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Laplacian of a Graph Covering and Its Applications

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Abstract Let G be a finite graph and $\overline{G}$ be any graph covering over G. By applying the representation theory of symmetric groups, the Laplacian characteristic polynomial and the normalized Laplacian characteristic polynomial of $\overline{G}$ are investigated. As applications, adopting the algebra method the Kirchhoff index, the multiplicative degree-Kirchhoff index and the complexity of any connected covering over a connected graph are derived.

Keywords Graph covering · Laplacian matrix · Normalized Laplacian matrix · Resistance distance · Kirchhoff index · Complexity

Mathematics Subject Classification 05C50

1 Introduction

Let $G = (V(G), E(G))$ be a finite graph with vertex set $V(G)$ and edge set $E(G)$, which may have multiple edges and loops. A *homomorphism* from a finite graph $\overline{G}$ to G is a map that sends vertices to vertices, edges to edges, and preserves incidence. A homomorphism map $\pi : \overline{G} \to G$ is a *covering projection* if it is surjection such

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that $\pi|_{N(\overline{v})}: N(\overline{v}) \to N(v)$ is a bijection for all vertices $v \in V(G)$ and $\overline{v} \in \pi^{-1}(v)$, where $N(v)$ denotes the neighborhood of the vertex v. The graph $\overline{G}$ is called a *graph covering* over G, and $\overline{G}$ is a *k-fold covering* if π is k-to-one. As an important class of graphs, covering graphs have been studied in many literatures [5,6,15,18]. Especially, Feng, Kwark and Lee discussed the characteristic polynomials of a graph covering in [6]. Deng et al. introduced the concept of ramified coverings over a digraph as a generalization of branched coverings and derived a decomposition formula for the adjacency characteristic polynomial of a ramified covering in [5].

The resistance distance, a novel distance function defined on graphs, is put forward by Klein and Randić [13]. The *resistance distance* between two vertices v_i and v_j of a connected graph G, denoted by $r_{v_i v_j}(G)$, is the electrical resistance between v_i and v_j when placing a unit resistor on every edge and a battery is attached at v_i and v_j. As an important branch of electrical circuit theory, the resistance distance has been much studied in Physics and Engineering. Consequently, analogous to distance-based graph invariants, various graph invariants based on resistance distance have been defined and investigated. In particular, the two most famous graph invariants are the Kirchhoff index and the multiplicative degree-Kirchhoff index. The *Kirchhoff index* of a connected graph G [13] is defined as the sum of resistance distances between all pairs of vertices of G, i.e.,

$$\boldsymbol{Kf}(G) = \sum_{i<j} r_{v_i v_j}(G).$$

The Kirchhoff index is closely related to the spectrum on the Laplacian matrix of G with $n(n \geq 2)$ vertices, that is $\boldsymbol{Kf}(G) = n \sum_{i=2}^{n} \frac{1}{\lambda_i}$, where $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ are all eigenvalues of the Laplacian matrix of G [9]. Another related descriptor that incorporate the degree d_{v_i} of vertex $v_i \in V(G)$ is the *multiplicative degree-Kirchhoff index*, introduced by Chen and Zhang [3] and defined as:

$$\boldsymbol{Kf}^*(G) = \sum_{i<j} d_{v_i} d_{v_j} r_{v_i v_j}(G).$$

A lot of attention has been given in recent years to these two indices. They are currently researched in a good deal of literature; see the recent papers [7,12,17,19], and the references cited therein.

The rest of the present paper is organized as follows. In Sect. 2, some auxiliary results are then given. In Sect. 3 based on the relationship between graph coverings and permutation voltage graphs, the Laplacian and the normalized Laplacian characteristic polynomials of any graph covering $\overline{G}$ over a finite graph G are obtained. This is done by applying the representation theory of symmetric groups. As applications, the formulae for the Kirchhoff index, the multiplicative degree-Kirchhoff index and the complexity of any connected graph covering over a connected graph are derived in Sect. 4.

2 Preliminary

We use the block diagonal sum $A_1 \oplus \cdots \oplus A_n$ or $\oplus_{i=1}^{n} A_i$ for the block diagonal sum of some square matrices $A_1, \ldots, A_n$ and we write as $n \circ A$ for short when $A_1 = \cdots = A_n = A$. The *Kronecker product* $A \otimes B$ of matrices A and B is considered as the matrix B having the element b_{ij} replaced by the matrix $b_{ij}A$. $\mathbf{I}_n$ denotes the identity matrix of order n.

In this section, we recall some known results which are needed in next two sections.

Suppose that $\mathbb{C}$ is the complex field. The *complex general linear group* of degree r, denoted by $GL(r, \mathbb{C})$, is the group of all $r \times r$ invertible matrices over $\mathbb{C}$ with respect to multiplication. Let H be a finite group. A *representation* ρ of the group H over the complex field $\mathbb{C}$ is a homomorphism from H to $GL(r, \mathbb{C})$, and r is also called the degree of the representation ρ. If the representation $\rho : H \to GL(r, \mathbb{C})$ sends each $g \in H$ to a permutation matrix, then ρ is called the *permutation representation* of H.

Let S_k be the symmetric group on $[k] = \{1, 2, \ldots, k\}$. The permutation representation $\mathbf{P}$ of S_k sends each $g \in S_k$ to the $k \times k$ permutation matrix $\mathbf{P}_g = (p_{ij}^{(g)})$, where

$$p_{ij}^{(g)} = \begin{cases} 1 & \text{if } i = g(j), \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Lemma 1 [16] *Let S_k be the symmetric group on $[k]$. Suppose that $\rho_1 = \mathbf{I}_1, \rho_2, \ldots, \rho_s$ form a complete set of inequivalent irreducible representations of S_k and f_i is the degree of ρ_i for each $i \in [s]$, where $f_1 = 1$. Then, there exists a nonsingular matrix M such that for all $g \in S_k$*

$$M^{-1}\mathbf{P}_g M = m_1 \circ \mathbf{I}_1 \oplus m_2 \circ \rho_2(g) \oplus \cdots \oplus m_s \circ \rho_s(g),$$

where for each $i \in [s]$ m_i is the multiplicity of the inequivalent irreducible representation ρ_i in the permutation representation $\mathbf{P}$ *which is defined by Eq.* (1).

Let $G = (V(G), E(G))$ be a finite graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. A digraph $\overleftrightarrow{G}$ is obtained from the graph G by changing every edge of G to a pair of oppositely directed edges. We denote the directed edge from vertex v_i to vertex v_j as ordered pair (v_i, v_j). By $e^{-1} = (v_j, v_i)$, we mean the reverse directed edge to $e = (v_i, v_j)$. The function $\phi : E(\overleftrightarrow{G}) \to S_k$ is called a *permutation voltage assignment* on G if $\phi(e)^{-1} = \phi(e^{-1})$ for each $e \in E(\overleftrightarrow{G})$. The pair (G, ϕ) is called a *permutation voltage graph*. A *derived graph* G^ϕ of (G, ϕ) is a graph with vertex set $V(G^\phi) = V(G) \times [k] = \{(v_i, u) | v_i \in V(G), u \in [k]\}$, and (v_i, u) and (v_j, v) are adjacent if and only if $e = (v_i, v_j) \in E(\overleftrightarrow{G})$ and $v = \phi(e)u$. Gross and Tucker [8] showed that every k-fold covering $\overline{G}$ over G is a one-to-one correspondence with some derived graph G^ϕ of (G, ϕ).

The adjacency matrix of the graph G is written as $\mathbf{A}_G = (a_{ij})_{n \times n}$, where a_{ij} equals to the number of edges between vertices v_i and v_j if $i \neq j$, and otherwise, it equals to twice the number of loops. $\mathbf{D}_G = diag(d_G(v_1), d_G(v_2), \ldots, d_G(v_n))$ means the degree matrix of G, where the degree $d_G(v_i)$ of the vertex v_i is the number of neighbors of v_i (each loop at v_i contributes 2 units to the degree $d_G(v_i)$). By $\mathbf{L}_G = \mathbf{D}_G - \mathbf{A}_G$ and $\mathfrak{L}_G = \mathbf{D}_G^{-\frac{1}{2}}\mathbf{L}_G\mathbf{D}_G^{-\frac{1}{2}} = \mathbf{I}_n - \mathbf{D}_G^{-\frac{1}{2}}\mathbf{A}_G\mathbf{D}_G^{-\frac{1}{2}}$, we denote the Laplacian matrix and the normalized Laplacian matrix of G, respectively. Note that the isolated vertex is not permitted in G in order to $\mathfrak{L}_G$ can be well-defined. Let $\widetilde{G}$ be a signed graph [10] whose underlying graph is the graph G and let $\sigma : E(\widetilde{G}) \to \{+, -\}$ be the sign function of $\widetilde{G}$. The adjacency matrix of $\widetilde{G}$ is the matrix $\mathbf{A}_{\widetilde{G}}$ whose off-diagonal entries are $\widetilde{a}_{ij} = d^+(v_i, v_j) - d^-(v_i, v_j)$ for $i \neq j$, and whose diagonal entries are $\widetilde{a}_{ii} = 2d^+(v_i, v_i) - 2d^-(v_i, v_i)$, where $d^\varepsilon(v_i, v_j)(\varepsilon = +, -)$ is the number of edges between vertices v_i and v_j labeled ε (the number of loops, if $i = j$). $\mathbf{L}_{\widetilde{G}} = \mathbf{D}_G - \mathbf{A}_{\widetilde{G}}$ and $\mathfrak{L}_{\widetilde{G}} = \mathbf{D}_G^{-\frac{1}{2}}\mathbf{L}_{\widetilde{G}}\mathbf{D}_G^{-\frac{1}{2}}$ are the Laplacian matrix and the normalized Laplacian matrix of $\widetilde{G}$. More results on signed graphs can be found in [11,20].

Lemma 2 (The Matrix-Tree Theorem) [1,4] *If $\mathbf{L}_G$ is the Laplacian matrix of a connected graph G, then the complexity $\kappa(G)$ (the number of spanning trees in G) is*

$$\kappa(G) = \det \mathbf{L}_G(i),$$

where $\mathbf{L}_G(i)$ is the matrix obtained from $\mathbf{L}_G$ by deleting its i-th row and i-th column.

Lemma 3 [1,2] *The resistance distance of a connected graph G between any pairs of the vertices v_i and v_j is*

$$r_{v_i v_j}(G) = \frac{\det \mathbf{L}_G(i, j)}{\det \mathbf{L}_G(i)},$$

where $\mathbf{L}_G(i, j)$ is the matrix obtained from the Laplacian matrix $\mathbf{L}_G$ of G by deleting its i-th and j-th rows and columns.

According to the famous Matrix-Tree Theorem and Lemma 3, the Kirchhoff index of a connected graph G can be represented as:

$$\mathbf{K}f(G) = \sum_{1\le i<j\le n} \frac{\det \mathbf{L}_G(i, j)}{\kappa(G)}. \tag{2}$$

Let $\Phi_{\mathbf{L}}(G; x) = \det(x\mathbf{I}_n - \mathbf{L}_G)$ and $\Phi_{\mathfrak{L}}(G; x) = \det(x\mathbf{I}_n - \mathfrak{L}_G)$ be the characteristic polynomials of the Laplacian matrix and the normalized Laplacian matrix of G, respectively.

Lemma 4 [17] *Let G be a connected graph with $n(n \geq 2)$ vertices. Then, the Kirchhoff index of G is given by*

$$\mathbf{K}f(G) = -\frac{n}{2} \cdot \frac{\Phi''_{\mathbf{L}}(G; 0)}{\Phi'_{\mathbf{L}}(G; 0)}.$$

An analogy between the Kirchhoff and multiplicative degree-Kirchhoff indices is:

Lemma 5 [3] *Let* $Spec_{\mathfrak{L}}(G) = \{0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_n\}$ *be the normalized Laplacian spectrum of a connected graph* G *with* $n(n \geq 2)$ *vertices and* m *edges. Then, the multiplicative degree-Kirchhoff index of* G *is*

$$\mathbf{K}f^*(G) = 2m \sum_{i=2}^{n} \frac{1}{\mu_i}.$$

3 The Laplacian and the Normalized Laplacian Characteristic Polynomials of Any *k*-Fold Covering

In this section, let G be a finite graph which multiple edges and loops are allowed, and $\overline{G}$ a k-fold covering over G. Suppose that vertex set $V(G) = \{\nu_1, \nu_2, \ldots, \nu_n\}$ and $|E(G)| = m$. It is apparent that $|V(\overline{G})| = kn$ and $|E(\overline{G})| = km$. Furthermore, we can find a permutation voltage assignment $\phi : E(\overleftrightarrow{G}) \to S_k$ on G and $\overline{G}$ brings into correspondence with G^{ϕ}. Suppose $\Gamma = \langle \phi(e) | e \in E(\overleftrightarrow{G}) \rangle$ is the subgroup of S_k generated by $\{\phi(e) | e \in E(\overleftrightarrow{G})\}$, whose elements act as permutation on the set $[k]$. For any $g \in \Gamma$, the $n \times n$ matrix $\mathbf{A}_g = (a_{ij}^{(g)})$ is defined as:

$$a_{ij}^{(g)} = \left| \left\{ e \in E(\overleftrightarrow{G}) | \phi(e) = g,\ e \in (\nu_i, \nu_j) \right\} \right|. \tag{3}$$

It is easy to check that

$$\mathbf{A}_G = \sum_{g \in \Gamma} \mathbf{A}_g. \tag{4}$$

Arrange the vertices of the graph covering $\overline{G}$ into k blocks:

$$\nu_1^{(1)}, \nu_2^{(1)}, \ldots \nu_n^{(1)}; \quad \nu_1^{(2)}, \nu_2^{(2)}, \ldots, \nu_n^{(2)}; \quad \ldots; \quad \nu_1^{(k)}, \nu_2^{(k)}, \ldots, \nu_n^{(k)}.$$

We consider the adjacency matrix $\mathbf{A}_{\overline{G}}$, degree matrix $\mathbf{D}_{\overline{G}}$, Laplacian matrix $\mathbf{L}_{\overline{G}}$, and normalized Laplacian matrix $\mathfrak{L}_{\overline{G}}$ of $\overline{G}$ under this order.

Theorem 6 *Let* G *be a finite graph (isolated vertex is excluded when the normalized Laplacian characteristic polynomial is considered). Suppose that* $\phi : E(\overleftrightarrow{G}) \to S_k$ *is a permutation voltage assignment on* G *and* $\Gamma = \langle \phi(e) | e \in E(\overleftrightarrow{G}) \rangle$*.* $\mathbf{P}$ *is a permutation representation of* Γ *associated with the set* $[k]$ *which is defined as Eq. (1). Let* $\rho_1 = I_1, \rho_2, \ldots, \rho_s$ *be all inequivalent irreducible representations of* Γ*, and the multiplicity of* ρ_i *in* $\mathbf{P}$ *and the degree of* ρ_i *be denoted by* m_i *and* f_i *for each* $i \in [s]$*, respectively. The Laplacian and the normalized Laplacian characteristic polynomials of the k-fold covering* $\overline{G}$ *over* G *associated with* ϕ *can be decomposed to the follows:*

$$\Phi_{\mathbf{L}}(\overline{G}; x) = \Phi_{\mathbf{L}}^{m_1}(G; x) \prod_{i=2}^{s} \left[\det(x\mathbf{I}_{nf_i} - \mathbf{D}_G \otimes \mathbf{I}_{f_i} + \Sigma_{g \in \Gamma}(\mathbf{A}_g \otimes \rho_i(g))) \right]^{m_i},$$

and

$$\Phi_{\mathfrak{L}}(\overline{G};x)=\Phi_{\mathfrak{L}}^{m_1}(G;x)\prod_{i=2}^{s}\left[\det\left((x-1)\mathbf{I}_{nf_i}+\Sigma_{g\in\Gamma}\left(\mathbf{D}_G^{-\frac{1}{2}}\mathbf{A}_g\mathbf{D}_G^{-\frac{1}{2}}\otimes\rho_i(g)\right)\right)\right]^{m_i},$$

where $\mathbf{A}_g$ *is given by Eq.* (3).

Proof Kwak and Lee [14] showed the adjacency matrix $\mathbf{A}_{\overline{G}}$ of the k-fold covering $\overline{G}$ is

$$\mathbf{A}_{\overline{G}}=\sum_{g\in\Gamma}(\mathbf{A}_g\otimes\mathbf{P}_g). \tag{5}$$

From the definition of graph coverings, the degree matrix $\mathbf{D}_{\overline{G}}$ of $\overline{G}$ can be expressed as:

$$\mathbf{D}_{\overline{G}}=\mathbf{D}_G\otimes\mathbf{I}_k. \tag{6}$$

By Lemma 1, there exists a nonsingular matrix M such that for all $g\in\Gamma$

$$M^{-1}\mathbf{P}_gM=m_1\circ\rho_1(g)\oplus m_2\circ\rho_2(g)\oplus\cdots\oplus m_s\circ\rho_s(g), \tag{7}$$

where $\rho_1(g)=\mathbf{I}_1$.

Set $T=\mathbf{I}_n\otimes M$, then we have

$$\begin{aligned}
T^{-1}\mathbf{A}_{\overline{G}}T&=(\mathbf{I}_n\otimes M)^{-1}\sum_{g\in\Gamma}(\mathbf{A}_g\otimes\mathbf{P}_g)(\mathbf{I}_n\otimes M)\\
&=\sum_{g\in\Gamma}(\mathbf{A}_g\otimes M^{-1}\mathbf{P}_gM)\\
&=\sum_{g\in\Gamma}[\mathbf{A}_g\otimes(\oplus_{i=1}^{s}m_i\circ\rho_i(g))]\quad\text{(by Eq. (7))}\\
&=\bigoplus_{i=1}^{s}\left\{\Sigma_{g\in\Gamma}[\mathbf{A}_g\otimes(m_i\circ\rho_i(g))]\right\}\\
&=\left[\Sigma_{g\in\Gamma}(\mathbf{A}_g\otimes\mathbf{I}_{m_1})\right]\bigoplus\bigoplus_{i=2}^{s}\left\{\Sigma_{g\in\Gamma}[\mathbf{A}_g\otimes(m_i\circ\rho_i(g))]\right\}\\
&=(\mathbf{A}_G\otimes\mathbf{I}_{m_1})\bigoplus\bigoplus_{i=2}^{s}\left\{m_i\circ\left[\Sigma_{g\in\Gamma}(\mathbf{A}_g\otimes\rho_i(g))\right]\right\}\quad\text{(by Eq. (4))}.
\end{aligned}$$

From Eq. (6), it is obtained that $T^{-1}\mathbf{D}_{\overline{G}}T=(\mathbf{I}_n\otimes M^{-1})(\mathbf{D}_G\otimes\mathbf{I}_k)(\mathbf{I}_n\otimes M)=\mathbf{D}_G\otimes\mathbf{I}_k$ and notice that $f_1=1$ and $\sum_{i=1}^{s}m_if_i=k$. According to the above facts, it follows:

$$
\begin{aligned}
&T^{-1}\mathbf{L}_{\overline{G}}T\\
&= T^{-1}(\mathbf{D}_{\overline{G}} - \mathbf{A}_{\overline{G}})T\\
&= \mathbf{D}_G \otimes \mathbf{I}_k - (\mathbf{A}_G \otimes \mathbf{I}_{m_1}) \bigoplus \bigoplus_{i=2}^{s} \{m_i \circ [\Sigma_{g\in\Gamma}(\mathbf{A}_g \otimes \rho_i(g))]\}\\
&= (\mathbf{D}_G \otimes \mathbf{I}_{m_1}) \bigoplus (\mathbf{D}_G \otimes \mathbf{I}_{\Sigma_{i=2}^{s} m_i f_i})\\
&\quad -(\mathbf{A}_G \otimes \mathbf{I}_{m_1}) \bigoplus \bigoplus_{i=2}^{s} \{m_i \circ [\Sigma_{g\in\Gamma}(\mathbf{A}_g \otimes \rho_i(g))]\}\\
&= [(\mathbf{D}_G - \mathbf{A}_G) \otimes \mathbf{I}_{m_1}] \bigoplus \left\{ \bigoplus_{i=2}^{s} [m_i \circ (\mathbf{D}_G \otimes \mathbf{I}_{f_i})] \right.\\
&\quad \left. - \bigoplus_{i=2}^{s} [m_i \circ (\Sigma_{g\in\Gamma}(\mathbf{A}_g \otimes \rho_i(g)))] \right\}\\
&= (\mathbf{L}_G \otimes \mathbf{I}_{m_1}) \bigoplus \bigoplus_{i=2}^{s} \{m_i \circ [\mathbf{D}_G \otimes \mathbf{I}_{f_i} - \Sigma_{g\in\Gamma}(\mathbf{A}_g \otimes \rho_i(g))]\}. \qquad (8)
\end{aligned}
$$

Thus, the Laplacian characteristic polynomial $\Phi_{\mathbf{L}}(\overline{G}; x)$ is obtained from Eq. (8).

Now, we consider the normalized Laplacian characteristic polynomial $\Phi_{\mathfrak{L}}(\overline{G}; x)$ of $\overline{G}$. By Eq. (4), the normalized Laplacian matrix $\mathfrak{L}_G$ of G can be written as:

$$
\mathfrak{L}_G = \mathbf{I}_n - \mathbf{D}_G^{-\frac{1}{2}} \mathbf{A}_G \mathbf{D}_G^{-\frac{1}{2}} = \mathbf{I}_n - \sum_{g\in\Gamma} \mathbf{D}_G^{-\frac{1}{2}} \mathbf{A}_g \mathbf{D}_G^{-\frac{1}{2}}. \qquad (9)
$$

Moreover, by using Eqs. (5) and (6), the normalized Laplacian matrix $\mathfrak{L}_{\overline{G}}$ of $\overline{G}$ can be expressed as:

$$
\begin{aligned}
\mathfrak{L}_{\overline{G}} &= \mathbf{I}_{nk} - \mathbf{D}_{\overline{G}}^{-\frac{1}{2}} \mathbf{A}_{\overline{G}} \mathbf{D}_{\overline{G}}^{-\frac{1}{2}}\\
&= \mathbf{I}_n \otimes \mathbf{I}_k - (\mathbf{D}_G \otimes \mathbf{I}_k)^{-\frac{1}{2}} \sum_{g\in\Gamma} (\mathbf{A}_g \otimes \mathbf{P}_g)(\mathbf{D}_G \otimes \mathbf{I}_k)^{-\frac{1}{2}}\\
&= \mathbf{I}_n \otimes \mathbf{I}_k - \sum_{g\in\Gamma} \left(\mathbf{D}_G^{-\frac{1}{2}} \mathbf{A}_g \mathbf{D}_G^{-\frac{1}{2}} \otimes \mathbf{P}_g \right). \qquad (10)
\end{aligned}
$$

Therefore, $T^{-1}\mathfrak{L}_{\overline{G}}T$ takes the matrix $\mathfrak{L}_{\overline{G}}$ to the following form applying Eq. (10):

$$
\begin{aligned}
&T^{-1}\mathfrak{L}_{\overline{G}}T\\
&= \mathbf{I}_n \otimes \mathbf{I}_k - \sum_{g\in\Gamma} \left(\mathbf{D}_G^{-\frac{1}{2}} \mathbf{A}_g \mathbf{D}_G^{-\frac{1}{2}} \otimes M^{-1} \mathbf{P}_g M \right)\\
&= \mathbf{I}_n \otimes \mathbf{I}_k - \sum_{g\in\Gamma} \left[\mathbf{D}_G^{-\frac{1}{2}} \mathbf{A}_g \mathbf{D}_G^{-\frac{1}{2}} \otimes (\mathbf{I}_{m_1} \oplus \oplus_{i=2}^{s} (m_i \circ \rho_i(g))) \right] \quad \text{(by Eq. (7))}
\end{aligned}
$$

$$
\begin{aligned}
&= (\mathbf{I}_n \otimes \mathbf{I}_{m_1}) \oplus \oplus_{i=2}^{s} (\mathbf{I}_n \otimes \mathbf{I}_{m_i f_i}) \\
&\quad - \sum_{g \in \Gamma} \left\{ \left(\mathbf{D}_G^{-\frac{1}{2}} \mathbf{A}_g \mathbf{D}_G^{-\frac{1}{2}} \otimes \mathbf{I}_{m_1} \right) \oplus \oplus_{i=2}^{s} \left[m_i \circ \left(\mathbf{D}_G^{-\frac{1}{2}} \mathbf{A}_g \mathbf{D}_G^{-\frac{1}{2}} \otimes \rho_i(g) \right) \right] \right\} \\
&= \left[\mathbf{I}_n \otimes \mathbf{I}_{m_1} - \Sigma_{g \in \Gamma} \left(\mathbf{D}_G^{-\frac{1}{2}} \mathbf{A}_g \mathbf{D}_G^{-\frac{1}{2}} \otimes \mathbf{I}_{m_1} \right) \right] \bigoplus \bigoplus_{i=2}^{s} \{ \mathbf{I}_{nf_i} \otimes \mathbf{I}_{m_i} \\
&\quad - \Sigma_{g \in \Gamma} \left[\left(\mathbf{D}_G^{-\frac{1}{2}} \mathbf{A}_g \mathbf{D}_G^{-\frac{1}{2}} \otimes \rho_i(g) \right) \otimes \mathbf{I}_{m_i} \right] \Big\} \\
&= (\mathfrak{L}_{\overline{G}} \otimes \mathbf{I}_{m_1}) \bigoplus \bigoplus_{i=2}^{s} \left\{ \left[\mathbf{I}_{nf_i} - \Sigma_{g \in \Gamma} \left(\mathbf{D}_G^{-\frac{1}{2}} \mathbf{A}_g \mathbf{D}_G^{-\frac{1}{2}} \otimes \rho_i(g) \right) \right] \otimes \mathbf{I}_{m_i} \right\} \text{ (by Eq. (9))}.
\end{aligned}
$$

Thus, the normalized Laplacian characteristic polynomial $\Phi_{\mathfrak{L}}(\overline{G}, x)$ of $\overline{G}$ is derived from above equation. □

If both G and $\overline{G}$ are connected, then we can obtain the following decomposition formulae immediately.

Corollary 7 *Let G be a connected graph. Keeping the notation as in Theorem 6, then the Laplacian characteristic polynomial and the normalized Laplacian characteristic polynomial of any connected k-fold covering $\overline{G}$ over G associated with a permutation voltage assignment ϕ can be decomposed as:*

$$
\Phi_{\mathbf{L}}(\overline{G}; x) = \Phi_{\mathbf{L}}(G; x) \prod_{i=2}^{s} [\det(x\mathbf{I}_{nf_i} - \mathbf{D}_G \otimes \mathbf{I}_{f_i} + \Sigma_{g \in S_k} (\mathbf{A}_g \otimes \rho_i(g)))]^{m_i}, \quad (11)
$$

and

$$
\Phi_{\mathfrak{L}}(\overline{G}; x) = \Phi_{\mathfrak{L}}(G; x) \prod_{i=2}^{s} \left[\det \left((x-1)\mathbf{I}_{nf_i} + \Sigma_{g \in S_k} \left(\mathbf{D}_G^{-\frac{1}{2}} \mathbf{A}_g \mathbf{D}_G^{-\frac{1}{2}} \otimes \rho_i(g) \right) \right) \right]^{m_i}. \quad (12)
$$

Proof By the representation theory of symmetric groups, it is easy to see $\Gamma = S_k$ and $m_1 = 1$ if $\overline{G}$ is connected. The results hold from Theorem 6. □

4 Some Applications

Let G be a connected graph. In this section, we apply algebraic method to consider the Kirchhoff index, the multiplicative degree-Kirchhoff index and the complexity of a connected k-fold covering $\overline{G}$ over G.

The following lemma is obtained easily by Lemma 2:

Lemma 8 *Let G be a connected graph with n vertices. Then, we have*

$$
\Phi'_{\mathbf{L}}(G; 0) = (-1)^{n-1} \cdot n\kappa(G), \quad (13)
$$

and

$$\Phi''_{\mathbf{L}}(G;0) = (-1)^{n-2} \cdot 2 \sum_{i<j} \det \mathbf{L}_G(i,j). \tag{14}$$

Theorem 9 *Let G be a connected graph with n vertices, and $\overline{G}$ a connected k-fold covering over G associated with the permutation voltage assignment ϕ. $\mathbf{P}$ is a permutation representation of S_k which is defined as Eq. (1). Let $\rho_1 = I_1, \rho_2, \ldots, \rho_s$ be all inequivalent irreducible representations of S_k, and the multiplicity of ρ_i in $\mathbf{P}$ and the degree of ρ_i are denoted by m_i and f_i for each $i \in [s]$, respectively. Then, the Kirchhoff index of $\overline{G}$ is*

$$\mathbf{K}f(\overline{G}) = k \cdot \mathbf{K}f(G) + kn \cdot \sum_{i=2}^{s} \frac{m_i \Sigma_{j=1}^{nf_i} \det M_i(j)}{\det M_i},$$

where $M_i = \mathbf{D}_G \otimes \mathbf{I}_{f_i} - \Sigma_{g \in S_k}(\mathbf{A}_g \otimes \rho_i(g))$ for $i \in [s]\backslash\{1\}$, and $\mathbf{A}_g$ is given by Eq. (3).

Proof Take $h_i(x) = x\mathbf{I}_{nf_i} - \mathbf{D}_G \otimes \mathbf{I}_{f_i} + \Sigma_{g \in S_k}(\mathbf{A}_g \otimes \rho_i(g))$ for each $i \in [s]\backslash\{1\}$, then Eq. (11) can be rewritten as $\Phi_{\mathbf{L}}(\overline{G};x) = \Phi_{\mathbf{L}}(G;x)\Pi_{i=2}^{s} \det h_i(x)^{m_i}$. Noting that $\Phi_{\mathbf{L}}(G;0) = 0$ and using Eq. (13), we have

$$\begin{aligned} \Phi'_{\mathbf{L}}(\overline{G};x)|_{x=0} &= \left[\Phi'_{\mathbf{L}}(G;x)\prod_{i=2}^{s}\det h_i(x)^{m_i} + \Phi_{\mathbf{L}}(G;x)\left(\prod_{i=2}^{s}\det h_i(x)^{m_i}\right)'\right]|_{x=0} \\ &= (-1)^{n-1} \cdot n\kappa(G)\prod_{i=2}^{s}\det h_i(0)^{m_i}. \end{aligned} \tag{15}$$

Furthermore, combining Eqs. (13) and (14), it follows that

$$\begin{aligned} \Phi''_{\mathbf{L}}(\overline{G};x)|_{x=0} &= \left[\Phi'_{\mathbf{L}}(G;x)\prod_{i=2}^{s}\det h_i(x)^{m_i} + \Phi_{\mathbf{L}}(G;x)\left(\prod_{i=2}^{s}\det h_i(x)^{m_i}\right)'\right]'\Bigg|_{x=0} \\ &= \left[\Phi''_{\mathbf{L}}(G;x)\prod_{i=2}^{s}\det h_i(x)^{m_i} + 2\Phi'_{\mathbf{L}}(G;x)\left(\prod_{i=2}^{s}\det h_i(x)^{m_i}\right)'\right. \\ &\quad \left. + \Phi_{\mathbf{L}}(G;x)\left(\prod_{i=2}^{s}\det h_i(x)^{m_i}\right)''\right]\Bigg|_{x=0} \\ &= (-1)^{n-2} \cdot 2 \sum_{1 \le i<j \le n} \det \mathbf{L}_G(i,j) \cdot \prod_{i=2}^{s}\det h_i(0)^{m_i} \\ &\quad + (-1)^{n-1} \cdot 2n\kappa(G) \cdot \left(\prod_{i=2}^{s}\det h_i(x)^{m_i}\right)'\Bigg|_{x=0}. \end{aligned} \tag{16}$$

Notice that $|V(\overline{G})| = nk$ and by Lemma 4, we obtain that

$$\begin{aligned}
\boldsymbol{Kf}(\overline{G}) &= -\frac{nk}{2} \cdot \frac{\Phi''_{\mathbf{L}}(\overline{G};0)}{\Phi'_{\mathbf{L}}(\overline{G};0)} = -\frac{nk}{2} \cdot \frac{1}{(-1)^{n-1} \cdot n\kappa(G)\Pi_{i=2}^{s} \det h_i(0)^{m_i}} \\
&\quad \cdot \left[(-1)^{n-2} \cdot 2 \sum_{1\le i<j\le n} \det \mathbf{L}_G(i,j) \right. \\
&\quad \cdot \prod_{i=2}^{s} \det h_i(0)^{m_i} + (-1)^{n-1} \cdot 2n\kappa(G) \\
&\quad \left. \cdot \left(\prod_{i=2}^{s} \det h_i(x)^{m_i} \right)' \Bigg|_{x=0} \right] \quad \text{(by Eqs. (15), (16))} \\
&= \frac{k}{\kappa(G)} \cdot \sum_{1\le i<j\le n} \det \mathbf{L}_G(i,j) - kn \cdot \frac{(\Pi_{i=2}^{s} \det h_i(x)^{m_i})'|_{x=0}}{\Pi_{i=2}^{s} \det h_i(0)^{m_i}} \\
&= k \cdot \boldsymbol{Kf}(G) - kn \cdot \left(\log \prod_{i=2}^{s} \det h_i(x)^{m_i} \right)' \Bigg|_{x=0} \quad \text{(by Eq. (2))}. \qquad (17)
\end{aligned}$$

Set $M_i = -h_i(0) = \mathbf{D}_G \otimes \mathbf{I}_{f_i} - \Sigma_{g\in S_k}(\mathbf{A}_g \otimes \rho_i(g))$ for $i \in [s]\backslash\{1\}$, then it is easy to see

$$\det h_i(0) = (-1)^{nf_i} \cdot \det M_i, \qquad (18)$$

and

$$(\det h_i(x))'|_{x=0} = \sum_{j=1}^{nf_i} \det(-M_i(j)) = (-1)^{nf_i-1} \cdot \sum_{j=1}^{nf_i} \det M_i(j).$$

Since the multiplicity of the Laplacian eigenvalue 0 for the connected graph $\overline{G}$ is 1, so we have $\det M_i \neq 0$ for $i \in [s]\backslash\{1\}$ by Eq. (11). Thus, by the above two equations, we get

$$\begin{aligned}
\left(\log \prod_{i=2}^{s} \det h_i(x)^{m_i} \right)' \Bigg|_{x=0} &= \sum_{i=2}^{s} m_i \left[\frac{(\det h_i(x))'}{\det h_i(x)} \right] \Bigg|_{x=0} \\
&= -\sum_{i=2}^{s} \frac{m_i \Sigma_{j=1}^{nf_i} \det M_i(j)}{\det M_i}. \qquad (19)
\end{aligned}$$

This theorem is proved by Eqs. (17) and (19). □

Let G be a connected graph with the permutation voltage assignment $\phi : E(\overleftrightarrow{G}) \to S_2$, and $\widetilde{G}$ a signed graph taking G as the underlying graph of it. Denote e_{ij} the edge connecting vertices v_i and v_j. The sign function $\sigma : E(\widetilde{G}) \to \{+,-\}$ is defined as:

$$\sigma(e_{ij}) = \begin{cases} + & \text{if } \phi(e) = (1), \quad e = (v_i, v_j) \in E(\overleftrightarrow{G}); \\ - & \text{if } \phi(e) = (12), \quad e = (v_i, v_j) \in E(\overleftrightarrow{G}). \end{cases} \tag{20}$$

Then, the adjacency matrix $\mathbf{A}_{\widetilde{G}}$ of $\widetilde{G}$ can be expressed as $\mathbf{A}_{\widetilde{G}} = \mathbf{A}_{(1)} - \mathbf{A}_{(12)}$. Thus, the Kirchhoff index of any connected 2-fold covering over G is as follows:

Corollary 10 *Let G be a connected graph with n vertices, and $\overline{G}$ a 2-fold connected covering over G associated with the permutation voltage assignment ϕ. Besides, let $\widetilde{G}$ be a signed graph which the underlying graph is G and the sign function σ of it is given by Eq. (20). Then, the Kirchhoff index of $\overline{G}$ can be expressed as:*

$$\mathbf{K}f(\overline{G}) = 2 \cdot \mathbf{K}f(G) + 2n \sum_{i=1}^{n} \frac{1}{\mu_i},$$

where $\mu_1, \mu_2, \ldots, \mu_n$ are all of Laplacian eigenvalues of $\widetilde{G}$.

Proof Notice that the permutation representation $\mathbf{P}$ of S_2 associated with ϕ is equivalent to $I_1 \oplus \rho_2$, where $\rho_2((1)) = 1$ and $\rho_2((12)) = -1$, and the multiplicity of ρ_2 in $\mathbf{P}$ is 1. By using Theorem 9 the Kirchhoff index of $\overline{G}$ is

$$\boldsymbol{K}f(\overline{G}) = 2 \cdot \boldsymbol{K}f(G) + 2n \cdot \frac{\Sigma_{j=1}^{n} \det \mathbf{L}_{\widetilde{G}}(j)}{\det \mathbf{L}_{\widetilde{G}}}.$$

The corollary holds by using Vieta formula. □

Theorem 11 *Let G be a connected graph with n vertices, and $\overline{G}$ a connected k-fold covering over G associated with the permutation voltage assignment ϕ. $\mathbf{P}$ is a permutation representation of S_k which is defined as Eq. (1). Let $\rho_1 = I_1, \rho_2, \ldots, \rho_s$ be all inequivalent irreducible representations of S_k, and the multiplicity of ρ_i in $\mathbf{P}$ and the degree of ρ_i are denoted by m_i and f_i for each $i \in [s]$, respectively. Then the complexity of $\overline{G}$ over G associated with ϕ is:*

$$\kappa(\overline{G}) = \frac{1}{k} \cdot \kappa(G) \cdot \prod_{i=2}^{s} (\det M_i)^{m_i}, \tag{21}$$

where $M_i = \mathbf{D}_G \otimes \mathbf{I}_{f_i} - \Sigma_{g \in S_k}(\mathbf{A}_g \otimes \rho_i(g))$.

Proof For one thing Eq. (15) holds, and for another thing by Lemma 8 we have

$$\Phi'_{\mathbf{L}}(\overline{G}; 0)| = (-1)^{nk-1} \cdot nk \cdot \kappa(\overline{G}). \tag{22}$$

Note that $\sum_{i=1}^{s} m_i f_i = k$ and $m_1 = f_1 = 1$. Combining Eqs. (15) and (22), we obtain

$$\begin{aligned}\kappa(\overline{G}) &= (-1)^{n(1-k)} \cdot \frac{\kappa(G)}{k} \cdot \prod_{i=2}^{s} (\det h_i(0))^{m_i} \\ &= (-1)^{n(1-k)} \cdot \frac{\kappa(G)}{k} \cdot (-1)^{n\sum_{i=2}^{s} m_i f_i} \cdot \prod_{i=2}^{s} (\det M_i)^{m_i} \quad \text{(by Eq. (18))} \\ &= \frac{1}{k} \cdot \kappa(G) \cdot \prod_{i=2}^{s} (\det M_i)^{m_i}.\end{aligned}$$

□

Finally, let us consider the multiplicative degree-Kirchhoff index of $\overline{G}$ with the same method as Theorem 9.

Similar to the proof of Lemma 4, the following Lemma 12 is gained by applying Lemma 5.

Lemma 12 *Let G be a connected graph with n vertices and m edges. Then we have*

$$\mathbf{K}f^*(G) = -m \cdot \frac{\Phi''_{\mathfrak{L}}(G;0)}{\Phi'_{\mathfrak{L}}(G;0)}.$$

Lemma 13 *Let G be a connected graph with n vertices and m edges. Then the multiplicative degree-Kirchhoff index can be expressed as:*

$$\mathbf{K}f^*(G) = 2m \cdot \frac{\Sigma_{1\le i<j\le n} \det \mathfrak{L}_G(i,j)}{\Sigma_{i=1}^{n} \det \mathfrak{L}_G(i)}.$$

Proof It is easy to see

$$\Phi'_{\mathfrak{L}}(G;0) = \sum_{i=1}^{n} \det(x\mathbf{I}_{n-1} - \mathfrak{L}_G(i))|_{x=0} = (-1)^{n-1} \cdot \sum_{i=1}^{n} \det \mathfrak{L}_G(i), \tag{23}$$

and

$$\Phi''_{\mathfrak{L}}(G;0) = \left[\sum_{i=1}^{n} \det(x\mathbf{I}_{n-1} - \mathfrak{L}_G(i))\right]' |_{x=0} = (-1)^{n-2} \cdot 2 \sum_{1\le i<j\le n} \det \mathfrak{L}_G(i,j). \tag{24}$$

So, the proof is completed by Lemma 12. □

Theorem 14 *Let G be a connected graph with n vertices and m edges, and $\overline{G}$ a connected k-fold covering over G associated with the permutation voltage assignment ϕ. $\mathbf{P}$ is a permutation representation of S_k which is defined as Eq. (1). Let $\rho_1 = I_1, \rho_2, \ldots, \rho_s$ be all inequivalent irreducible representations of S_k, and the multiplicity of ρ_i and the degree of ρ_i are denoted by m_i and f_i for each $i \in [s]$, respectively. Then the multiplicative degree-Kirchhoff index of $\overline{G}$ is*

$$\mathbf{K}f^*(\overline{G}) = k \cdot \mathbf{K}f^*(G) + 2km \cdot \sum_{i=2}^{s} \frac{m_i \Sigma_{j=1}^{nf_i} \det N_i(j)}{\det N_i},$$

where $N_i = \mathbf{I}_{nf_i} - \Sigma_{g\in S_k}(\mathbf{D}_G^{-\frac{1}{2}}\mathbf{A}_g\mathbf{D}_G^{-\frac{1}{2}} \otimes \rho_i(g))$ *for each* $i \in [s]\backslash\{1\}$.

Proof The proof of this theorem is similar to that of Theorem 9.

Set $l_i(x) = (x-1)\mathbf{I}_{nf_i} + \Sigma_{g\in S_k}(\mathbf{D}_G^{-\frac{1}{2}}\mathbf{A}_g\mathbf{D}_G^{-\frac{1}{2}} \otimes \rho_i(g))$ for each $i \in [s]\backslash\{1\}$. So, Eq. (12) can be rewritten as $\Phi_{\mathfrak{L}}(\overline{G}; x) = \Phi_{\mathfrak{L}}(G; x)\Pi_{i=2}^{s} \det l_i(x)^{m_i}$. Noting that $\Phi_{\mathfrak{L}}(G; 0) = 0$ and by Eq. (23), then

$$\begin{aligned}
\Phi'_{\mathfrak{L}}(\overline{G}; 0) &= \left[\Phi'_{\mathfrak{L}}(G; x)\prod_{i=2}^{s}\det l_i(x)^{m_i} + \Phi_{\mathfrak{L}}(G; x)\left(\prod_{i=2}^{s}\det l_i(x)^{m_i}\right)'\right]\Bigg|_{x=0} \\
&= (-1)^{n-1} \cdot \sum_{i=1}^{n}\det \mathfrak{L}_G(i) \cdot \prod_{i=2}^{s}\det l_i(0)^{m_i}. \qquad (25)
\end{aligned}$$

On the other hand, since $\det \mathfrak{L}_G = 0$, it follows that

$$\begin{aligned}
\Phi''_{\mathfrak{L}}(\overline{G}; 0) &= \left[\Phi'_{\mathfrak{L}}(G; x)\prod_{i=2}^{s}\det l_i(x)^{m_i} + \Phi_{\mathfrak{L}}(G; x)\left(\prod_{i=2}^{s}\det l_i(x)^{m_i}\right)'\right]'\Bigg|_{x=0} \\
&= \left[\Phi''_{\mathfrak{L}}(G; x)\prod_{i=2}^{s}\det l_i(x)^{m_i} + 2\Phi'_{\mathfrak{L}}(G; x)\left(\prod_{i=2}^{s}\det l_i(x)^{m_i}\right)'\right. \\
&\quad \left. + \Phi_{\mathfrak{L}}(G; x)\left(\prod_{i=2}^{s}\det l_i(x)^{m_i}\right)''\right]\Bigg|_{x=0} \\
&= (-1)^{n-2} \cdot 2\sum_{1\le i<j\le n}\det \mathfrak{L}_G(i, j) \cdot \prod_{i=2}^{s}\det l_i(0)^{m_i} \\
&\quad + (-1)^{n-1} \cdot 2\sum_{i=1}^{n}\det \mathfrak{L}_G(i) \cdot \left(\prod_{i=2}^{s}\det l_i(x)^{m_i}\right)'\Bigg|_{x=0}. \\
&\quad \text{(by Eqs. (23), (24))}. \qquad (26)
\end{aligned}$$

Since $|E(\overline{G})| = mk$, and Eqs. (25) and (26), we get

$$\begin{aligned}
\boldsymbol{K}f^*(\overline{G}) &= -km \cdot \frac{\Phi''_{\mathfrak{L}}(\overline{G}; 0)}{\Phi'_{\mathfrak{L}}(\overline{G}; 0)} \quad \text{(by Lemma 12)} \\
&= -\frac{km}{(-1)^{n-1} \cdot \Sigma_{i=1}^{n}\det \mathfrak{L}_G(i) \cdot \Pi_{i=2}^{s}\det l_i(0)^{m_i}}\left[(-1)^{n-2} \cdot 2\sum_{1\le i<j\le n}\det \mathfrak{L}_G(i, j)\right.
\end{aligned}$$

$$
\begin{aligned}
&\cdot \prod_{i=2}^{s} \det l_i(0)^{m_i} + (-1)^{n-1} \cdot 2 \sum_{i=1}^{n} \det \mathfrak{L}_G(i) \cdot \left(\prod_{i=2}^{s} \det l_i(x)^{m_i} \right)' \Bigg|_{x=0} \Bigg] \\
&= 2km \cdot \frac{\Sigma_{1 \le i < j \le n} \det \mathfrak{L}_G(i, j)}{\Sigma_{i=1}^{n} \det \mathfrak{L}_G(i)} - 2km \cdot \frac{(\Pi_{i=2}^{s} \det l_i(x)^{m_i})'|_{x=0}}{\Pi_{i=2}^{s} \det l_i(0)^{m_i}} \\
&= k \cdot \boldsymbol{K} f^*(G) - 2km \cdot \left(\log \prod_{i=2}^{s} \det l_i(x)^{m_i} \right)' \Bigg|_{x=0} \quad \text{(by Lemma 13).} \qquad (27)
\end{aligned}
$$

Let $N_i = -l_i(0) = \mathbf{I}_{nf_i} - \Sigma_{g \in S_k} (\mathbf{D}_G^{-\frac{1}{2}} \mathbf{A}_g \mathbf{D}_G^{-\frac{1}{2}} \otimes \rho_i(g))$ for each $i \in [s] \backslash \{1\}$. Similar to Eq. (19), we can obtain that

$$
\left(\log \prod_{i=2}^{s} \det l_i(x)^{m_i} \right)' \Bigg|_{x=0} = - \sum_{i=2}^{s} \frac{m_i \, \Sigma_{j=1}^{nf_i} \det N_i(j)}{\det N_i}. \qquad (28)
$$

The theorem is proved by Eqs. (27) and (28). □

Similar to the proof of Corollary 10, we get

Corollary 15 *Let G be a connected graph with n vertices and m edges, and $\overline{G}$ a connected 2-fold covering over G associated with the permutation voltage assignment ϕ. Suppose $\widetilde{G}$ is a signed graph taking G as the underlying graph and the sign function σ of it is given by Eq. (20). Then, the multiplicative degree-Kirchhoff index of $\overline{G}$ is*

$$
\mathbf{K} f^*(\overline{G}) = 2 \cdot \mathbf{K} f^*(G) + 4m \sum_{i=1}^{n} \frac{1}{\theta_i},
$$

where $\theta_1, \theta_2, \ldots, \theta_n$ are all of the normalized Laplacian eigenvalues of $\widetilde{G}$.

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