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The Jacobian Conjecture and Injectivity Conditions

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Abstract One of the aims of this article is to provide a class of polynomial mappings for which the Jacobian conjecture is true. Also, we state and prove several global univalence theorems and present a couple of applications of them.

Keywords Univalent · Injectivity · Jacobian · Polynomial map · Keller map · Jacobian conjecture

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1 Introduction and Main Results

This article mainly concerns with mappings $f : \mathbb{C}^n \to \mathbb{C}^n$, written in coordinates as

$$f(Z) = (f_1(Z), \ldots, f_n(Z)), \quad Z = (z_1, \ldots, z_n).$$

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We say that f is a *polynomial map* if each component function $f_i : \mathbb{C}^n \to \mathbb{C}$ is a polynomial in n-variables $z_1, \ldots, z_n$, for $1 \le i \le n$. A polynomial map $f : \mathbb{C}^n \to \mathbb{C}^n$ is called *invertible* if it has an inverse map which is also a polynomial map.

Let $Df := \left(\frac{\partial f_j}{\partial z_i}\right)_{n \times n}$, $1 \le i, j \le n$, be the Jacobian matrix of f. The Jacobian determinant is denoted by $\det Df$. If a polynomial map f is invertible and $g = f^{-1}$, then $g \circ f = \mathrm{id}$, and because $\det Df \cdot \det Dg = 1$, $\det Df$ must be a nonzero complex constant. However, the converse question is more difficult. Then, the Jacobian conjecture (**JC**) asserts that every polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ is globally invertible if $\det Df$ is identically equal to a nonzero complex constant. This conjecture remains open for any dimension $n \ge 2$. We remark that the **JC** was originally formulated by Keller [13] in 1939 for polynomial maps with integer coefficients. In the case of dimension one, it is simple. Polynomial map f is called a *Keller map*, if $\det Df$ is a nonzero complex constant. In fact, Bialynicki-Birula and Rosenlicht [4] proved that a polynomial map is invertible if it is injective.

It is a simple exercise to see that the **JC** is true if it holds for polynomial mappings whose Jacobian determinant is 1, and thus, after suitable normalization, one can assume that $\det Df = 1$. The **JC** is attractive because of the simplicity of its statement. Moreover, because there are so many ways to approach and making it useful, the **JC** has been studied extensively from calculus to complex analysis to algebraic topology, and from commutative algebra to differential algebra to algebraic geometry. Indeed, some faulty proofs have even been published. The **JC** is stated as one of the eighteen challenging problems for the twenty-first century proposed with brief details by Field medalist Steve Smale [19]. For the importance, history, a detailed account of the research on the **JC** and equivalent conjectures, and related investigations, we refer, for example, to [3] and the excellent book of van den Essen [11] and the references therein. See also [5–10,14,23,24]. We would like to point out that in 1980, Wang [22] showed that every Keller map of degree less than or equal to 2 is invertible. In 1982, Bass et al. [3] (see also [5,25]) showed that it suffices to prove the **JC** for all $n \ge 2$ and all Keller mappings of the form $f(Z) = Z + H(Z)$, where $Z = (z_1, \ldots, z_n)$, and H is cubic homogeneous, i.e., $H = (H_1, \ldots, H_n)$ with $H_i(Z) = (L_i(Z))^3$ and $L_i(Z) = a_{i1}z_1 + \cdots + a_{in}z_n$, $1 \le i \le n$. Cubic homogeneous map f of this form is called a Drużkowski or cubic linear map. Moreover, polynomial mappings from $\mathbb{C}^n$ to $\mathbb{C}^n$ are well behaved than the polynomial mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$. Indeed, Pinchuk [17] constructed an explicit example to show that there exists a non-invertible polynomial map $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $\det Df(X) \ne 0$ for all $X \in \mathbb{R}^2$. In any case, the study of the **JC** has given rise to several surprising results and interesting relations in various directions and in different perspectives. For instance, Abdesselam [1] formulated the **JC** as a question in perturbative quantum field theory and pointed out that any progress on this question will be beneficial not only for mathematics, but also for theoretical physics as it would enhance our understanding of perturbation theory.

The main purpose of this work is to identify the Keller maps for which the **JC** is true.

Theorem 1 *The Jacobian conjecture is true for mappings* $F(X) = (A \circ f \circ B)(X)$, *where* $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$, *$A$ and B are linear such that* $\det A . \det B \ne 0$, $f = (u_1, \ldots, u_n)$,

$$u_k(X) = x_k + \gamma_k \Big[\alpha_2(x_1 + \cdots + x_n)^2 + \alpha_3(x_1 + \cdots + x_n)^3 + \cdots + \alpha_m(x_1 + \cdots + x_n)^m\Big]$$

for $k = 1, \ldots, n$, $\alpha_j, \gamma_k \in \mathbb{R}$ *with* $\sum_{k=1}^{n} \gamma_k = 0$ *and* $m \in \mathbb{N}$.

Often it is convenient to identify X in $\mathbb{C}^n$ (resp. $\mathbb{R}^n$) as an $n \times 1$ matrix with entries as complex (resp. real) numbers. It is interesting to know whether there are other polynomial mappings for which the **JC** is true in $\mathbb{C}^n$ (resp. $\mathbb{R}^n$). In this connection, we will notice that for the case $n = 2$ of Theorem 1 it is possible to prove the following:

Theorem 2 *With* $X = (x, y) \in \mathbb{R}^2$*, consider* $f(X) = (u_1(X), u_2(X))$*, where* $u_k(X)$ *for* $k = 1, 2$ *are as in Theorem 1 and*

$$\tilde{f}(X) = (u_1(X) + W(X), u_2(X) + w(X)),$$

where W *and* w *are homogeneous polynomials of degree* $(m + 1)$ *in* x *and* y*. If* $\det D\tilde{f}(X) \equiv 1$*, then* $\tilde{f} = A^{-1} \circ F \circ A$*, where* A *is linear homogeneous non-degenerate mapping and*

$$F(X) = \Big(u_1(X) + \alpha_{m+1}(x + y)^{m+1}, u_2(X) - \alpha_{m+1}(x + y)^{m+1}\Big),$$

for some real constant α_{m+1}*. The Jacobian conjecture is true for the mapping* $\tilde{f}$*.*

Remark 1 It follows from the proof of Theorem 2 that A equals the identity matrix I if $f(X) \not\equiv X$.

In connection with Theorems 1 and 2, it is interesting to note that in the case $n = 2$, the mappings F defined in Theorem 1 provide a complete description of the Keller mappings F for which $\deg F \le 3$ (see [21]).

Next, we denote by $\widehat{\mathcal{P}}_n(m)$, the set of all polynomial mappings $F : \mathbb{R}^n \to \mathbb{R}^n$ of degree less than or equal to m such that $DF(0) = I$ and $F(0) = 0$. Let $P_n(m)$ be a subset consisting of mappings $f \in \widehat{\mathcal{P}}_n(m)$ which satisfy the conditions of Theorem 1. Also, we introduce

$$\mathcal{P}_n(m) = \{F \in \widehat{\mathcal{P}}_n(m) : \ F \text{ is injective}\}.$$

If $f, g \in P_n(m)$, then

$$f(X) = X + u(X)\gamma \ \text{ and } \ g(X) = X + v(X)\delta,$$

where $X = (x_1, \ldots, x_n)$, $\gamma = (\gamma_1, \ldots, \gamma_n)$, $\delta = (\delta_1, \ldots, \delta_n)$,

$$u(X) = \sum_{k=2}^{m} \alpha_k (x_1 + \cdots + x_n)^k \ \text{ and } \ v(X) = \sum_{k=2}^{m} \beta_k (x_1 + \cdots + x_n)^k,$$

such that $\sum_{k=1}^{n} \delta_k = \sum_{k=1}^{n} \gamma_k = 0$. Here α_k's and β_k's are some constants. It is obvious that $f \circ g$ is injective, but it is unexpected that the composition $f \circ g$ also belongs to $\mathcal{P}_n(m)$. This circumstance allows us to generalize Theorem 1 significantly.

Theorem 3 *For $n \geq 3$, consider the mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ defined by $F(X) = (u_1, \ldots, u_n)$, where*

$$u_k(X) = x_k + p_k^{(2)}(x_1 + \cdots + x_n)^2 + \cdots + p_k^{(m)}(x_1 + \cdots + x_n)^m \quad (k = 1, \ldots, n),$$

and $p_k^{(l)}$ are constants satisfying the condition $\sum_{k=1}^{n} p_k^{(l)} = 0$ for all $l = 2, \ldots, m$. Then the Jacobian conjecture is true for F, where $F = f_1 \circ \cdots \circ f_N$ and each f_j $(j = 1, \ldots, N)$ is a polynomial map of the form f given by Theorem 1.

From the proofs of Theorems 1, 2 and 3, it is easy to see that these results continue to hold even if we replace $\mathbb{R}^n$ by $\mathbb{C}^n$. The proofs of Theorems 1, 2 and 3 will be presented in Sect. 3. In Sect. 2, we present conditions for injectivity of functions defined on a convex domain.

2 Injectivity Conditions on Convex Domains

One can find discussion and several sufficient conditions for global injectivity [2,16]. In the following, we state and prove several results on injectivity on convex domains.

Theorem 4 *Let $D \subset \mathbb{R}^n$ be convex and $f : D \to \mathbb{R}^n$ belong to $C^1(D)$. Then $f = (f_1, \ldots, f_n)$ is injective in D if for every $X_1, X_2 \in D$ $(X_1 \neq X_2)$ and $\gamma(t) = X_1 + t(X_2 - X_1)$ for $t \in [0, 1]$,* det $A \neq 0$, *where*

$$A = (a_{ij})_{n \times n} \;\; with \;\; a_{ij} = \int_0^1 \frac{\partial f_j}{\partial x_i}(\gamma(t))\, \mathrm{d}t,\; 1 \leq i, j \leq n.$$

Proof Let $X_1, X_2 \in D$ be two distinct points. Since D is convex, the line segment $\gamma(t) \in D$ for $t \in (0, 1)$, and thus, we have

$$f(X_2) - f(X_1) = \int_\gamma \mathrm{d}f(\zeta) = \int_\gamma Df(\zeta)\, \mathrm{d}\zeta = \int_0^1 (Df)(\gamma(t))(X_2 - X_1)\, \mathrm{d}t.$$

Taking into account of the assumptions, we deduce that $f(X_2) - f(X_1) \neq 0$ for each $X_1, X_2 \in D$ $(X_1 \neq X_2)$ if for every $X \in \mathbb{R}^n \setminus \{0\}$,

$$\int_0^1 Df(\gamma(t))\, \mathrm{d}t \times X \neq 0$$

which holds whenever det $A \neq 0$. The proof is complete. □

We remark that Theorem 4 has obvious generalization for functions defined on convex domains $D \subseteq \mathbb{C}^n$. In this case, the Jacobian matrix of f, i.e., $Df = \left(\frac{\partial f_j}{\partial z_i}\right)_{n \times n}$,

$1 \leq i, j \leq n$, will be used. Moreover, using Theorem 4, we may easily obtain the following simple result.

Corollary 1 *Let $D \subseteq \mathbb{R}^n$ be a convex domain and $f = (f_1, \ldots, f_n)$ belong to $C^1(D)$. If for every line L in $\mathbb{R}^n$, with $L \cap D \neq \emptyset$,*

$$\det\left(\frac{\partial f_j}{\partial x_i}(X_{i,j})\right) \neq 0 \text{ for every } X_{i,j} \in L \cap D,$$

then f is injective in D.

Proof Let $X_1, X_2 \in D$ be two distinct points and $\gamma(t) = X_1 + t(X_2 - X_1)$ be the line segment joining X_1 and X_2, $t \in [0, 1]$. Then, for every $i, j = 1, \ldots, n$, we have

$$\int_0^1 \frac{\partial f_j}{\partial x_i}(\gamma(t))\,\mathrm{d}t = \frac{\partial f_j}{\partial x_i}(X_{i,j}),$$

where $X_{i,j} \in (X_1, X_2)$. The desired conclusion follows from Theorem 4. □

Corollary 2 *Let $D \subseteq \mathbb{C}$ be a convex domain, $z = x+iy$ and $f(z) = u(x, y)+iv(x, y)$ be analytic in D. Then f is injective in D whenever for every $z_1, z_2 \in D$, we have*

$$\det\begin{bmatrix} u_x(z_1) & -v_x(z_2) \\ v_x(z_2) & u_x(z_1) \end{bmatrix} = u_x^2(z_1) + v_x(z_2)^2 \neq 0.$$

Corollary 2 shows that if $u_x \neq 0$ or $v_x \neq 0$ in a convex domain D, then the analytic function $f = u + iv$ is univalent (injective) in D. Thus, it is a sufficient condition for the univalency and is different from the necessary condition $f'(z) \neq 0$, the fact that in the latter case u_x and v_x have no common zeros in D. The reader may compare with the well-known Noshiro–Warschawski theorem which asserts that if f is analytic in a convex domain D in $\mathbb{C}$ and Re $f'(z) > 0$ in D, then f is univalent in D. See also Corollary 4.

Throughout we let $\mathbb{R}^n_+ = \{X = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 > 0\}$ and $S_{n-1} = \{X \in \mathbb{R}^n : \|X\| = 1\}$, the unit sphere in the Euclidean space $\mathbb{R}^n$.

Theorem 5 *Let $D \subseteq \mathbb{R}^n$ be a convex domain, $f : D \to \mathbb{R}^n$ belong to $C^1(D)$ and $\Omega = f(D)$. Then f is injective if and only if there exists a $\phi \in C^1(\Omega)$, $\phi : \Omega \to \mathbb{R}^n$, satisfying the following property: for every $X_0 \in S_{n-1}$ there exists a unitary matrix $U = U(X_0)$ with*

$$U \times (D\phi)(f(X)) \times Df(X)X_0 \in \mathbb{R}^n_+ \tag{1}$$

for every $X \in D$.

Proof Let $X_1, X_2 \in D$. Then the line segment $\gamma(t)$ connecting these points given by $\gamma(t) = X_1 + t(X_2 - X_1)$ belongs to the convex domain D for every $t \in [0, 1]$. We denote $\psi = \phi \circ f$ and observe that

$$\psi(X_2) - \psi(X_1) = \int_\gamma d\psi(\zeta) = \int_0^1 d(\psi \circ \gamma)(t) = \int_0^1 (D\psi)(\gamma(t)) \times (X_2 - X_1)\,\mathrm{d}t.$$

If $X_2 \neq X_2$, we may let

$$X_0 = \frac{X_2 - X_1}{||X_2 - X_1||} \in S_{n-1}.$$

Sufficiency ($\Leftarrow$): Now we assume (1) and show that f is injective on D. Because of the truth of (1), it follows that

$$U(\psi(X_2) - \psi(X_1)) = \|X_2 - X_1\| \int_0^1 U(D\psi)(\gamma(t))X_0 \, dt \neq 0.$$

Then the first component $a_1(t)$ of $U(D\psi)(\gamma(t))X_0 = (a_1(t), \dots, a_n(t)) \in \mathbb{R}^n_+$, by definition, satisfies the positivity condition $a_1(t) > 0$ for each t, and thus, $\int_0^1 a_1(t)\, dt \neq 0$. Consequently, $f(X_2) \neq f(X_1)$ for every X_1, X_2 ($X_1 \neq X_2$) in D.

Necessity ($\Rightarrow$): Assume that f is injective in D. Then we may let $\phi = f^{-1}$ and assume that U is an unitary matrix such that $UX_0 = (1, 0, \dots, 0)$. This implies that $\psi(X) \equiv X$, and thus,

$$U \times (D\psi)(\gamma(t))X_0 \equiv (1, 0, \dots, 0)$$

and (1) holds. □

It is now appropriate to state a several complex variables analog of Theorem 5. As with standard practice, for $D \subset \mathbb{C}^n$, we consider $f \in C^1(D)$, $\Omega = f(D) \subset \mathbb{C}^n$, $\phi \in C^1(\Omega)$, $\phi : \Omega \longrightarrow \mathbb{C}^n$, $Z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $\overline{Z} = (\bar{z}_1, \dots, \bar{z}_n)$, and $\psi = \phi \circ f$. We frequently, write down these mappings as functions of the independent complex variables Z and $\overline{Z}$, namely as $f(Z, \overline{Z})$, $\phi(W, \overline{W})$ and $\psi(Z, \overline{Z})$. Denote as usual

$$\partial\psi = \frac{\partial(\psi_1, \dots, \psi_n)}{\partial(z_1, \dots, z_n)} = \begin{pmatrix} \frac{\partial\psi_1}{\partial z_1} & \cdots & \frac{\partial\psi_1}{\partial z_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial\psi_n}{\partial z_1} & \cdots & \frac{\partial\psi_n}{\partial z_n} \end{pmatrix} \text{ and } \overline{\partial}\psi = \frac{\partial(\psi_1, \dots, \psi_n)}{\partial(\bar{z}_1, \dots, \bar{z}_n)}.$$

At this place, it is convenient to use $\partial\psi$ instead of $D\psi$. Then for $\gamma(t) = Z_1 + t(Z_2 - Z_1)$, $t \in (0, 1)$, we have

$$\begin{aligned} \psi(Z_2, \overline{Z_2}) - \psi(Z_1, \overline{Z_1}) &= \int_\gamma d\psi(Z, \overline{Z}) \\ &= \int_0^1 \left[\partial\psi(\gamma(t), \overline{\gamma(t)}) \cdot (Z_2 - Z_1) + \overline{\partial}\psi(\gamma(t), \overline{\gamma(t)}) \cdot \overline{(Z_2 - Z_1)}\right] dt. \end{aligned}$$

Thus, Theorem 5 takes the following form.

Theorem 6 *Let $D \subseteq \mathbb{C}^n$ be a convex domain, $f : D \to \mathbb{C}^n$ belong to $C^1(D)$ and $\Omega = f(D)$. Then f is injective in D if and only if there exists a $\phi \in C^1(\Omega)$, $\phi : \Omega \to \mathbb{C}^n$ satisfying the following property with $\psi = \phi \circ f$: for every $Z_0 \in \mathbb{C}^n$, $\|Z_0\| = 1$, there exists a unitary complex matrix $U = U(Z_0)$ such that for every $Z \in D$, one has*

$$U\left[\partial\psi(Z, \overline{Z})Z_0 + \overline{\partial}\psi(Z, \overline{Z})\overline{Z}_0\right] \in \{Z \in \mathbb{C}^n : \operatorname{Re} z_1 > 0\}.$$

In particular, Theorem 6 is applicable to pluriharmonic mappings. In the case of planar harmonic mappings $f = h + \overline{g}$, where h and g are analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, Theorem 6 takes the following form–another criterion for injectivity–harmonic analog of Φ-like mappings, see [12, Theorem 1].

Corollary 3 *Let $f = h + \overline{g}$ be harmonic on a convex domain $D \subset \mathbb{C}$ and $\Omega = f(D)$. Then f is univalent in D if and only if there exists a complex-valued function $\phi = \phi(w, \overline{w})$ in $C^1(\Omega)$ and such that for every ϵ with $|\epsilon| = 1$, there exists a real number $\gamma = \gamma(\epsilon)$ satisfying*

$$\operatorname{Re}\left\{e^{i\gamma}\left(\partial\phi(f(z), \overline{f(z)}) + \epsilon\overline{\partial}\phi(f(z), \overline{f(z)})\right)\right\} > 0 \text{ for all } z \in D,$$

where $\partial = \frac{\partial}{\partial z}$ and $\overline{\partial} = \frac{\partial}{\partial \overline{z}}$.

Several consequences and examples of Corollary 3 are discussed in [12], and they seem to be very useful. Another univalence criterion for harmonic mappings of $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ was obtained in [20]. Moreover, using Corollary 3, it is easy to obtain the following sufficient condition for the univalency of C^1 functions.

Corollary 4 ([15]) *Let D be a convex domain in $\mathbb{C}$ and f be a complex-valued function of class $C^1(D)$. Then f is univalent in D if there exists a real number γ such that*

$$\operatorname{Re}\left(e^{i\gamma} f_z(z)\right) > \left|f_{\overline{z}}(z)\right| \text{ for all } z \in D.$$

For example, if $f = h + \overline{g}$ is a planar harmonic mapping in the unit disk $\mathbb{D}$ and if there exists a real number γ such that

$$\operatorname{Re}\left\{e^{i\gamma}h'(z)\right\} > |g'(z)| \text{ for } z \in \mathbb{D}, \tag{2}$$

then f is univalent in $\mathbb{D}$. In [18], it was shown that harmonic functions $f = h + \overline{g}$ satisfying condition (2) in $\mathbb{D}$ are indeed univalent and *close-to-convex* in $\mathbb{D}$, i.e., the complement of the image region $f(\mathbb{D})$ is the union of non-intersecting rays (except that the origin of one ray may lie on another one of the rays).

Moreover, using Theorem 6, one can also obtain a sufficient condition for p-valent mappings.

Corollary 5 *Suppose that $D \subseteq \mathbb{R}^n$ is a domain such that $D = \cup_{m=1}^{p} D_m$, where D_m's are convex for $m = 1, \ldots, p$. Furthermore, let $f \in C^1(D)$, $\Omega = f(D) \subset \mathbb{R}^n$, $\phi \in$*

$C^1(\Omega)$ such that for every $X_m \in S_{n-1}$ there exists a unitary matrix $U_m := U_m(X_m)$ for each $m = 1, \dots, p$, such that

$$U_m \times (D\phi)(f(X)) \times (Df)(X) \times X_m \in \mathbb{R}^n_+$$

for every $X \in D_m$, $m = 1, \dots, p$. Then f is no more than p-valent in D.

3 Proofs of Theorems 1, 2 and 3

Proof of Theorem 1 It is enough to prove the Theorem in the case $A = B = I$. For convenience, we let $z = x_1 + \cdots + x_n$. Then

$$u_k(X) = x_k + \gamma_k \left[\alpha_2 z^2 + \alpha_3 z^3 + \cdots + \alpha_m z^m\right]$$

for $k = 1, \dots, n$, $\alpha_j, \gamma_k \in \mathbb{R}$ with $\sum_{k=1}^n \gamma_k = 0$ and $X = (x_1, \dots, x_n)$.

We first prove that $\det Df(X) \equiv 1$. To do this, we begin to introduce

$$L(X) = 2\alpha_2 z + 3\alpha_3 z^2 + \cdots + m\alpha_m z^{m-1}.$$

Then a computation gives

$$\frac{\partial u_k}{\partial x_j} = \delta_k^j + \gamma_k L.$$

and

$$\det Df = \begin{vmatrix} 1+\gamma_1 L & \gamma_1 L & \dots & \gamma_1 L \\ \gamma_2 L & 1+\gamma_2 L & \dots & \gamma_2 L \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_n L & \gamma_n L & \dots & 1+\gamma_n L \end{vmatrix} = I_1 + \gamma_1 L I_2,$$

where

$$I_1 = \begin{vmatrix} 1 & 0 & \dots & 0 \\ \gamma_2 L & 1+\gamma_2 L & \dots & \gamma_2 L \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_n L & \gamma_n L & \dots & 1+\gamma_n L \end{vmatrix} \text{ and } I_2 = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \gamma_2 L & 1+\gamma_2 L & \dots & \gamma_2 L \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_n L & \gamma_n L & \dots & 1+\gamma_n L \end{vmatrix}.$$

We will now show by induction that

$$\det Df(X) = 1 + \left(\sum_{k=1}^n \gamma_k\right) L(X).$$

Obviously, for $n = 2$, we have $\det Df(X) = 1 + (\gamma_1 + \gamma_2)L$ for $\gamma_1, \gamma_2 \in \mathbb{R}$.

Next, we suppose that $\det Df(X) = 1 + (\gamma_1 + \cdots + \gamma_p)L$ holds for $p = n - 1$, where $\gamma_1, \ldots, \gamma_p \in \mathbb{R}$. We need to show that it is true for $p = n$. Clearly,

$$I_1 = \begin{vmatrix} 1+\gamma_2 L & \ldots & \gamma_2 L \\ \vdots & \vdots & \vdots \\ \gamma_n L & \ldots & 1+\gamma_n L \end{vmatrix} = 1 + (\gamma_2 + \cdots + \gamma_n)L$$

and

$$I_2 = \begin{vmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{vmatrix} = 1.$$

Since $\det Df = I_1 + \gamma_1 L I_2$, using the above and the hypothesis $\sum_{k=1}^{n} \gamma_k = 0$ that

$$\det Df(X) = 1 + (\gamma_1 + \gamma_2 + \cdots + \gamma_n)L(X) \equiv 1.$$

Applying Theorem 4, we will finally show that f is indeed a univalent mapping. Now, for convenience, we denote $L_* = \displaystyle\int_0^1 L[\gamma(t)]\,\mathrm{d}t$ and obtain that

$$\begin{aligned} \det\left(\int_0^1 \frac{\partial u_k}{\partial x_j}(\gamma(t))\,\mathrm{d}t\right)_{j,k=1}^{n} &= \begin{vmatrix} 1+\gamma_1 L_* & \gamma_1 L_* & \ldots & \gamma_1 L_* \\ \gamma_2 L_* & 1+\gamma_2 L_* & \ldots & \gamma_2 L_* \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_n L_* & \gamma_n L_* & \ldots & 1+\gamma_n L_* \end{vmatrix} \\ &= 1 + (\gamma_1 + \cdots + \gamma_n)L_* \\ &= 1 \end{aligned}$$

for all $\gamma(t)$ as in Theorem 4. Thus, by Theorem 4, f is univalent in $\mathbb{R}^n$. □

Remark 2 The injective mapping f in Theorem 1 can be obtained directly from this kind of display and without the use of Theorem 4 but by a method of contradiction. Indeed, if $X = (x_1, \ldots, x_n)$, $Y = (y_1, \ldots, y_n)$, $z = \sum_{j=1}^{n} x_j$, $\varphi(z) = \alpha_2 z^2 + \ldots + \alpha_m z^m$, $f = (u_1, \ldots, u_n)$, where

$$u_k(X) = x_k + \gamma_k \varphi(z)$$

for $k = 1, \ldots, n$, and α_j, γ_k are constants with

$$\sum_{k=1}^{n} \gamma_k = 0, \tag{3}$$

then f is a globally injective mapping. This is because $f(X) = f(Y)$ implies

$$x_k + \gamma_k \varphi\left(\sum_{j=1}^{n} x_j\right) = y_k + \gamma_k \varphi\left(\sum_{j=1}^{n} y_j\right) \quad \text{for } k = 1, \ldots, n, \tag{4}$$

which, because of (3), obviously gives that $\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k$. Finally, from (4), it follows that $x_k = y_k$, and hence, $X = Y$.

Proof of Theorem 2 Consider $f(X) = (u_1(X), u_2(X))$, where $X = (x, y) \in \mathbb{R}^2$ and

$$u_1(x, y) = x + \left[\alpha_2(x + y)^2 + \alpha_3(x + y)^3 + \cdots + \alpha_m(x + y)^m\right]$$

and

$$u_2(x, y) = y - \left[\alpha_2(x + y)^2 + \alpha_3(x + y)^3 + \cdots + \alpha_m(x + y)^m\right]$$

for $\alpha_j \in \mathbb{R}$, $j = 2, \ldots n$. Let

$$\tilde{f}(X) = (u_1(X) + W(X), u_2(X) + w(X)),$$

where W and w are as mentioned in the statement.

As in the proof of Theorem 1, we see easily that

$$\begin{aligned} \det D\tilde{f} &= \begin{vmatrix} 1 + L + W_x & L + W_y \\ -L + w_x & 1 - L + w_y \end{vmatrix} \\ &= 1 + (1 + L)w_y + (1 - L)W_x + LW_y - Lw_x + (W_x w_y - w_x W_y) \end{aligned} \tag{5}$$

which is identically 1, by the hypothesis of Theorem 2. Moreover, allowing $\|X\| \to \infty$ in (5), it follows that $W_x w_y - w_x W_y = 0$. We now show that this gives the relation

$$w = \lambda W$$

for some constant λ. We observe that both w and W are not identically zero simultaneously. If $w \equiv 0$ and $W \not\equiv 0$, then we choose $\lambda = 0$. Because of the symmetry, equality holds in the last relation when $W \equiv 0$ and $w \not\equiv 0$. Therefore, it suffices to consider the case $W \not\equiv 0 \not\equiv w$. We denote $t = x/y$. Using the definition of $w(x)$ and $W(X)$ in the statement, we may conveniently write

$$w(X) = \sum_{k=0}^{m+1} \alpha_k x^k y^{m+1-k} = y^{m+1} \sum_{k=0}^{m+1} \alpha_k t^k =: y^{m+1} p(t)$$

and similarly,

$$W(X) = \sum_{k=0}^{m+1} \beta_k x^k y^{m+1-k} =: y^{m+1} q(t). \tag{6}$$

Using these, we find that

$$w_x(X) = \sum_{k=1}^{m+1} k\alpha_k x^{k-1} y^{m+1-k} = y^m p'(t), \quad \text{and}$$
$$w_y(X) = \sum_{k=0}^{m} (m+1-k)\alpha_k x^k y^{m-k} = y^m((m+1)p(t) - tp'(t)).$$

Similarly, we see that

$$W_x(X) = y^m q'(t) \text{ and } W_y(X) = y^m((m+1)q(t) - tq'(t)).$$

Then

$$\frac{W_y(X)}{W_x(X)} = \frac{w_y(X)}{w_x(X)} \iff \frac{(m+1)q(t) - tq'(t)}{q'(t)} = \frac{(m+1)p(t) - tp'(t)}{p'(t)}$$
$$\iff \frac{q'(t)}{q(t)} = \frac{p'(t)}{p(t)},$$

and the last relation, by integration, gives $q(t) = \lambda p(t)$ for some constant λ. Thus, we have the desired claim $w = \lambda W$. Consequently, by (5), det $D\tilde{f}(X) \equiv 1$ implies that

$$(1+L)w_y + (1-L)W_x + LW_y - Lw_x \equiv 0$$

which, by the relation $w = \lambda W$, becomes

$$(\lambda + L\lambda + L)W_y + (1 - L - L\lambda)W_x \equiv 0. \tag{7}$$

Allowing $\|X\| \to 0$ in (7), we see that $\lambda W_y(X) + W_x(X) = 0$ is equivalent to

$$\sum_{k=1}^{m+1} \left[\lambda(m+1-(k-1))\beta_{k-1} + k\beta_k\right] x^{k-1} y^{m+1-k} = 0.$$

This gives the condition $\lambda(m+2-k)\beta_{k-1} + k\beta_k = 0$ for $k = 1, \dots, m+1$. From the last relation, it follows easily that

$$\beta_k = \frac{(-\lambda)^k (m+1)!}{k!(m+1-k)!}\beta_0 \ \text{ for } k = 1, \dots, m+1$$

and thus, using this and (6), we obtain that

$$W(X) = \beta_0 (y - \lambda x)^{m+1}.$$

Since

$$W_x(X) = -\lambda(m+1)\beta_0(y-\lambda x)^m \text{ and } W_y(X) = (m+1)\beta_0(y-\lambda x)^m,$$

allowing $\|X\| \to \infty$ in (7) (and making use of the expression L in the proof of Theorem 1 for $n = 2$), we have in case $L \not\equiv 0$ and $\alpha_m \neq 0$:

$$\alpha_m(1+\lambda)^2 = 0,$$

which gives the condition $\lambda = -1$. Note that if $\alpha_m = 0$, we choose the leading largest j for which $\alpha_j \neq 0$. However, the last condition gives that $\lambda = -1$. Consequently, we end up with the forms

$$W(X) = \beta_0(x+y)^{m+1} = \alpha_{m+1}(x+y)^{m+1}$$

and

$$w(X) = -\beta_0(x+y)^{m+1} = -\alpha_{m+1}(x+y)^{m+1}.$$

If $L \equiv 0$, then

$$\tilde{f}(X) = (x + \beta_0(y-\lambda x)^{m+1}, y + \lambda\beta_0(y-\lambda x)^{m+1}).$$

If at the same time $\lambda \neq 0$, then we denote

$$A = \begin{pmatrix} -\lambda & 0 \\ 0 & 1 \end{pmatrix}, \; F(X) = A\tilde{f}(A^{-1}(X)) \text{ and } \alpha_{m+1} = -\lambda\beta_0.$$

Thus, we have $\tilde{f} = A^{-1} \circ F \circ A$ and

$$F(X) = \left(x + \alpha_{m+1}(x+y)^{m+1}, y - \alpha_{m+1}(x+y)^{m+1}\right). \tag{8}$$

If $L \equiv 0$ and $\lambda = 0$, then $w \equiv 0$ and $W(X) = \beta_0 y^{m+1}$ so that

$$\tilde{f}(X) = (x + \beta_0 y^{m+1}, y).$$

Now, we denote

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \text{ and } \alpha_{m+1} = \beta_0.$$

This gives $\tilde{f} = A^{-1} \circ F \circ A$ and F has form (8). The proof is complete. □

Proof of Theorem 3 requires some preparation.

Lemma 1 *Suppose that $G_j \in P_n(m)$ for $j = 1, \ldots, N$ and $G_j(X) = X + u^{(j)}(X)\gamma^{(j)}$, where $u^{(j)}(X) = \sum_{l=2}^m a_l^{(j)} z^l$ with $z = x_1 + \cdots + x_n$, $\gamma^{(j)} = (\gamma_1^{(j)}, \ldots, \gamma_n^{(j)})$, $\sum_{k=1}^n \gamma_k^{(j)} = 0$ for all j, and $a_l^{(j)}$ are some constants. Then $G_1 \circ \cdots \circ G_N =: F \in \mathcal{P}_n(m)$, and*

$$F(X) = X + u^{(1)}(X)\gamma^{(1)} + \cdots + u^{(N)}(X)\gamma^{(N)}.$$

Proof At first, we would like to prove the lemma for $N = 2$ and then extend it for the composition of N mappings, $N \geq 2$. Let $G_j \in P_n(m)$ for $j = 1, 2$ and

$$G_j(X) = X + u^{(j)}(X)\gamma^{(j)}.$$

Also, introduce

$$G_2(X) = (g_1^{(2)}, g_2^{(2)}, \ldots, g_n^{(2)}) \text{ and } L_p(X) = \sum_{l=2}^m l a_l^{(p)} z^{l-1} \text{ for } p = 1, 2.$$

We observe that

$$g_1^{(2)}(X) + \cdots + g_n^{(2)}(X) = z + \sum_{l=2}^m \left[a_l^{(2)} \sum_{k=1}^n \gamma_k^{(2)} z^l \right] = z$$

and therefore,

$$L_1(G_2(X)) = L_1(X) \text{ and } DG_1(G_2(X)) = DG_1(X).$$

Finally, for the case of $N = 2$, one has

$$\begin{aligned} DF(X) &= DG_1(G_2(X))DG_2(X) \\ &= DG_1(X)DG_2(X) = A^{(1)}(L_1)A^{(2)}(L_2) = (r_{ij})_{n\times n}, \end{aligned}$$

where $A^{(p)}(L_p)$ for $p = 1, 2$ are the two $n \times n$ matrices given by

$$A^{(p)}(L_p) = \begin{pmatrix} 1 + \gamma_1^{(p)} L_p & \gamma_1^{(p)} L_p & \cdots & \gamma_1^{(p)} L_p \\ \gamma_2^{(p)} L_p & 1 + \gamma_2^{(p)} L_p & \cdots & \gamma_2^{(p)} L_p \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_n^{(p)} L_p & \gamma_n^{(p)} L_p & \cdots & 1 + \gamma_n^{(p)} L_p \end{pmatrix}, \quad p = 1, 2.$$

If $i \neq j$, then a computation gives

$$r_{ij} = \gamma_i^{(1)} L_1 L_2 \sum_{k=1}^n \gamma_k^{(2)} + \gamma_i^{(1)} L_1 + \gamma_i^{(2)} L_2 = \gamma_i^{(1)} L_1 + \gamma_i^{(2)} L_2;$$

Similarly, for $i = j$, we have

$$r_{ii} = \gamma_i^{(1)} L_1 L_2 \sum_{k=1}^{n} \gamma_k^{(2)} + 1 + \gamma_i^{(1)} L_1 + \gamma_i^{(2)} L_2 = 1 + \gamma_i^{(1)} L_1 + \gamma_i^{(2)} L_2.$$

Consequently, we obtain that

$$DF(X) = \begin{pmatrix} (1+\gamma_1^{(1)} L_1 + \gamma_1^{(2)} L_2) & \gamma_1^{(1)} L_1 + \gamma_1^{(2)} L_2 & \cdots & \gamma_1^{(1)} L_1 + \gamma_1^{(2)} L_2 \\ \gamma_2^{(1)} L_1 + \gamma_2^{(2)} L_2 & (1+\gamma_2^{(1)} L_1 + \gamma_2^{(2)} L_2) & \cdots & \gamma_2^{(1)} L_1 + \gamma_2^{(2)} L_2 \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_n^{(1)} L_1 + \gamma_n^{(2)} L_2 & \gamma_n^{(1)} L_1 + \gamma_n^{(2)} L_2 & \cdots & (1+\gamma_n^{(1)} L_1 + \gamma_n^{(2)} L_2) \end{pmatrix}.$$

Moreover, it follows easily that

$$\begin{aligned} F(X) = (G_1 \circ G_2)(X) &= \begin{pmatrix} x_1 + \gamma_1^{(1)} u^{(1)}(X) + \gamma_1^{(2)} u^{(2)}(X) \\ x_2 + \gamma_2^{(1)} u^{(1)}(X) + \gamma_2^{(2)} u^{(2)}(X) \\ \vdots \\ x_n + \gamma_n^{(1)} u^{(1)}(X) + \gamma_n^{(2)} u^{(2)}(X) \end{pmatrix} \\ &= X + u^{(1)}(X)\gamma^{(1)} + u^{(2)}(X)\gamma^{(2)}. \end{aligned}$$

Similarly in the case of the composition of three mappings G_1, G_2 and G_3 (i.e., for the case of $N = 3$), the Jacobian matrix of $F = G_1 \circ G_2 \circ G_3$ is given by

$$\begin{aligned} DF(X) &= DG_1((G_2 \circ G_3)(X)) DG_2(G_3(X)) DG_3(X) \\ &= DG_1(X) DG_2(X) DG_3(X) \\ &= (r_{ij})_{n \times n}, \end{aligned}$$

where for $i \neq j$,

$$r_{ij} = \gamma_i^{(1)} L_1 + \gamma_i^{(2)} L_2 + \gamma_i^{(3)} L_3, \;\; L_3(X) = \sum_{l=2}^{m} l a_l^{(3)} z^{l-1};$$

and

$$r_{ii} = 1 + \gamma_i^{(1)} L_1 + \gamma_i^{(2)} L_2 + \gamma_i^{(3)} L_3.$$

Moreover, we find that

$$F(X) = (G_1 \circ G_2 \circ G_3)(X) = X + u^{(1)}(X)\gamma^{(1)} + u^{(2)}(X)\gamma^{(2)} + u^{(3)}(X)\gamma^{(3)}.$$

The above process may be continued to complete the proof. □

Remark 3 We remark that the lemma is of interest only when the dimension of the space is greater than or equal to 3. For $n = 2$, it does not give anything new in comparison with Theorem 1. However, for $n > 2$, we obtain from Lemma 1 new mappings from $\mathcal{P}_n(m)$, not belonging to $P_n(m)$.

Example 1 Let $m = 3 = n$, $N = 2$, $a_l^{(1)} = l - 1$, $a_l^{(2)} = 2 + l$, $\gamma^{(1)} = (1, 2, -3)$, $\gamma^{(2)} = (-3, 1, 2)$ and apply Lemma 1 for the mappings $G_1, G_2 \in P_3(3)$.

Define $F(X) = (G_1 \circ G_2)(X) = (u_1(X), u_2(X), u_3(X))$, where $X = (x_1, x_2, x_3)$, $z = x_1 + x_2 + x_3$. Since

$$\begin{aligned} F(X) &= X + u^{(1)}(X)\gamma^{(1)} + u^{(2)}(X)\gamma^{(2)} \\ &= X + (z^2 + 2z^3)(1, 2, -3) + (4z^2 + 5z^3)(-3, 1, 2), \end{aligned}$$

a comparison gives

$$u_1(X) = x_1 - 11z^2 - 13z^3,\ u_2(X) = x_2 + 6z^2 + 9z^3 \text{ and } u_3(X) = x_3 + 5z^2 + 4z^3.$$

According to Lemma 1, we obtain that $F \in \mathcal{P}_3(3)$. We now show that $F \notin P_3(3)$. For its proof, it is enough to show that there are no such vectors $\Gamma = (\Gamma_1, \Gamma_2)$ and $A = (A_1, A_2)$ from $\mathbb{R}^2$ that

$$\Gamma_1 A = (-11, -13),\ \Gamma_2 A = (6, 9) \text{ and } (-\Gamma_1 - \Gamma_2)A = (5, 4).$$

It is easy to see that the above system of equations has no solution which means that $F \notin P_3(3)$. □

Now, we prove our final result which offers functions from $\mathcal{P}_n(m)$ and generalizes Theorem 1 significantly in a natural way.

Proof of Theorem 3 The idea of the proof is in presenting F as a composition of mappings $G_j \in P_n(m)$ for $j = 1, \ldots, N$. For G_j, we may write $G_j(X) = (u_1^{(j)}, \ldots, u_n^{(j)})$, where

$$u_k^{(j)}(X) = x_k + \gamma_k^{(j)} \sum_{l=2}^{m} A_l^{(j)} z^l, \quad \sum_{k=1}^{n} \gamma_k^{(j)} = 0 \text{ for all } j = 1, \ldots, N,$$

and $A_l^{(j)}$ are some constants.

From Lemma 1, the possibility of choosing $F = G_1 \circ \cdots \circ G_N$ means the following: there exist two sets of numbers

$$\left\{\gamma_k^{(j)}\right\}_{k=1,\, j=1}^{n,\, N} \text{ with } \sum_{k=1}^{n} \gamma_k^{(j)} = 0 \text{ for all } j = 1, \ldots, N,$$

and

$$\left\{A_l^{(j)}\right\}_{l=2,\, j=1}^{m,\, N}$$

such that for all $k = 1, \ldots, n$,

$$\gamma_k^{(1)} A_l^{(1)} + \gamma_k^{(2)} A_l^{(2)} + \cdots + \gamma_k^{(N)} A_l^{(N)} = p_k^{(l)} \tag{9}$$

holds for each $l = 2, \ldots, m$. For the solution of this task, for every $j = 1, \ldots, N$, we fix a set of nonzero numbers $\left\{\gamma_k^{(j)}\right\}_{k=1}^{n}$ such that $\sum_{k=1}^{n} \gamma_k^{(j)} = 0$ and the rank of the matrix $\left\{\gamma_k^{(j)}\right\}_{k=1,\, j=1}^{n,\, N}$ is equal to $(n-1)$ (we can consider $N > n$). Then in each of the linear system of equations (9) for $l = 2, \ldots, m$, the corresponding matrix $\left(\gamma_k^{(j)}\right)_{k=1,\, j=1}^{n,\, N}$ of the system is one and the same and has also the rank $(n-1)$, and $A_l^{(1)}, \ldots, A_l^{(N)}$ play a role of variables in the system of equations (9). At the same time in each system (9), the rank of an expanded matrix $\left(\gamma_k^{(j)} \cup p_k^{(l)}\right)_{k=1,\, j=1}^{n,\, N}$ as well as the rank of $\left(\gamma_k^{(j)}\right)_{k=1,\, j=1}^{n,\, N}$ will be equal to $(n-1)$ in each system, since

$$\sum_{k=1}^{n} p_k^{(l)} = 0 = \sum_{k=1}^{n} \gamma_k^{(j)} \quad \text{for each } l \text{ and } j.$$

Therefore, according to the theorem of Kronecker, each of the system of equations (9) ($l = 2, \ldots, m$) will have the solution $(A_l^{(1)}, \ldots, A_l^{(N)})$. It finishes the proof of Theorem 3.

Remark 4 The proof that F in Theorem 3 is injective can also be easily obtained from the form F analogous to that of Remark 2 to Theorem 1.

Remark 5 For $n \geq 3$, Theorem 3 significantly expands the set $P_n(m)$ of mappings for which **JC** is fair. Really, without parameters of matrices A and B from Theorem 1, the set $P_n(m)$ has $[(m-1)+(n-1)]$ free parameters, and the set of polynomial mappings from Theorem 3 has $(m-1)(n-1)$ such parameters.

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