In this paper, we will use standard notations from the value distribution theory of meromorphic functions (see [12,14]). We suppose that $f(z)$ is a meromorphic function in the whole complex plane $\mathbb{C}$ and denote its order by $\sigma(f)$ and hyper-order by

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r}.$$

Consider the second-order homogeneous linear differential equation

$$f'' + P(e^z)f' + Q(e^z)f = 0, \tag{1}$$

where $P(z)$ and $Q(z)$ are polynomials in z and not both constants. It is well known that every solution f of (1) is entire.

Suppose $f \not\equiv 0$ is a solution of (1). If f satisfies the condition

$$\limsup_{r\to\infty} \frac{\log T(r, f)}{r} = 0,$$

then we say that f is a nontrivial subnormal solution of (1).

Wittich [13], Gundersen and Steinbart [10], Chen and Shon [4,6], etc., have investigated the subnormal solutions of (1) and obtained some excellent results.

Wittich [13] gave the detailed form of all subnormal solutions of (1). Gundersen and Steinbart [10] refined Wittich's result by analyzing the degree of the coefficients $P(z)$ and $Q(z)$. They also considered the form of subnormal solutions of the more general differential equation

$$f'' + P(e^z)f' + Q(e^z)f = R_1(e^z) + R_2(e^{-z}), \tag{2}$$

where $P(z)$, $Q(z)$, and $R_d(z)$ $(d = 1, 2)$ are polynomials in z.

For the higher-order linear homogeneous differential equation

$$f^{(k)} + P_{k-1}(e^z)f^{(k-1)} + \cdots + P_0(e^z)f = 0, \tag{3}$$

where $P_j(z)$ $(j = 0, \cdots, k-1)$ are polynomials in z, many papers were devoted to the investigation of the nonexistence of its nontrivial subnormal solutions under certain conditions, see, e.g., [3,5,10]. In [7], Chen and Shon gave an example to show that in other case the Eq. (3) might have subnormal solutions. Further, they estimated the number and growth of this kind of solutions and all other solutions and obtained the following result.

Theorem A ([7]) *Let* $P_j(z)(j = 0, \cdots, k-1)$ *be polynomials in* z *and* $\deg P_j = m_j$. *Suppose that there exists* $m_s (s \in \{0, \ldots, k-1\})$ *satisfying*

$$m_s > \max\{m_j : j = 0, \ldots, s-1, s+1, \ldots, k-1\}.$$

Then

(i) *every subnormal solution* f_0 *of* (3) *satisfies* $\sigma(f_0) = 1$ *or is a polynomial with* $\deg f \le s-1$*, and any other solution* f *not of the above two forms satisfies* $\sigma_2(f) = 1$*;*
(ii) *Equation* (3) *possesses at most* s *linearly independent subnormal solutions.*

We set

$$A(z) = d_n z^n + d_{n-1} z^{n-1} + \cdots + d_0,\ d_n \neq 0,\ n(\geq 1) \text{ is an integer},$$

throughout the rest of this paper. Then it is natural to ask what will happen if we change $\exp\{z\}$ in the coefficients of (3) into $\exp\{A(z)\}$?

In this paper, we consider the above problem and go on studying the solutions of the following higher-order linear differential equation

$$f^{(k)} + P_{k-1}(e^{A(z)}) f^{(k-1)} + \cdots + P_0(e^{A(z)}) f = 0, \tag{4}$$

where $P_j(z) = a_{jm_j} z^{m_j} + \cdots + a_{j1} z + a_{j0}\ (j = 0, 1, \ldots, k-1)$, $a_{jm_j}, \ldots, a_{j0}$, are complex constants such that $a_{jm_j} \neq 0$, m_j are nonnegative integers, and obtain Theorem 1.1.

We need the following definition in order to state our results. Suppose $f \not\equiv 0$ is a solution of differential equation. If f satisfies the condition

$$\limsup_{r \to \infty} \frac{\log T(r, f)}{r^n} = 0, \tag{5}$$

then we say that the equation has a nontrivial n-subnormal solution.

Theorem 1.1 *Let* $P_j(z)\ (j = 0, \cdots, k-1)$, $A(z)$ *be polynomials in* z *with* deg $P_j = m_j$ *and* deg $A = n$*. Suppose that there exists an integer* s $(s \in \{0, \cdots, k-1\})$ *satisfying*

$$m_s > \max\{m_j : j = 0, \ldots, k-1, j \neq s\} = \widetilde{m}, \tag{6}$$

then

(i) *every* n*-subnormal solution* f_0 *of* (4) *satisfies* $\sigma(f_0) = n$ *or is a polynomial with* $\deg f \le s-1$*, and any other solution* f *not of the above two forms satisfies* $\sigma_2(f) = n$*;*
(ii) *Equation* (4) *possesses at most* s *linearly independent* n*-subnormal solutions.*

In paper [10], Gundersen and Steinbart also activated a new direction for the study on the nontrivial subnormal solutions to Eq. (2) and raised the question that can the results they obtained be generalized to equation

$$f'' + [P_1(e^z) + P_2(e^{-z})] f' + [Q_1(e^z) + Q_2(e^{-z})] f = R_1(e^z) + R_2(e^{-z}), \tag{7}$$

where $P_j(z)$, $Q_j(z)$, $R_j(z)\ (j = 1, 2)$ are polynomials in z?

Many papers focus on the above problem, see, e.g., [2,4,6,11]. In [4], Chen and Shon considered this problem and investigated the existence of subnormal solutions of Eq. (7) and its corresponding homogeneous form under certain conditions on the degree of the coefficients $P_j(z)$ and $Q_j(z)\,(j=1,2)$. In [6], they generalized their results to the higher-order case. They considered the linear homogeneous equation

$$f^{(k)} + \left(P_{k-1}(e^z) + Q_{k-1}(e^{-z})\right) f^{(k-1)} + \cdots + \left(P_0(e^z) + Q_0(e^{-z})\right) f = 0, \quad (8)$$

and linear nonhomogeneous equation

$$f^{(k)}+\left(P_{k-1}(e^z)+Q_{k-1}(e^{-z})\right) f^{(k-1)}+\cdots+\left(P_0(e^z)+Q_0(e^{-z})\right) f = R_1(e^z)+R_2(e^{-z}), \quad (9)$$

where $P_j,\ Q_j\ (j=0,1,\ldots,k-1),\ R_1,\ R_2$ are polynomials in z, and obtained the following theorems.

Theorem B ([6]) *Let $P_j(z),\ Q_j(z)\,(j=0,\ldots,k-1)$ be polynomials in z with* $\deg P_j = m_j$, $\deg Q_j = n_j$, *and $P_0 + Q_0 \not\equiv 0$. If there exist $m_s,\ n_d\ (s,d \in \{0,\ldots,k-1\})$ satisfying both of the inequalities*

$$\begin{cases} m_s > \max\{m_j : j = 0,\ldots,s-1,s+1,\ldots,k-1\}, \\ n_d > \max\{n_j : j = 0,\ldots,d-1,d+1,\ldots,k-1\}, \end{cases} \quad (10)$$

then the linear homogeneous Eq. (8) has no nontrivial subnormal solution, and every solution of (8) is of hyper-order $\sigma_2(f)=1$.

Theorem C ([6]) *Let $P_j(z),\ Q_j(z)\,(j=0,\cdots,k-1)$ be defined as in Theorem B; let $R_i(z)\,(i=1,2)$ be polynomials in z. If there exist $m_s,\ n_d\ (s,d\in\{0,\cdots,k-1\})$ satisfying both of the inequalities in (10), then*

(i) *Equation (9) has at most one nontrivial subnormal solution f_0, and f_0 is of form*

$$f(z) = S_1(e^z) + S_2(e^{-z}),$$

where $S_1(z)$ and $S_2(z)$ are polynomials in z;

(ii) *all other solutions f of (9) satisfy $\sigma_2(f)=1$ except the possible subnormal solution in (i).*

In [11], Huang and Sun changed $\exp\{z\}$ in the coefficients of (8) into $\exp\{R(z)\}$, where $R(z)$ is a nonconstant polynomial, and obtained the following result.

Theorem D ([11]) *Let $A_j = P_j\left(e^{R(z)}\right) + Q_j\left(e^{-R(z)}\right)$ for $j=1,\ldots,k-1$, where $P_j(z), Q_j(z)$, and $R(z) = c_s z^s + \cdots + c_1 z + c_0$ ($s(\geq 1)$ is an integer) are polynomials. Suppose that $P_0(z)+Q_0(z)\not\equiv 0$ and there exists $d\ (0\leq d\leq k-1)$ such that for $j\neq d$,* $\deg P_d > \deg P_j$ *and* $\deg Q_d > \deg Q_j$. *Then every solution $f(z)$ of*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_0 f = 0, k\geq 2,$$

is of infinite order and satisfies $\sigma_2(f) = s$.

It is natural to ask can the condition "deg $P_d >$ deg P_j and deg $Q_d >$ deg Q_j $(j \neq d)$" in Theorem D be weakened?

In this paper, we consider the above problem, investigate the solutions of the linear homogeneous equation

$$f^{(k)} + \left(P_{k-1}\left(e^{A(z)}\right) + Q_{k-1}\left(e^{-A(z)}\right)\right) f^{(k-1)} + \cdots + \left(P_0\left(e^{A(z)}\right) + Q_0\left(e^{-A(z)}\right)\right) f = 0, \tag{11}$$

where $P_j\left(e^{A(z)}\right) + Q_j\left(e^{-A(z)}\right) = a_{jm_j} e^{m_j A(z)} + \cdots + a_{j1} e^{A(z)} + c_{j0} + b_{j1} e^{-A(z)} + \cdots + b_{jn_j} e^{-n_j A(z)}$, $a_{jm_j}, \ldots, a_{j1}, c_{j0}, b_{jn_j}, \ldots, b_{j1}$ are constants, $m_j, n_j\ (\geq 0)$ are integers, $a_{jm_j} \neq 0$, $b_{jn_j} \neq 0$, and obtain the following results.

Theorem 1.2 *Let* $P_j(z),\ Q_j(z)\ (j = 0, \ldots, k-1)$ *and* $A(z)$ *be polynomials in* z *with* deg $P_j = m_j$, deg $Q_j = n_j$, deg $A = n$, *and* $P_0 + Q_0 \not\equiv 0$. *If there exist* $m_s,\ n_d\ (s, d \in \{0, \ldots, k-1\})$ *satisfying both of the inequalities*

$$\begin{cases} m_s > \max\{m_j : j = 0, \ldots, k-1,\ j \neq s\} = \widetilde{m}, \\ n_d > \max\{n_j : j = 0, \ldots, k-1,\ j \neq d\} = \widetilde{n}, \end{cases} \tag{12}$$

then the linear homogeneous Eq. (11) has no nontrivial n*-subnormal solution, and every solution of* (11) *is of* $\sigma_2(f) = n$.

It is obvious from $\limsup_{r\to\infty} \frac{\log T(r,f)}{r} = 0$ that we can deduce $\limsup_{r\to\infty} \frac{\log T(r,f)}{r^n} = 0$ easily. So if (11) has no nontrivial n-subnormal solutions, we can obtain that (11) has no nontrivial subnormal solutions. Thus, we get the following corollary.

Corollary 1.3 *Under the assumption of Theorem 1.2, the linear homogeneous Eq.* (11) *has no nontrivial subnormal solutions, and every solution of* (11) *is of* $\sigma_2(f) = n$.

Obviously, Theorem 1.2 and Corollary 1.3 improved Theorems B and D.

For the nonhomogeneous linear differential equation

$$f^{(k)} + \left(P_{k-1}\left(e^{A(z)}\right) + Q_{k-1}\left(e^{-A(z)}\right)\right) f^{(k-1)} + \cdots + \left(P_0\left(e^{A(z)}\right) + Q_0\left(e^{-A(z)}\right)\right) f = R(z), \tag{13}$$

we obtain

Theorem 1.4 *Let* $P_j(z),\ Q_j(z)\ (j = 0, \ldots, k-1)$ *and* $A(z)$ *be defined as in Theorem 1.2. Let* $R(z)$ *be entire function with* $\sigma_2(R) \leq n$. *If there exist* $m_s,\ n_d\ (s, d \in \{0, \ldots, k-1\})$ *satisfying both of the inequalities in* (12)*, then*

(i) *Equation* (13) *has at most one nontrivial* n*-subnormal solution* f_0*;*
(ii) *all other solutions* f *of* (13) *satisfy* $\sigma_2(f) = n$ *except the possible* n*-subnormal solution in (i).*

2 Preliminary Lemmas

Recall that

$$A(z) = d_n z^n + d_{n-1} z^{n-1} + \cdots + d_1 z + d_0,\ d_l = \alpha_l e^{i\theta_l},\ z = re^{i\theta},$$

we set $\delta_l(A, \theta) = \mathrm{Re}(d_l(e^{i\theta})^l) = \alpha_l \cos(\theta_l + l\theta)$, and

$$H_{l,0} = \{\theta \in [0, 2\pi) : \delta_l(A, \theta) = 0\},\ H_{l,+} = \{\theta \in [0, 2\pi) : \delta_l(A, \theta) > 0\},$$
$$H_{l,-} = \{\theta \in [0, 2\pi) : \delta_l(A, \theta) < 0\},$$

for $l = 1, \ldots, n$, throughout the rest of this paper.

Obviously, if $\delta_n(A, \theta) \neq 0$, as $r \to \infty$, we get

$$\left|e^{A(z)}\right| = e^{\delta_n(A,\theta)r^n + \cdots + \delta_1(A,\theta)r + \mathrm{Re} d_0} = e^{\delta_n(A,\theta)r^n(1+o(1))}.$$

Therefore, for $j = 0, 1, \ldots, k-1$, as $r \to \infty$, the coefficients in Eq. (4), we have

$$\left|P_j(e^{A(z)})\right| = \begin{cases} |a_{jm_j}| e^{m_j \delta_n(A,\theta)(1+o(1))r^n}(1 + o(1)), & \delta_n(A, \theta) > 0, \\ O(1), & \delta_n(A, \theta) < 0; \end{cases} \tag{14}$$

and the coefficients in equations (11) and (13), we have

$$\left|P_j(e^{A(z)}) + Q_j(e^{-A(z)})\right| = \begin{cases} |a_{jm_j}| e^{m_j \delta_n(A,\theta)(1+o(1))r^n}(1+o(1)), & \delta_n(A, \theta) > 0, \\ |b_{jn_j}| e^{-n_j \delta_n(A,\theta)(1+o(1))r^n}(1+o(1)), & \delta_n(A, \theta) < 0. \end{cases} \tag{15}$$

The following lemma plays an important role in uniqueness problems of meromorphic functions.

Lemma 2.1 ([14]) *Let $f_j(z)$ $(j = 1, \ldots, n)$ $(n \geq 2)$ be meromorphic functions, and let $g_j(z)$ $(j = 1, \ldots, n)$ be entire functions satisfying*

(i) $\sum_{j=1}^n f_j(z) e^{g_j(z)} \equiv 0$;
(ii) *when $1 \leq j < k \leq n$, then $g_i(z) - g_k(z)$ is not a constant;*
(iii) *when $1 \leq j \leq n$, $1 \leq h < k \leq n$, then*

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\} \quad (r \to \infty, r \notin E),$$

where $E \subset (1, \infty)$ is of finite linear measure or logarithmic measure.

Then, $f_j(z) \equiv 0$ $(j = 1, \ldots, n)$.

The following three lemmas are of great importance in estimating the growth of the ratio of two derivatives of a meromorphic function.

Lemma 2.2 ([9]) *Let $f(z)$ be an entire function, and suppose that $|f^{(k)}(z)|$ is unbounded on some ray $\arg z = \theta$. Then, there exists an infinite sequence of points $z_n = r_n e^{i\theta}$ $(n = 1, 2, \ldots)$, where $r_n \to \infty$, such that $f^{(k)}(z_n) \to \infty$ and*

$$\left|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}\right| \leq |z_n|^{(k-j)}(1 + o(1)), \quad j = 0, \ldots, k-1. \tag{16}$$

Lemma 2.3 ([8]) *Let f be a transcendental meromorphic function and $\alpha > 1$ be a given constant. Then there exist a set $E \subset (1, \infty)$ with finite logarithmic measure and a constant $B > 0$ that depends only on α and i, j $(0 \leq i < j)$, such that for all z satisfying $|z| = r \notin E \cup [0, 1]$,*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B\left(\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\right)^{j-i}. \tag{17}$$

Remark 1 From the proof of Lemma 2.3 ([8, Theorem 3]), we can see that the exceptional set E equals $\{|z| : z \in (\cup_{n=1}^{+\infty} O(a_n))\}$, where $a_n (n = 1, 2, \ldots)$ denote all zeros and poles of $f^{(i)}$, and $O(a_n)$ denote sufficiently small neighborhoods of a_n. Hence, if $f(z)$ is a transcendental entire function and z is a point that satisfies $|f(z)|$ to be sufficiently large, then the point $z \notin E$ thus (17) holds for these kinds of z.

Particularly, when $\sigma(f) < \infty$, then we have the following result.

Lemma 2.4 ([8]) *Let $f(z)$ be a transcendental meromorphic function with $\sigma(f) = \sigma < \infty$, and let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset [0, 2\pi)$ of linear measure zero (or a set $E_1 \subset (1, \infty)$ of finite logarithmic measure) such that for all $z = re^{i\theta}$ with r sufficiently large and $\theta \in [0, 2\pi) \setminus E$ (or $r \notin E_1 \cup [0, 1]$), and for all $k, j, 0 \leq j \leq k$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}. \tag{18}$$

The following lemma is often used to prove that a function is polynomial and to determine the degree of this polynomial.

Lemma 2.5 ([3]) *Let $f(z)$ be an entire function with $\sigma(f) = \sigma < \infty$. Let there exist a set $E \subset [0, 2\pi)$ with linear measure zero such that for any $\arg z = \theta_0 \in [0, 2\pi) \setminus E$, $|f(re^{i\theta_0})| \leq Mr^k$ ($M = M(\theta_0) > 0$ is a constant and $k(> 0)$ is constant independent of θ_0). Then $f(z)$ is a polynomial of $\deg f \leq k$.*

The following lemma, which is a revised version of [7, Lemma 7], can estimate the central index of an entire function.

Lemma 2.6 ([7]) *Let $f(z)$ be an entire function that satisfies $\sigma(f) = \sigma (n < \sigma < \infty)$; or $\sigma(f) = \infty$ and $\sigma_2 = 0$; or $\sigma_2 = \alpha (0 < \alpha < \infty)$, and a set $E \subset [1, \infty)$ has a finite logarithmic measure. Then, there exists $\{z_k = r_k e^{i\theta_k}\}$, such that $|f(z_k)| = M(r_k, f)$, $\theta_k \in [0, 2\pi)$, $\lim_{k\to\infty} \theta_k = \theta_0 \in [0, 2\pi)$, $r_k \notin E$, and $r_k \to \infty$, such that*

(i) *if* $\sigma(f)=\sigma(n<\sigma<\infty)$*, then for any given* $\varepsilon_1(0<\varepsilon_1<\frac{\sigma-n}{2})$*,*

$$r_k^{\sigma-\varepsilon_1}<\nu(r_k)<r_k^{\sigma+\varepsilon_1}; \tag{19}$$

(ii) *if* $\sigma(f)=\infty$ *and* $\sigma_2(f)=0$*, then for any given* $\varepsilon_2(0<\varepsilon_2<\frac{1}{2})$*, and any large* $M(>0)$*, we have, as* r_k *is sufficiently large,*

$$r_k^M<\nu(r_k)<\exp\{r_k^{\varepsilon_2}\}; \tag{20}$$

(iii) *if* $\sigma_2(f)=\alpha(0<\alpha<\infty)$*, then for any given* $\varepsilon_3(0<\varepsilon_3<\alpha)$*,*

$$\exp\{r_k^{\alpha-\varepsilon_3}\}<\nu(r_k)<\exp\{r_k^{\alpha+\varepsilon_3}\}. \tag{21}$$

Proof By $\sigma(f)=\sigma$, we have

$$\limsup_{r\to\infty}\frac{\log\nu(r)}{\log r}=\sigma.$$

Thus, there exists a sequence $\{r_k'\}$ $(r_k'\to\infty)$ satisfying

$$\lim_{r_k'\to\infty}\frac{\log\nu(r_k')}{\log r_k'}=\sigma.$$

Set $lmE=\delta<\infty$. Then the interval $[r_k',(1+e^\delta)r_k']$ meets the complement of E since

$$\int_{r_k'}^{(1+e^\delta)r_k'}\frac{dt}{t}=\log(1+e^\delta)r_k'-\log r_k'=\log(1+e^\delta)>\delta.$$

Therefore, there exists a point $r_k\in[r_k',(1+e^\delta)r_k']\setminus E$. Because

$$\frac{\log\nu(r_k)}{\log r_k}\geq\frac{\log\nu(r_k')}{\log[(1+e^\delta)r_k']}=\frac{\log\nu(r_k')}{\left(1+\frac{\log(1+e^\delta)}{\log r_k'}\right)\log r_k'},$$

and

$$\frac{\log\nu(r_k)}{\log r_k}\leq\frac{\log\nu[(1+e^\delta)r_k']}{\log r_k'}=\frac{\log\nu[(1+e^\delta)r_k']}{\left(1-\frac{\log(1+e^\delta)}{\log[(1+e^\delta)r_k']}\right)\log[(1+e^\delta)r_k']},$$

we have

$$\lim_{r_k\to\infty}\frac{\log\nu(r_k)}{\log r_k}=\sigma. \tag{22}$$

Now take $z_k = r_k e^{i\theta_k}$, $\theta_k \in [0, 2\pi)$ such that $|f(z_k)| = M(r_k, f)$. Thus, there exists a subset of $\{\theta_k\}$, for convenience, we still denote it by $\{\theta_k\}$, and it satisfies $\lim_{k\to\infty} \theta_k = \theta_0 \in [0, 2\pi)$. So we obtain (19) by (22).

Using a similar method as above, we can prove (ii) and (iii). □

The following two lemmas give relationships between the solutions and the coefficients of some given linear differential equations.

Lemma 2.7 ([7]) *Let $A_0, \ldots, A_{k-1}$ be entire functions of finite order. If $f(z)$ is a solution of equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_0 f = 0,$$

then $\sigma_2(f) \leq \max\{\sigma(A_j) : j = 0, \ldots, k-1\}$.

Lemma 2.8 ([14]) *Let $f_1, \ldots, f_n$ be linearly independent meromorphic solutions of*

$$f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_0 f = 0$$

with meromorphic coefficients. Then the Wronskian determinant $W(f_1, \ldots, f_n)$ satisfies the differential equation $W' + a_{n-1}(z)W = 0$. In particular, if a_{n-1} is an entire function, then for some $C \in \mathbb{C}\backslash\{0\}$, $W(f_1, \ldots, f_n) = C \exp\varphi$, where φ is a primitive function of $-a_{n-1}$.

The following lemma is often used to exclude an exceptional set.

Lemma 2.9 ([1,12]) *Let $g : (0, +\infty) \to R$ and $h : (0, +\infty) \to R$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite logarithmic measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ holds for all $r > r_0$.*

Lemmas 2.10 and 2.11 are used to prove Theorem 1.1.

Lemma 2.10 *Let P_j, m_j, m_s satisfy the hypotheses of Theorem 1.1. If f is a solution of (4) and $\sigma(f) < n$, then*

(i) *if $P_0 \not\equiv 0$, then $f \equiv 0$;*
(ii) *if $P_0 \equiv P_1 \equiv \cdots \equiv P_{d-1} \equiv 0$ and $P_d \not\equiv 0\,(d < s)$, then f is a polynomial with $\deg f \leq d - 1$.*

Proof (i) Suppose that f is a solution of (4) with $\sigma(f) < n$, then f is an entire function. For convenience, we denote

$$P_j(e^{A(z)}) = a_{jm_s} e^{m_s A(z)} + \cdots + a_{j(m_j+1)} e^{(m_j+1)A(z)} + a_{jm_j} e^{m_j A(z)} + \cdots + a_{j1} e^{A(z)} + a_{j0},$$

where $a_{jm_j} \neq 0$ and $a_{jm_s} = \cdots = a_{j(m_j+1)} = 0$. Thus, (4) can be rewritten as

$$Q_{m_s}(z) e^{m_s A(z)} + Q_{m_s-1}(z) e^{(m_s-1)A(z)} + \cdots + Q_1(z) e^{A(z)} + Q_0(z) = 0, \quad (23)$$

where

$$\begin{cases} Q_{m_s}(z) = a_{sm_s} f^{(s)}(z), \\ Q_j(z) = \sum_{t=0}^{k-1} a_{tj} f^{(t)}(z), \ \ j = 1, \dots, m_s - 1, \\ Q_0(z) = f^{(k)}(z) + \sum_{t=0}^{k-1} a_{t0} f^{(t)}(z). \end{cases} \tag{24}$$

Obviously, Q_γ ($\gamma = 0, 1, \dots, m_s$) satisfy $\sigma(Q_\gamma) < n$. Since $e^{(\alpha-\beta)A(z)}$ ($\alpha, \beta \in \{0, \dots, m_s\}, 0 \le \beta < \alpha \le m_s$) is of regular growth, we have

$$T(r, Q_\gamma) = o\{T(r, e^{(\alpha-\beta)A(z)})\}, \ \gamma = 0, \dots, m_s.$$

Thus, by applying Lemma 2.1 to (23), we have

$$f^{(s)}(z) \equiv 0, \quad Q_{m_s-1}(z) \equiv \cdots \equiv Q_1(z) \equiv Q_0(z) \equiv 0. \tag{25}$$

By $f^{(s)} \equiv 0$, we get that f is a polynomial with $\deg f \le s - 1$ if $s > 0$ and $f \equiv 0$ if $s = 0$. From $P_0 \not\equiv 0$, we get that f cannot be a nonzero constant from (4). Let $\deg f = h$ $(1 \le h \le s - 1)$. It follows from (6), Equations (24), (25), and the facts

$$\deg f > \deg f' > \cdots > \deg f^{(h)}, \quad f^{(h+1)} \equiv f^{(h+2)} \equiv \cdots \equiv f^{(k)} \equiv 0,$$

that $a_{00} = a_{01} = \cdots = a_{0m_0} = 0$ holds. Therefore, $P_0 \equiv 0$, which contradicts our assumption $P_0 \not\equiv 0$. Hence, $f \equiv 0$.

(ii) Let $f^{(d)} = F$, since $P_0 \equiv P_1 \equiv \cdots \equiv P_{d-1} \equiv 0$ and $P_d \not\equiv 0\,(d < s)$, from (4) we get

$$F^{(k-d)} + P_{k-1}(e^{A(z)})F^{(k-d-1)} + \cdots + P_d(e^{A(z)})F = 0.$$

By $\sigma(F) = \sigma(f^{(d)}) = \sigma(f) < n$ and $P_d \not\equiv 0$, we apply the result from Lemma 2.10 (i) to conclude that $f^{(d)} = F \equiv 0$. So, f is a polynomial with $\deg f \le d - 1$. □

Lemma 2.11 *Let $P_j(z)$ $(j = 0, \dots, k-1)$ be polynomials in z with* $\deg P_j = m_j$. *Suppose that there exists an integer s $(s \in \{0, \dots, k-1\})$ satisfying (6), and let $\{f_1, \dots, f_k\}$ be a fundamental solution set of (4). If each f_j satisfies either $\sigma(f_j) \le n$ or $\sigma_2(f_j) = n$, then there are at most s solutions, say $f_1, \dots, f_s$, satisfy $\sigma(f_j) \le n$ $(j = 1, \dots, s)$.*

Proof Assume that $f_1, \dots, f_s, f_{s+1}$ satisfy $\sigma(f_j) \le n$ $(j = 1, \dots, s+1)$, and $f_{s+2}, \dots, f_k$ satisfy $\sigma_2(f_j) = n$ $(j = s+2, \dots, k)$. Now we apply the order reduction procedure and deduce a contradiction. For convenience, we use the notation u_k instead of f in equation (4), $u_{k,1}, \dots, u_{k,k}$ instead of $f_1, \dots, f_k$, $P_{k,0}, \dots, P_{k,(k-1)}$ instead of $P_0, \dots, P_{k-1}$, respectively. Thus,

$$\sigma(u_{k,j}) = \sigma(f_j) \le n, \ j = 1, \dots, s+1, \ \ \sigma_2(u_{k,j}) = \sigma_2(f_j) = n, \ j = s+2, \dots, k.$$

Set

$$u_{k-1}(z) = \frac{d}{dz}\left(\frac{u_k(z)}{u_{k,1}(z)}\right), \ u_{k-1,j}(z) = \frac{d}{dz}\left(\frac{u_{k,j}(z)}{u_{k,1}(z)}\right), \ \ j = 2, \dots, k.$$

We denote $u_{k-1}^{(-1)}$ to be the primitive function of u_{k-1}. Thus, $\left(u_{k-1}^{(-1)}\right)' = u_{k-1}$, $u_k = u_{k,1}u_{k-1}^{(-1)}$, and

$$u_k^{(j)} = \sum_{t=0}^{j} \mathrm{C}_j^t u_{k,1}^{(t)} u_{k-1}^{(j-1-t)}, \quad j = 0, \ldots, k, \tag{26}$$

where C_j^t are the binomial coefficients. Substituting (26) into (4), we obtain

$$\sum_{t=0}^{k} \mathrm{C}_k^t u_{k,1}^{(t)} u_{k-1}^{(k-1-t)} + \sum_{j=1}^{k-1} P_{k,j} \sum_{t=0}^{j} \mathrm{C}_j^t u_{k,1}^{(t)} u_{k-1}^{(j-1-t)} + P_{k,0} u_{k,1} u_{k-1}^{(-1)} = 0. \tag{27}$$

Rearranging the sums of (27), we obtain

$$\begin{aligned} &u_{k,1} u_{k-1}^{(k-1)} + (k u_{k,1}' + P_{k,k-1} u_{k,1}) u_{k-1}^{(k-2)} \\ &\quad + \sum_{j=0}^{k-3} \Bigg(\sum_{t=0}^{k-j-1} \mathrm{C}_{j+1+t}^t P_{k,j+1+t} u_{k,1}^{(t)} \Bigg) u_{k-1}^{(j)} \\ &\quad + u_{k-1}^{(-1)} \left(u_{k,1}^{(k)} + P_{k,k-1} u_{k,1}^{(k-1)} + \cdots + P_{k,0} u_{k,1} \right) = 0. \end{aligned} \tag{28}$$

Since $u_{k,1} \not\equiv 0$ is a solution of (4), by (28) we obtain

$$u_{k-1}^{(k-1)} + P_{k-1,k-2}(z) u_{k-1}^{(k-2)} + \cdots + P_{k-1,0}(z) u_{k-1} = 0, \tag{29}$$

where

$$P_{k-1,j}(z) = P_{k,j+1}(z) + \sum_{t=1}^{k-j-1} \mathrm{C}_{j+1+t}^t P_{k,j+1+t}(z) \frac{u_{k,1}^{(t)}(z)}{u_{k,1}}, \; j = 0, \ldots, k-2.$$

Now we examine the growth of $P_{k-1,j}$ ($j = 0, 1, \ldots, k-2$), particularly $P_{k-1,s-1}$. Since $\sigma(u_{k,1}) \le n$, by Lemma 2.4, there exists a set $E \subset (1, +\infty)$ with finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E$,

$$\left| \frac{u_{k,1}^{(t)}(z)}{u_{k,1}(z)} \right| \le r^{kn}, \; t = 1, \ldots, k-1.$$

Take a ray $\arg z = \theta \in H_{n,+}$. From (14), as $r \to \infty$,

$$\begin{aligned} |P_{k,s}(z)| &= |a_{sm_s}| e^{m_s \delta_n(A,\theta)(1+o(1)) r^n} (1 + o(1)), \\ |P_{k,j}(z)| &= O(e^{\widetilde{m} \delta_n(A,\theta)(1+o(1)) r^n}), \quad j \neq s. \end{aligned}$$

Therefore, we get that for all z satisfying $\arg z = \theta \in H_{n,+}$, as $r \to \infty$ and $r \notin [0, 1] \cup E$,

$$\begin{cases} |P_{k-1,s-1}(z)| = |a_{sm_s}| e^{m_s \delta_n(A,\theta)(1+o(1))r^n}(1+o(1)), \\ |P_{k-1,j}(z)| = O(r^{kn} e^{\widetilde{m}\delta_n(A,\theta)(1+o(1))r^n}), \quad j = s, \dots, k-2; \\ |P_{k-1,j}(z)| = O(r^{kn} e^{m_s \delta_n(A,\theta)(1+o(1))r^n}), \quad j = 0, \dots, s-2. \end{cases} \tag{30}$$

We now consider the growth order of $u_{k-1,j} = \frac{d}{dz}(\frac{u_{k,j}}{u_{k,1}})$ $(j = 2, \dots, k)$. Since $\sigma(u_{k,j}) \le n$ $(j = 1, \dots, s+1)$ and $\sigma_2(u_{k,j}) = n$ $(j = s+2, \dots, k)$, we see that

$$\sigma(u_{k-1,j}) \le n,\ j = 2, \dots, s+1,\ \sigma_2(u_{k-1,j}) = n,\ j = s+2, \dots, k. \tag{31}$$

Suppose that $c_2, \dots, c_k$ are constants such that

$$c_2 u_{k-1,2} + \cdots + c_k u_{k-1,k} = c_2 \left(\frac{u_{k,2}}{u_{k,1}}\right)' + \cdots + c_k \left(\frac{u_{k,k}}{u_{k,1}}\right)' = 0; \tag{32}$$

by integrating both sides of (32), we get

$$c_2 u_{k,2} + \cdots + c_k u_{k,k} + c_1 u_{k,1} = 0,$$

where c_1 is a constant. Since $u_{k,1}, \dots, u_{k,k}$ are linearly independent, $c_1 = \ldots = c_k = 0$; hence, $\{u_{k-1,2}, \dots, u_{k-1,k}\}$ is a fundamental solution set of (29).

Next, we repeat the order reduction procedure as above to Eq. (29). After s order reduction procedures, we get

$$u_{k-s}^{(k-s)} + P_{k-s,k-s-1} u_{k-s}^{(k-s-1)} + \cdots + P_{k-s,0} u_{k-s} = 0. \tag{33}$$

On a ray $\arg z = \theta \in H_{n,+}$, as $r \to \infty$ and $r \notin [0, 1] \cup E$,

$$\begin{cases} |P_{k-s,0}(z)| = |a_{sm_s}| e^{m_s \delta_n(A,\theta)(1+o(1))r^n}(1+o(1)), \\ |P_{k-s,j}(z)| = O(r^{kn} e^{\widetilde{m}\delta_n(A,\theta)(1+o(1))r^n}),\ j = 1, \dots, k-s-1. \end{cases} \tag{34}$$

Also,

$$u_{k-s,j} = \frac{d}{dz}\left(\frac{u_{k-(s-1),j}}{u_{k-(s-1),s}}\right),\ j = s+1, \dots, k,$$

are $k - s$ linearly independent solutions of (33) that satisfy

$$\sigma(u_{k-s,s+1}) \le n, \quad \sigma_2(u_{k-s,j}) = n,\ j = s+2, \dots, k. \tag{35}$$

On the other hand, for a solution u_{k-s} of (33), by Lemma 2.3, there exists a set $E_0 \subset (1, \infty)$ with a finite logarithmic measure such that for all z satisfying $r \notin [0, 1] \cup E_0$,

$$\left|\frac{u_{k-s}^{(t)}(z)}{u_{k-s}(z)}\right| \le M[T(2r, u_{k-s})]^{k-s+1},\ t = 1, \dots, k-s, \tag{36}$$

where $M(>0)$ is a constant. So by (33), (34), and (36), we obtain that for all z satisfying $\arg z = \theta \in H_{n,+}$, as $r \to \infty$ and $r \notin [0,1] \cup E_0$,

$$\begin{aligned}
&|a_{sm_s}| e^{m_s \delta_n(A,\theta)(1+o(1))r^n}(1+o(1)) = |P_{k-s,0}(z)| \\
&\leq \left|\frac{u_{k-s}^{(k-s)}(z)}{u_{k-s}(z)}\right| + \left|P_{k-s,k-s-1}(z)\frac{u_{k-s}^{(k-s-1)}(z)}{u_{k-s}(z)}\right| + \cdots + \left|P_{k-s,1}(z)\frac{u_{k-s}'(z)}{u_{k-s}(z)}\right| \\
&\leq MM'[T(2r,u_{k-s})]^{k-s+1} r^{kn} e^{\widetilde{m}\delta_n(A,\theta)(1+o(1))r^n}, \ \ M'(>0) \text{ is a constant.}
\end{aligned}$$

Thus, we have

$$|a_{sm_s}| r^{-kn} e^{(m_s-\widetilde{m})\delta_n(A,\theta)(1+o(1))r^n}(1+o(1)) \leq MM'[T(2r,u_{k-s})]^{k-s+1}.$$

The above inequality and Lemma 2.9 imply that $\sigma_2(u_{k-s}) \geq n$, i.e., all solutions u_{k-s} of (33) satisfy $\sigma_2(u_{k-s}) \geq n$, which contradicts $\sigma(u_{k-s,s+1}) \leq n$. Thus, Lemma 2.11 is proved. □

3 Proof of Theorem 1.1

(i) *First step* We prove that every transcendental n-subnormal solution is of $\sigma(f_0) = n$. By Lemma 2.10, we see that if $\sigma(f_0) < n$, then f_0 is a polynomial with $\deg f_0 \leq s-1$. So $\sigma(f_0) \geq n$.

Suppose that $\sigma(f_0) > n$, and we will show that this supposition will lead to a contradiction next.

By Lemma 2.3, there exists a subset $E_1 \subset (1,\infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin E_1 \cup [0,1]$,

$$\left|\frac{f_0^{(j)}(z)}{f_0^{(i)}(z)}\right| \leq M[T(2r,f_0)]^{k+1}, \ i,j \in \{0,1,\ldots,k\}, \ i<j, \tag{37}$$

where $M(>0)$ is a constant.

By the Wiman-Valiron theory, there is a set $E_2 \subset (1,\infty)$ with finite logarithmic measure, such that for all z satisfying $|z| = r \notin E_2 \cup [0,1]$, and $|f_0(z)| = M(r,f_0)$,

$$\frac{f_0^{(j)}(z)}{f_0(z)} = \left(\frac{\nu(r)}{z}\right)^j (1+o(1)), \ j=1,\ldots,k. \tag{38}$$

By Lemma 2.6 and $\sigma(f_0) > n$, we see that there exists a sequence $\{z_t = r_t e^{i\theta_t}\}$ such that $|f_0(z_t)| = M(r_t,f_0)$, $\theta_t \in [0,2\pi)$, $\lim_{t\to\infty}\theta_t = \theta_0 \in [0,2\pi)$, with $r_t \notin E_1 \cup E_2 \cup [0,1]$, $r_t \to \infty$, $\{z_t\}$ satisfies (38), and for any given ε_1 $(0 < \varepsilon_1 < \min\{\frac{1}{4}(\sigma-n),1\})$,

$$\nu(r_t) > r_t^{\sigma-\varepsilon_1} > r_t^{n+\varepsilon_1} \geq r_t^{1+\varepsilon_1} > r_t. \tag{39}$$

Since θ_0 may belong to $H_{n,+}$, $H_{n,-}$, or $H_{n,0}$, we divide this proof into three cases.

Case 1 Suppose $\theta_0 \in H_{n,+}$. Since $\delta_n(A,\theta) = \alpha_n \cos(\theta_n + n\theta)$ is a continuous function of θ, and $\theta_t \to \theta_0$, thus we have $\lim_{t\to\infty} \delta_n(A,\theta_t) = \delta_n(A,\theta_0) > 0$. Therefore, there exists a constant $N(>0)$, such that as $t > N$,

$$\delta_n(A,\theta_t) \geq \frac{1}{2}\delta_n(A,\theta_0) > 0.$$

By (5), for any given ε_4 $(0 < \varepsilon_4 < \frac{1}{2^{n+2}(k+1)}\delta_n(A,\theta_0))$,

$$[T(2r_t, f_0)]^{k+1} \leq e^{\varepsilon_4(k+1)(2r_t)^n} \leq e^{\frac{1}{2}\delta_n(A,\theta_t)r_t^n} \tag{40}$$

holds for $t > N$.

By (37), (38), and (40), we see that

$$\left(\frac{\nu(r_t)}{r_t}\right)^{k-s}(1+o(1)) = \left|\frac{f_0^{(k-s)}(z_t)}{f_0(z_t)}\right| \leq M[T(2r_t, f_0)]^{k+1} \leq Me^{\frac{1}{2}\delta_n(A,\theta_t)r_t^n}. \tag{41}$$

By (4), we get

$$-\frac{f_0^{(s)}(z_t)}{f_0(z_t)}P_s(e^{A(z_t)}) = \frac{f_0^{(k)}(z_t)}{f_0(z_t)} + \sum_{j=0, j\neq s}^{k-1} P_j(e^{A(z_t)})\frac{f_0^{(j)}(z_t)}{f_0(z_t)}. \tag{42}$$

Because $\delta_n(A,\theta_t) > 0$ as $t > N$, from (14) we get that

$$|P_s(e^{A(z_t)})| = |a_{sm_s}|e^{m_s\delta_n(A,\theta_t)(1+o(1))r_t^n}(1+o(1)), \tag{43}$$

and

$$|P_j(e^{A(z_t)})| \leq M_0 e^{\widetilde{m}\delta_n(A,\theta_t)(1+o(1))r_t^n},\ j=0,\ldots,k-1,\ j\neq s,\ M_0(>0) \text{is a constant}. \tag{44}$$

Substituting (38), (43), and (44) into (42), we get that for sufficiently large r_t,

$$\begin{aligned}\left(\frac{\nu(r_t)}{r_t}\right)^{s}|a_{sm_s}|e^{m_s\delta_n(A,\theta_t)(1+o(1))r_t^n}(1+o(1)) &\leq \left(\frac{\nu(r_t)}{r_t}\right)^{k}(1+o(1))\\ +M_0 e^{\widetilde{m}\delta_n(A,\theta_t)(1+o(1))r_t^n} \sum_{j=0, j\neq s}^{k-1}\left(\frac{\nu(r_t)}{r_t}\right)^{j}(1+o(1)).\end{aligned} \tag{45}$$

By (41), (45), and (39), we get

$$\begin{aligned}|a_{sm_s}|e^{(m_s-\widetilde{m})\delta_n(A,\theta_t)(1+o(1))r_t^n}(1+o(1)) &\leq kM_0\left(\frac{\nu(r_t)}{r_t}\right)^{k-s}(1+o(1))\\ &\leq kM_0 M e^{\frac{1}{2}\delta_n(A,\theta_t)r_t^n},\end{aligned}$$

which yields a contradiction by $m_s - \widetilde{m} \geq 1 > \frac{1}{2}$, and $\delta_n(A, \theta_t) > 0$.

Case 2 Suppose $\theta_0 \in H_{n,-}$. Then $\delta_n(A, \theta_0) < 0$. By using the similar method as in Case 1, we get that for sufficiently large t, $\delta_n(A, \theta_t) < 0$ as $\theta_t \to \theta_0$. From (14), there exists a constant M_1 (> 0), such that

$$|P_j(e^{A(z_t)})| \leq M_1, \ \ j = 0, \ldots, k-1. \tag{46}$$

By (4), (38), (39), and (46), we get

$$\left(\frac{\nu(r_t)}{r_t}\right)^k (1 + o(1)) = \left|\frac{f_0^{(k)}(z_t)}{f_0(z_t)}\right| \leq kM_1 \left(\frac{\nu(r_t)}{r_t}\right)^{k-1} (1 + o(1)),$$

i.e.,

$$\nu(r_t)(1 + o(1)) \leq kM_1 r_t (1 + o(1)),$$

which also yields a contradiction by (39).

Case 3 Suppose $\theta_0 \in H_{n,0}$. Since $\theta_t \to \theta_0$, for any given ε_5 $(0 < \varepsilon_5 < \frac{1}{10n})$, we see that there exists an integer N_1 (> 0), as $t > N_1$, $\theta_t \in [\theta_0 - \varepsilon_5, \theta_0 + \varepsilon_5]$, and

$$z_t = r_t e^{i\theta_t} \in \overline{\Omega} = \{z : \theta_0 - \varepsilon_5 \leq \arg z \leq \theta_0 + \varepsilon_5\}.$$

Now, we consider the growth of $f_0(re^{i\theta})$ on a ray $\arg z = \theta \in \overline{\Omega} \setminus \{\theta_0\}$.

By the properties of cosine function, we can easily see that when $\theta_1 \in [\theta_0 - \varepsilon_5, \theta_0)$ and $\theta_2 \in (\theta_0, \theta_0 + \varepsilon_5]$, then $\delta_n(A, \theta_1)\delta_n(A, \theta_2) < 0$. Without loss of generality, we suppose that $\delta_n(A, \theta) > 0$ for $\theta \in [\theta_0 - \varepsilon_5, \theta_0)$ and $\delta_n(A, \theta) < 0$ for $\theta \in (\theta_0, \theta_0 + \varepsilon_5]$.

For a fixed $\theta \in [\theta_0 - \varepsilon_5, \theta_0)$, we have $\delta_n(A, \theta) > 0$. By (5), for any given ε_6 satisfying $0 < \varepsilon_6 < \frac{1}{2^{n+1}(k+1)}\delta_n(A, \theta)$,

$$[T(2r, f_0)]^{k+1} \leq e^{\varepsilon_6(k+1)(2r)^n} \leq e^{\frac{1}{2}\delta_n(A,\theta)r^n}. \tag{47}$$

We assert that $|f_0^{(s)}(re^{i\theta})|$ is bounded on the ray $\arg z = \theta$. If $|f_0^{(s)}(re^{i\theta})|$ is unbounded on it, then by Lemma 2.2, there exists a sequence $\{y_j = R_j e^{i\theta}\}$, such that as $R_j \to \infty$, $f_0^{(s)}(y_j) \to \infty$ and

$$\left|\frac{f_0^{(d)}(y_j)}{f_0^{(s)}(y_j)}\right| \leq (R_j)^{s-d}(1 + o(1)), \ d = 0, \ldots, s-1. \tag{48}$$

By Remark 1, $f_0^{(s)}(y_j) \to \infty$, we know that y_j satisfies (37). By (37) and (47), we see that for sufficiently large j,

$$\left|\frac{f_0^{(d)}(y_j)}{f_0^{(s)}(y_j)}\right| \leq M[T(2R_j, f_0)]^{k+1} \leq Me^{\frac{1}{2}\delta_n(A,\theta)R_j^n}, \ d = s+1, \ldots, k. \tag{49}$$

By (4), (43), (44), (48), (49), and $\delta_n(A,\theta) > 0$, we deduce that

$$\begin{aligned}
&|a_{sm_s}|e^{m_s\delta_n(A,\theta)(1+o(1))R_j^n}(1+o(1)) = |P_s(e^{A(y_j)})|\\
&\le M_2 e^{(\widetilde{m}+\frac{1}{2})\delta_n(A,\theta)(1+o(1))R_j^n} + M_3 e^{\widetilde{m}\delta_n(A,\theta)(1+o(1))R_j^n} R_j^s\\
&\le \max\{M_2, M_3\} R_j^s e^{(\widetilde{m}+\frac{1}{2})\delta_n(A,\theta)(1+o(1))R_j^n},
\end{aligned}$$

where $M_2, M_3\ (>0)$ are constants, which yields a contradiction by the facts $m_s - \widetilde{m} \ge 1 > \frac{1}{2}$ and $\delta_n(A,\theta) > 0$. Thus, $|f_0^{(s)}(re^{i\theta})| \le M_4$ ($M_4 > 0$ is a constant) on the ray $\arg z = \theta$. Since

$$f_0^{(s-1)}(re^{i\theta}) = f_0^{(s-1)}(0) + \int_0^r f_0^{(s)}(te^{i\theta})dt,$$

we have $|f_0^{(s-1)}(re^{i\theta})| \le M_5 r$ ($M_5 > 0$ is a constant). By induction, we obtain

$$|f_0(re^{i\theta})| \le M_6 r^s,\ \ M_6\ (>0)\ \text{ is a constant}, \tag{50}$$

on the ray $\arg z = \theta \in [\theta_0 - \varepsilon_5, \theta_0)$.

On the other hand, since $\{z_t\}$ satisfies $|f_0(z_t)| = M(r_t, f_0)$ and $\sigma(f_0) > n$, we see that for sufficiently large r_t and $r_t \notin E_1 \cup E_2 \cup [0,1]$,

$$|f_0(z_t)| \ge \exp\{r_t^n\}. \tag{51}$$

By (50) and (51), we see that for sufficiently large t, $\theta_t \notin [\theta_0 - \varepsilon_5, \theta_0)$, i.e.,

$$\theta_t \in [\theta_0, \theta_0 + \varepsilon_5]\,. \tag{52}$$

Suppose there are infinitely many θ_t in $(\theta_0, \theta_0 + \varepsilon_5]$. Then we can choose a subsequence $\{\theta_{t_j}\}$ of $\{\theta_t\}$ and a corresponding subsequence $\{z_{t_j} = r_{t_j}e^{i\theta_{t_j}}\}$ of $\{z_t\}$. For the subsequence $\{z_{t_j}\} \subset \{z : \theta_0 < \arg z \le \theta_0 + \varepsilon_5)$, by using a similar method to that in the proof of Case 2, we can get

$$\nu(r_{t_j}) \le kM_1 r_{t_j},$$

which yields a contradiction by (39).

Hence, there are only finitely many $\theta_t \in (\theta_0, \theta_0 + \varepsilon_5]$, and for sufficiently large t, $\theta_t = \theta_0$ and $\delta_n(A, \theta_t) = 0$.

Next we consider the following three subcases. $\theta_0 \in H_{n-1,+}$; $\theta_0 \in H_{n-1,-}$; $\theta_0 \in H_{n-1,0}$. If $\theta_0 \in H_{n-1,+}$ or $\theta_0 \in H_{n-1,-}$, then by using a similar method as in Case 1 and Case 2, we can get a contradiction. If $\theta_0 \in H_{n-1,0}$, then from the similar reasoning as used in Case 3, the remaining case is $\theta_t = \theta_0$ for sufficiently large t. This gives that $\delta_{n-1}(A,\theta_t) = \alpha_{n-1}\cos(\theta_{n-1} + (n-1)\theta_t) = 0$ for sufficiently large t. On the analogue by this, the remaining case is that $\delta_j(A,\theta_t) = \alpha_j\cos(\theta_j + j\theta_t) = 0$ for sufficiently large t, where $j = 1, \ldots, n-2$.

Therefore, for sufficiently large t, we get

$$\begin{aligned}|P_j(e^{A(z_t)})| &= |a_{jm_j}e^{m_jA(z_t)} + \cdots + a_{j1}e^{A(z_t)} + a_{j0}| \\ &\le |a_{jm_j}|e^{m_j|d_0|} + \cdots + |a_{jm_1}|e^{|d_0|} + |a_{j0}| \\ &\le M_7,\ j = 0, \cdots, k-1, \end{aligned} \tag{53}$$

where M_7 (> 0) is a constant.

By (4), (38), (39), and (53), we get that

$$\left|-\left(\frac{\nu(r_t)}{z_t}\right)^k (1+o(1))\right| = \left|-\frac{f_0^{(k)}(z_t)}{f_0(z_t)}\right| \le kM_7\left(\frac{\nu(r_t)}{r_t}\right)^{k-1}(1+o(1)),$$

i.e.,

$$\nu(r_t)(1+o(1)) \le kM_7 r_t(1+o(1)),$$

which also yields a contradiction by (39).

Second step We prove that all other solutions f of (4) satisfy $\sigma_2(f) = n$. Suppose that f is not a n-subnormal solution, and $\sigma_2(f) < n$. Then clearly $\limsup_{r\to\infty} \frac{\log T(r,f)}{r^n} = 0$, i.e., f is a n-subnormal solution, a contradiction. So, $\sigma_2(f) \ge n$. By Lemma 2.7 and $\sigma(P_j) = n$ $(j = 0, \ldots, k-1)$, we have $\sigma_2(f) \le n$. Hence, $\sigma_2(f) = n$.

(ii) By the assertion of (i), we see that a solution of (4) either is a polynomial, or satisfies $\sigma(f) = n$ or $\sigma_2(f) = n$. Thus, by Lemma 2.11, we see (ii) holds.

4 Proof of Theorem 1.2

Suppose that $f \not\equiv 0$ is a solution of (11), then f is an entire function. Because $P_0 + Q_0 \not\equiv 0$, f cannot be a constant. Suppose that $f = b_h z^h + \cdots + b_1 z + b_0$ ($h \ge 1$, $b_h, \ldots, b_0$ are constants, $b_h \ne 0$) is a polynomial solution of (11), substituting this polynomial solution into (11), we get that the coefficient of the highest degree of z, i.e., z^n, is $\left(P_0\left(e^{A(z)}\right) + Q_0\left(e^{-A(z)}\right)\right)$ ($\not\equiv 0$), which yields a contradiction from (11). Thus, we get the conclusion that f is transcendental.

Step one We prove that $\sigma(f) = \infty$.

Suppose, to the contrary, that $\sigma(f) = \sigma < \infty$. By Lemma 2.4, for any given $\varepsilon > 0$, there exists a set $E \subset [0, 2\pi)$ with linear measure zero, such that if $\theta \in [0, 2\pi) \setminus E$, then there exists a constant $R_0 = R_0(\theta) > 1$, such that for all z satisfying $\arg z = \theta$ and $|z| = r > R_0$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right| \le r^{(\sigma-1+\varepsilon)(j-s)}, \quad j = s+1, \ldots, k. \tag{54}$$

Take a ray $\arg z = \theta \in H_{n,+} \setminus E$, then $\delta_n(A, \theta) > 0$. We assert that $|f^{(s)}(re^{i\theta})|$ is bounded on the ray $\arg z = \theta$.

If $|f^{(s)}(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, then, by Lemma 2.2, there exists a sequence $z_t = r_t e^{i\theta}$, such that as $r_t \to \infty$, $f^{(s)}(z_t) \to \infty$, and

$$\left|\frac{f^{(i)}(z_t)}{f^{(s)}(z_t)}\right| \leq r_t^{(s-i)}(1+o(1)), \quad i = 0, \ldots, s-1. \tag{55}$$

Because $\delta_n(A, \theta) > 0$, from (11), (15), (54), (55), we obtain that for sufficiently large r_t,

$$\begin{aligned}
&|a_{sm_s}| e^{m_s \delta_n(A,\theta)(1+o(1))r_t^n}(1+o(1)) = \left|P_s\left(e^{A(z_t)}\right) + Q_s\left(e^{-A(z_t)}\right)\right| \\
&\leq \left|\frac{f^{(k)}(z_t)}{f^{(s)}(z_t)}\right| + \sum_{j=0, j\neq s}^{k-1} \left|P_j\left(e^{A(z_t)}\right) + Q_j\left(e^{-A(z_t)}\right)\right| \times \left|\frac{f^{(j)}(z_t)}{f^{(s)}(z_t)}\right| \\
&\leq kMr_t^{k(\sigma+1)} e^{\widetilde{m}\delta_n(A,\theta)(1+o(1))r_t^n}, \ M > 0,
\end{aligned}$$

which yields a contradiction by $m_s > \widetilde{m}$ and $\delta_n(A, \theta) > 0$. So,

$$|f(re^{i\theta})| \leq M_1 r^s \leq M_1 r^k, \ M_1 > 0, \tag{56}$$

on the ray $\arg z = \theta \in H_{n,+} \setminus E$.

Now, we take a ray $\arg z = \theta \in H_{n,-} \setminus E$, then $\delta_n(A, \theta) < 0$. If $|f^{(d)}(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$, then, by Lemma 2.2, there exists a sequence $z_t' = r_t' e^{i\theta}$, such that as $r_t' \to \infty$, $f^{(d)}(z_t') \to \infty$, and

$$\left|\frac{f^{(i)}(z_t')}{f^{(d)}(z_t')}\right| \leq (r_t')^{(d-i)}(1+o(1)), \quad i = 0, \ldots, d-1.$$

Using a proof similar to above, we can obtain that for sufficiently large r_t',

$$|b_{dn_d}| e^{-n_d \delta_n(A,\theta)(1+o(1))(r_t')^n}(1+o(1)) \leq kM(r_t')^{k(\sigma+1)} e^{-\widetilde{n}\delta_n(A,\theta)(1+o(1))(r_t')^n}$$

which also yields a contradiction by $-n_d\delta_n(A, \theta) > -\widetilde{n}\delta_n(A, \theta) > 0$. Hence,

$$|f(re^{i\theta})| \leq M_1 r^d \leq M_1 r^k, \tag{57}$$

on the ray $\arg z = \theta \in H_{n,-} \setminus E$.

From Lemma 2.5, (56) and (57), we know that $f(z)$ is a polynomial, which contradicts the assertion that $f(z)$ is transcendental. Therefore, $\sigma(f) = \infty$.

Step two We prove that (11) has no nontrivial n-subnormal solutions. On the contrary, suppose that (11) has a nontrivial n-subnormal solution f_0. We will deduce a contradiction. By the conclusion in First step, f_0 satisfies (11) and $\sigma(f_0) = \infty$. By Lemma 2.7, we see that $\sigma_2(f_0) \leq n$. By Lemma 2.3, there exist a subset $E_1 \subset (1, \infty)$

with finite logarithmic measure and a constant $B > 0$, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have

$$\left| \frac{f_0^{(j)}(z)}{f_0^{(i)}(z)} \right| \leq B[T(2r, f_0)]^{k+1}, \ i, j \in \{0, 1, \ldots, k\}, \ i < j. \tag{58}$$

From the Wiman-Valiron theory, there exists a set $E_2 \subset (1, \infty)$ with finite logarithmic measure, so that we can choose z satisfying $|z| = r \notin [0, 1] \cup E_2$ and $|f_0(z)| = M(r, f_0)$. Selecting z this way, we obtain

$$\frac{f_0^{(j)}(z)}{f_0(z)} = \left(\frac{\nu(r)}{z} \right)^j (1 + o(1)), \quad j = 1, \ldots, k. \tag{59}$$

By Lemma 2.6 and $\sigma(f_0) = \infty$, there exists a sequence $\{z_t = r_t e^{i\theta_t}\}$, such that $|f_0(z_t)| = M(r_t, f_0)$, $\theta_t \in [0, 2\pi)$, and $\lim_{t \to \infty} \theta_t = \theta_0 \in [0, 2\pi)$, with $r_t \notin [0, 1] \cup E_1 \cup E_2$, $r_t \to \infty$, and for any sufficiently large M_2 $(> 2k + 3)$,

$$\nu(r_t) > r_t^{M_2} > r_t. \tag{60}$$

Case 1 Suppose $\theta_0 \in H_{n,+}$. Since $\delta_n(A, \theta) = \alpha_n \cos(\theta_n + n\theta)$ is a continuous function of θ, by $\theta_t \to \theta_0$ we get $\lim_{t \to \infty} \delta_n(A, \theta_t) = \delta_n(A, \theta_0) > 0$. Therefore, there exists a constant $N(> 0)$, such that as $t > N$,

$$\delta_n(A, \theta_t) \geq \frac{1}{2} \delta_n(A, \theta_0) > 0.$$

By (5), for any given ε_3 $(0 < \varepsilon_3 < \frac{1}{2^{n+2}(k+1)} \delta_n(A, \theta_0))$, and $t > N$,

$$[T(2r_t, f_0)]^{k+1} \leq e^{\varepsilon_3 (k+1)(2r_t)^n} \leq e^{\frac{1}{2}\delta_n(A, \theta_t) r_t^n}. \tag{61}$$

By (58), (59), and (61), we see that

$$\left(\frac{\nu(r_t)}{r_t} \right)^{k-s} (1 + o(1)) = \left| \frac{f_0^{(k-s)}(z_t)}{f_0(z_t)} \right| \leq B[T(2r_t, f_0)]^{k+1} \leq B e^{\frac{1}{2}\delta_n(A, \theta_t) r_t^n}. \tag{62}$$

Because $\delta_n(A, \theta_t) > 0$ as $t > N$, from (11), (15), and (59), we get, for sufficiently large r_t,

$$\begin{aligned} &\left(\frac{\nu(r_t)}{r_t} \right)^s |a_{sm_s}| e^{m_s \delta_n(A, \theta_t)(1+o(1)) r_t^n} (1 + o(1)) \\ &= \left| \frac{f_0^{(s)}(z_t)}{f_0(z_t)} \left(P_s \left(e^{A(z_t)} \right) + Q_s \left(e^{-A(z_t)} \right) \right) \right| \end{aligned}$$

$$\leq \left|\frac{f_0^{(k)}(z_t)}{f_0(z_t)}\right| + \sum_{j=0, j\neq s}^{k-1} \left|P_j\left(e^{A(z_t)}\right) + Q_j\left(e^{-A(z_t)}\right)\right| \times \left|\frac{f_0^{(j)}(z_t)}{f_0(z_t)}\right|$$

$$\leq 2\left(\frac{\nu(r_t)}{r_t}\right)^k + 2Me^{\widetilde{m}\delta_n(A,\theta_t)(1+o(1))r_t^n} \sum_{j=0, j\neq s}^{k-1} \left(\frac{\nu(r_t)}{r_t}\right)^j. \tag{63}$$

Further, from (60), (62), and (63), we get that as $r_t \to \infty$, and $r_t \notin [0, 1] \cup E_1 \cup E_2$,

$$|a_{sm_s}|e^{(m_s-\widetilde{m})\delta_n(A,\theta_t)(1+o(1))r_t^n} \leq 4kM\left(\frac{\nu(r_t)}{r_t}\right)^{k-s} \leq 8kMBe^{\frac{1}{2}\delta_n(A,\theta_t)r_t^n},$$

which yields a contradiction by $m_s - \widetilde{m} \geq 1 > \frac{1}{2}$ and $\delta_n(A, \theta_t) > 0$.

Case 2 Suppose $\theta_0 \in H_{n,-}$. Since $\delta_n(A, \theta)$ is a continuous function of θ, by $\theta_t \to \theta_0$ we get $\lim_{t\to\infty} \delta_n(A, \theta_t) = \delta_n(A, \theta_0) < 0$. Therefore, there exists a constant $N(> 0)$, such that as $t > N$,

$$\delta_n(A, \theta_t) \leq \frac{1}{2}\delta_n(A, \theta_0) < 0.$$

By (5), for any given ε_3' $(0 < \varepsilon_3' < \frac{-1}{2^{n+2}(k+1)}\delta_n(A, \theta_0))$, and $t > N$,

$$[T(2r_t, f_0)]^{k+1} \leq e^{\varepsilon_3'(k+1)(2r_t)^n} \leq e^{-\frac{1}{2}\delta_n(A,\theta_t)r_t^n}.$$

Using a proof similar to that in Case 1, we can obtain that as $r_t \to \infty$,

$$|b_{dn_d}|e^{-(n_d-\widetilde{n})\delta_n(A,\theta_t)(1+o(1))r_t^n} \leq 8kMBe^{-\frac{1}{2}\delta_n(A,\theta_t)r_t^n},$$

which also yields a contradiction by $-(n_d - \widetilde{n})\delta_n(A, \theta_t) > -\frac{1}{2}\delta_n(A, \theta_t) > 0$.

Case 3 Suppose $\theta_0 \in H_{n,0}$. Since $\theta_t \to \theta_0$, for any given ε_4 $\left(0 < \varepsilon_4 < \frac{1}{10n}\right)$, there exists an integer N (> 0), such that as $t > N$, $\theta_t \in [\theta_0 - \varepsilon_4, \theta_0 + \varepsilon_4]$, and

$$z_t = r_t e^{i\theta_t} \in \overline{\Omega} = \{z : \theta_0 - \varepsilon_4 \leq \arg z \leq \theta_0 + \varepsilon_4\}.$$

Now, we consider the growth of $f_0(re^{i\theta})$ on a ray $\arg z = \theta \in \overline{\Omega} \setminus \{\theta_0\}$.

By the properties of cosine function, as in the proof of Case 3 in Theorem 1.1, we suppose without loss of generality that $\delta_n(A, \theta) > 0$ for $\theta \in [\theta_0 - \varepsilon_4, \theta_0)$ and $\delta_n(A, \theta) < 0$ for $\theta \in (\theta_0, \theta_0 + \varepsilon_4]$.

Subcase 3.1 For a fixed $\theta \in (\theta_0, \theta_0 + \varepsilon_4]$, we have $\delta_n(A, \theta) < 0$. By (5), for any given ε_5 satisfying $0 < \varepsilon_5 < \frac{-1}{2^{n+1}(k+1)}\delta_n(A, \theta)$,

$$[T(2r, f_0)]^{k+1} \leq e^{-\frac{1}{2}\delta_n(A,\theta)r^n}. \tag{64}$$

We assert that $\left|f_0^{(d)}(re^{i\theta})\right|$ is bounded on the ray $\arg z = \theta$. If $\left|f_0^{(d)}(re^{i\theta})\right|$ is unbounded on it, then by Lemma 2.2, there exists a sequence $\{y_j = R_j e^{i\theta}\}$, such that as $R_j \to \infty$, $f_0^{(d)}(y_j) \to \infty$, and

$$\left|\frac{f_0^{(i)}(y_j)}{f_0^{(d)}(y_j)}\right| \le (R_j)^{d-i}(1+o(1)), \quad i = 0, \ldots, d-1. \tag{65}$$

By $f_0^{(d)}(y_j) \to \infty$ and Remark 1, we know that y_j satisfies (58). By (58) and (64), we see that, for sufficiently large R_j, and $R_j \notin [0, 1] \cup E_1$,

$$\left|\frac{f_0^{(i)}(y_j)}{f_0^{(d)}(y_j)}\right| \le B[T(2R_j, f_0)]^{k+1} \le Be^{-\frac{1}{2}\delta_n(A,\theta)R_j^n}, \ i = d+1, \cdots, k. \tag{66}$$

Because $\delta_n(A, \theta) < 0$, by (11), (15), (65), and (66), we deduce that for sufficiently large R_j, and $R_j \notin [0, 1] \cup E_1$,

$$\begin{aligned}|b_{dn_d}|e^{-n_d\delta_n(A,\theta)(1+o(1))R_j^n}(1+o(1)) &= \left|P_d\left(e^{A(y_j)}\right) + Q_d\left(e^{-A(y_j)}\right)\right| \\ &\le kMB(R_j)^k e^{-(\widetilde{n}+\frac{1}{2})\delta_n(A,\theta)(1+o(1))R_j^n},\end{aligned}$$

which yields a contradiction by $n_d - \widetilde{n} \ge 1 > \frac{1}{2}$ and $\delta_n(A, \theta) < 0$. So

$$\left|f_0\left(re^{i\theta}\right)\right| \le M_1 r^d, \ \ M_1 > 0, \tag{67}$$

on the ray $\arg z = \theta \in (\theta_0, \theta_0 + \varepsilon_4]$.

Subcase 3.2 For a fixed $\theta \in [\theta_0 - \varepsilon_4, \theta_0)$, we have $\delta_n(A, \theta) > 0$. Using a proof similar to that in Subcase 3.1, we obtain

$$\left|f_0\left(re^{i\theta}\right)\right| \le M_1 r^s, \tag{68}$$

on the ray $\arg z = \theta \in [\theta_0 - \varepsilon_4, \theta_0)$.

By (67) and (68), we see that on the ray $\arg z = \theta \in \overline{\Omega} \setminus \{\theta_0\}$,

$$\left|f_0\left(re^{i\theta}\right)\right| \le M_1 r^k. \tag{69}$$

But since $\sigma(f_0\left(re^{i\theta}\right)) = \infty$ and $\{z_t = r_t e^{i\theta_t}\}$ satisfies $|f_0(z_t)| = M(r_t, f_0)$, we see that, for any large M_3 $(> k)$, as t is sufficiently large,

$$|f_0(z_t)| = \left|f_0\left(r_t e^{i\theta_t}\right)\right| \ge \exp\{r_t^{M_3}\}. \tag{70}$$

Since $z_t \in \overline{\Omega}$, by (69) and (70), we see that $\theta_t = \theta_0$ as $t \to \infty$. Therefore, $\delta_n(A, \theta_t) = 0$ as $t \to \infty$. On the analogue by this, we get $\delta_j(A, \theta_t) = \alpha_j \cos(\theta_j + j\theta_t) = 0$ as $t \to \infty$, where $j = 1, \ldots, n-1$.

Thus, there exists M_4 (>0), for sufficiently large t,

$$\left|P_j\left(e^{A(z_t)}\right)+Q_j\left(e^{-A(z_t)}\right)\right|\le M_4,\ \ j=0,\ldots,k-1. \tag{71}$$

By (11) and (59), we obtain that as $r_t\to\infty$ and $r_t\notin[0,1]\cup E_1\cup E_2$,

$$-\left(\frac{\nu(r_t)}{z_t}\right)^k(1+o(1))=\sum_{j=0}^{k-1}\left(P_j\left(e^{A(z_t)}\right)+Q_j\left(e^{-A(z_t)}\right)\right)\left(\frac{\nu(r_t)}{z_t}\right)^j(1+o(1)). \tag{72}$$

By (60), (71), and (72), we obtain that as $r_t\to\infty$ and $r_t\notin[0,1]\cup E_1\cup E_2$,

$$\nu(r_t)\le 2kM_4r_t, \tag{73}$$

which yields a contradiction by (60). Hence, (11) has no nontrivial n-subnormal solutions.

Third step We prove that all solutions of (11) satisfy $\sigma_2(f)=n$. If there is a solution f_1 that satisfies $\sigma_2(f_1)<n$, then f_1 satisfies (5), i.e., f_1 is a n-subnormal solution. But this contradicts the conclusion in Second step. Hence, every solution f satisfies $\sigma_2(f)\ge n$, and by Lemma 2.7 and $\sigma_2(f)\le n$, we get that $\sigma_2(f)=n$. Thus, Theorem 1.2 is proved.

5 Proof of Theorem 1.4

(i) Suppose that f_1 and $f_2(\not\equiv f_1)$ are nontrivial n-subnormal solutions of equation (13). Then $f_1-f_2(\not\equiv 0)$ is a n-subnormal solution of the corresponding homogeneous equation (11). This contradicts the assertion of Theorem 1.2. Hence, equation (13) has at most one nontrivial n-subnormal solution f_0.

(ii) By Theorem 1.2, we see that all solutions of the corresponding homogeneous equation (11) are of hyper-order $\sigma_2(f)=n$. By variation of parameters, we assert that all solutions of (13) satisfy $\sigma_2(f)\le n$. Next, we will prove this assertion in detail. Let $f_1,\ldots,f_k$ be a solution base of (11). From Lemma 2.8, we have that the Wronskian of $f_1,\ldots,f_k$ satisfies $W(f_1,\ldots,f_k)=e^{-\Phi}$, where $\Phi(z)$ is a primitive function of $\left(P_{k-1}(e^{A(z)})+Q_{k-1}(e^{-A(z)})\right)$; hence, $W(f_1,\ldots,f_k)$ has no zeros. Therefore, the system of equations

$$\begin{cases} B_1'f_1+B_2'f_2+\cdots+B_k'f_k=0\\ B_1'f_1'+B_2'f_2'+\cdots+B_k'f_k'=0\\ \cdots\\ B_1'f_1^{(k-2)}+B_2'f_2^{(k-2)}+\cdots+B_k'f_k^{(k-2)}=0\\ B_1'f_1^{(k-1)}+B_2'f_2^{(k-1)}+\cdots+B_k'f_k^{(k-1)}=R(z)\end{cases} \tag{74}$$

defines uniquely entire functions $B_1', \cdots, B_k'$. In fact, by the classical Cramer rule, we obtain

$$B_j' = RG_j(f_1, \ldots, f_k)/W(f_1, \ldots, f_k), \quad j = 1, \ldots, k, \tag{75}$$

where each $G_j(f_1, \ldots, f_k)$ is a differential polynomial of $f_1, \ldots, f_k$ and of their derivatives, with constant coefficients. Take now some primitives $B_1, \ldots, B_k$ of $B_1', \ldots, B_k'$, and define $f_0 = B_1 f_1 + B_2 f_2 + \cdots + B_k f_k$. It follows by (74) that

$$\begin{cases} f_0 = B_1 f_1 + B_2 f_2 + \cdots + B_k f_k \\ f_0' = B_1 f_1' + B_2 f_2' + \cdots + B_k f_k' \\ \cdots \\ f_0^{(k-1)} = B_1 f_1^{(k-1)} + B_2 f_2^{(k-1)} + \cdots + B_k f_k^{(k-1)} \\ f_0^{(k)} = B_1 f_1^{(k)} + B_2 f_2^{(k)} + \cdots + B_k f_k^{(k)} + R(z). \end{cases} \tag{76}$$

Multiplying the equations of (76) with $(P_0(e^{A(z)}) + Q_0(e^{-A(z)})), \ldots, (P_{k-1}(e^{A(z)}) + Q_{k-1}(e^{-A(z)})), 1$, respectively, and adding all these equations together, we get that f_0 is an entire solution of (13). By the elementary theory of linear differential equations, all solutions of (13) can be represented in the form $f = f_0 + C_1 f_1 + \cdots + C_k f_k$ for $C_1, \ldots, C_k \in \mathbb{C}$, and so, by $\sigma_2(B_j') = \sigma_2(B_j)$ $(j = 1, \ldots, k)$ the assertion follows.

If $\sigma_2(f) < n$, then f is a n-subnormal solution. Hence, all other solutions f of (13) satisfy $\sigma_2(f) = n$ except the possible n-subnormal solution in (i).

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