Bull. Malays. Math. Sci. Soc. (2019) 42:2423–2442
https://doi.org/10.1007/s40840-018-0608-y

BULLETIN of the MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Existence of Solutions and Finite-Time Stability for Nonlinear Singular Discrete-Time Neural Networks

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Received: 4 October 2017 / Revised: 25 January 2018 / Published online: 13 February 2018

Abstract This paper investigates the problem of finite-time stability and control for a class of nonlinear singular discrete-time neural networks with time-varying delays and disturbances. First, based on the implicit function theorem and singular value decomposition method, a sufficient condition for the existence of the solution of such systems is established in terms of a linear matrix inequality (LMI). Then, using the Lyapunov functional approach combined with LMI technique we provide new delay-dependent sufficient conditions for robust H_∞ finite-time stability and control. Finally, some numerical examples are given to illustrate the efficiency of the proposed results.

Keywords Finite-time stability · Stabilization · Singularity · Discrete-time systems · Time-varying delays · Linear matrix inequalities

Mathematics Subject Classification 34D06 · 65L20 · 93D20 · 94D05

Communicated by Syakila Ahmad.

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1 Introduction

Over the past decades, the problem of stability and control for neural networks has attracted much attention due to its both practical and theoretical importance [13,14,20–22,31]. However, most of the results have been concerned with the asymptotic stability defined over an infinite-time interval. In many practical applications, the main concern is the behavior of the system over a fixed finite-time interval, for example, large values of the state are not acceptable in the presence of saturations. In this case, the traditional Lyapunov method is not applicable and the finite-time stability method is introduced [1,5,29]. Many valuable results on finite-time stability and control of continuous-time and discrete-time neural networks can be found in [2,16,23,30,32] and in the references therein. It should be noticed that in some systems we must consider their character of dynamic and state at the same time. Up to now, a wide variety of design methods for control of delayed neural networks have been studied mainly including stabilization, adaptive control, fuzzy control. The H_∞ control problem is to design state feedback controllers such that, in addition to the requirement of the robust finite-time stability of the closed-loop system, a specified performance level is also required to be achieved. In the literature, singular systems (also referred to as differential-algebraic equations, implicit systems, descriptor systems or generalized state-space systems) arise in a variety of practical systems such as biological systems, artificial electronic systems, system recognition, target tracking, static image processing and associative memory [4,9,29]. Both delay-independent and delay-dependent stability conditions for singular time delay systems have been extensively obtained by using the SVD approach and Lyapunov function method [7,8,10,15,24–26]. Meanwhile, considering the singular neural networks is of great significance [11,12,19]. Since the singular neural networks are usually described by nonlinear time delay equations, the results on stability and control of such systems are relative few. The main difficulty in studying singular neural networks is to solve the problem of existence and uniqueness of solutions. Some delay-dependent sufficient conditions for optimizing the size of singular neural networks using SVD approach can be found in [11]. The authors of Kumaresan and Balasubramaniam [12] provided solutions to optimal control for stochastic linear singular systems using neural networks with quadratic performance. More interesting criteria for stochastic stability of discrete-time singular neural networks with Markovian jump and time-varying delays were given in [19]. It is also worth mentioning that the problem of existence of the solution and the time-varying delays are not taken into account in the mentioned papers. For nonlinear discrete-time singular systems, since problems of existence and uniqueness of solutions and finite-time stability, regularity, causality need to be considered simultaneously, the finite-time stability analysis for such systems is more complicated and the methods of analyzing the existence and uniqueness of solution to singular systems in the mentioned papers are difficult to be applied. On the other hand, it should be noticed that almost the existing results for singular nonlinear discrete-time systems were developed in the context of Lyapunov asymptotic stability and control while very little attention has been paid to the finite-time stability and control of such systems. To the best of the our knowledge, the problems of the existence of solutions and the finite-time H_∞ control for singular discrete-time neural networks with delay have not been yet investigated, these problems are important and challenging in both theory and practice.

In this paper, we consider H_∞ finite-time stability and control of nonlinear singular discrete-time delay neurals network-based systems. First, by using the implicit function theorem and singular value decomposition method, LMI sufficient conditions are established which guarantees that the discrete-time singular neural networks are regular, causal and have unique solution in a neighborhood of the origin. Then, based on Lyapunov function method, delay-dependent sufficient conditions for designing state feedback controllers of H_∞ finite-time control are derived in terms of LMIs. The design of such controllers can be carried out in a systematic and computationally efficient manner via the use of LMI-based algorithms [6]. The result of this paper can be considered as a further development of the results obtained in [11,12,19]. Last, numerical examples are provided to illustrate the validity and effectiveness of the proposed results.

The structure of the paper is as follows. Section 2 presents problem statement and some technical propositions needed for the proof of the main results. Sufficient conditions for the existence and uniqueness of the solution and for designing state feedback controllers for robust H_∞ control problem are presented in Sect. 3. Numerical examples illustrated that the obtained results are given in Sect. 4.

Notation Z_+ denotes the set of all nonnegative integers; R^n denotes the n-dimensional space with the scalar product $x^\top y$; $R^{n\times r}$ denotes the space of $(n \times r)-$dimension matrices; $A^\top$ denotes the transpose of matrix A; A is positive definite ($A > 0$) if $x^\top Ax > 0$ for all $x \neq 0$; $A > B$ means $A - B > 0$. The notation diag$\{\ldots\}$ stands for a block-diagonal matrix. The symmetric term in a matrix is denoted by $*$.

2 Problem Formulation and Preliminaries

Consider the following discrete-time singular neural networks with time-varying delays and disturbances

$$\begin{cases} Ex(k+1) = Ax(k) + Wf(x(k)) + W_1 g(x(k-h(k))) + Bu(k) + C\omega(k), k \in Z_+, \\ z(k) = A_1 x(k) + Dx(k-h(k)) + B_1 u(k), \\ x(k) = \varphi(k),\ k \in \{-h_2, \ldots, 0\}, \end{cases} \tag{1}$$

where $x(k) \in R^n$ is the state; $u(k) \in R^m$ is the control input; $z(k) \in R^p$ is the observation output; n is the number of neural; $f(x(k)) = [f_1(x_1(k)), f_2(x_2(k)), \ldots, f_n(x_n(k))]$, $g(x(k-h(k))) = [g_1(x_1(k-h(k))), g_2(x_2(k-h(k))), \ldots, g_n(x_n(k-h(k)))]$ are activation functions, where $f_i, g_i, i = \overline{1,n}$, satisfy the following conditions

$$\begin{aligned} &\exists a_i > 0\colon\ |f_i(\xi)| \leqslant a_i|\xi|, \quad \forall i = \overline{1,n},\ \forall \xi \in R, \\ &\exists b_i > 0\colon\ |g_i(\xi)| \leqslant b_i|\xi|, \quad \forall i = \overline{1,n},\ \forall \xi \in R. \end{aligned} \tag{2}$$

The matrix $E \in R^{n\times n}$ is singular and $\underline{\text{rank}}(E) = r \leqslant n$. The diagonal matrix $A = \text{diag}\{\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n\}$, $|\overline{a}_i| < 1\ \forall i = \overline{1,n}$ represents the self-feedback term; the matrices $W, W_1 \in R^{n\times n}$ are the connection weight matrices; $B \in R^{n\times m}$, $B_1 \in R^{p\times m}$ are the control matrices; $C \in R^{n\times q}$ is the disturbance matrix; $A_1, D \in R^{p\times n}$ are the observation matrix; the time-varying delay functions $h(k)$ satisfy the condition

$$0 < h_1 \leqslant h(k) \leqslant h_2 \quad \forall k \in \mathbb{Z}_+, \tag{3}$$

where h_1, h_2 are given positive integers; $\varphi(k)$ is the initial function; the external disturbance $\omega(k) \in R^q$ satisfies the condition

$$\sum_{k=0}^{N} \omega^\top(k)\omega(k) < d, \tag{4}$$

where $d > 0$ is a given number.

Definition 1 [4] The pair (E, A) is said to be regular if characteristic polynomial $\det(sE - A)$, where $s \in C$, is not identical zero. The pair (E, A) is said to be causal if $\deg(\det(sE - A)) = \mathrm{rank}(E)$. System (1) with $u(k) = 0$ is said to be regular and causal if the pair (E, A) is regular and causal.

Definition 2 (*Robust finite-time stability* [5]) Given positive numbers $N, c_1, c_2, c_1 < c_2$, and a symmetric positive-definite matrix R, unforced system (1) ($u(k) = 0$) is robustly finite-time stable w.r.t. (c_1, c_2, R, N) if

$$\max_{k\in\{-h_2,\ldots,0\}} \varphi^\top(k)R\varphi(k) \leqslant c_1 \implies x^\top(k)Rx(k) < c_2 \quad \forall k = 1, 2, \ldots, N$$

for all disturbances $\omega(k)$ satisfying (4).

Definition 3 (*H_∞ finite-time stability* [1]) Given positive numbers $\gamma, N, c_1, c_2, c_1 < c_2$, and a symmetric positive-definite matrix R, unforced system (1) ($u(k) = 0$) is H_∞ finite-time stable w.r.t. (c_1, c_2, R, N) if the following two conditions hold:

(i) System (1) is robustly finite-time stable w.r.t. (c_1, c_2, R, N).
(ii) Under the zero initial condition (i.e., $\varphi(k) = 0\ \forall k \in \{-h_2, -h_2 + 1, \ldots, 0\}$), the output $z(k)$ satisfies

$$\sum_{k=0}^{N} z^\top(k)z(k) \leqslant \gamma \sum_{k=0}^{N} \omega^\top(k)\omega(k) \tag{5}$$

for all disturbances $\omega(k)$ satisfying (4).

Definition 4 (*H_∞ finite-time control*) Given positive numbers $\gamma, N, c_1, c_2, c_1 < c_2$, and a symmetric positive-definite matrix R, the finite-time H_∞ control problem for system (1) has a solution if there exists a state feedback controller $u(k) = Kx(k)$ such that the resulting closed-loop system is H_∞ finite-time stable w.r.t. (c_1, c_2, R, N).

Proposition 1 (Schur Complement Lemma [3]) *Given constant matrices X, Y, Z with appropriate dimensions satisfying $X = X^\top, Y = Y^\top > 0$, then*

$$X + Z^\top Y^{-1} Z < 0 \iff \begin{bmatrix} X & Z^\top \\ Z & -Y \end{bmatrix} < 0.$$

Proposition 2 (The Implicit Function Theorem [28]) *Suppose that V is open in R^{n+p}, and $F = (F_1, \ldots, F_n)\colon V \longrightarrow R^n$ is C^1 on V. Suppose further that $F(x_0, t_0) = 0$ for some $(x_0, t_0) \in V$, where $x_0 \in R^n$ and $t_0 \in R^p$. If Jacobian matrix*

$$\frac{\partial(F_1, \ldots, F_n)}{\partial(x_1, \ldots, x_n)}(x_0, t_0)$$

is nonsingular, then there is an open set $W \subset R^p$, containing t_0 and a unique continuously differentiable function $g\colon W \longrightarrow R^n$ such that $g(t_0) = x_0$, and $F(g(t), t) = 0$ for all $t \in W$.

3 Main Results

Consider singular discrete-time neural networks (1). Due to $\mathrm{rank}(E) = r \leq n$, there are two nonsingular matrices $M, G \in R^{n\times n}$ such that $MEG = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. Let us denote

$$\begin{aligned}
&M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix},\ \bar{M} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} M,\ MAG = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\\
&M^{-\top} P M^{-1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},\\
&F = \mathrm{diag}\{a_1, \ldots, a_n\},\ H = \mathrm{diag}\{b_1, \ldots, b_n\},\ \Phi_{12} = \Phi_{13} = 0,\ \Phi_{14} = A_1^\top D,\\
&\Phi_{11} = -\delta E^\top PE + (h_2 - h_1 + 1)Q + S_1 + A_1^\top A_1 + F^2 - P\bar{M}A - A\bar{M}^\top P,\\
&\Phi_{15} = -P\bar{M}W,\ \Phi_{16} = -P\bar{M}W_1,\ \Phi_{17} = -P\bar{M}C,\ \Phi_{2i} = 0, i = \overline{3, 8},\\
&\Phi_{22} = \delta^{h_1}(-S_1 + S_2),\\
&\Phi_{18} = AP,\ \Phi_{44} = -\delta^{h_1} Q + D^\top D + H^2,\ \Phi_{3i} = 0, i = \overline{4, 8},\ \Phi_{4i} = 0, i = \overline{5, 8},\\
&\Phi_{55} = \Phi_{66} = -I,\ \Phi_{33} = -\delta^{h_2} S_2,\ \Phi_{56} = \Phi_{57} = \Phi_{67} = 0,\ \Phi_{58} = W^\top P,\\
&\Phi_{68} = W_1^\top P,\ \Phi_{77} = \frac{-\gamma}{\delta^N} I,\ \Phi_{78} = C^\top P,\ \Phi_{88} = -P.
\end{aligned}$$

We first show the existence and uniqueness of the solution and the regularity and causality of system (1).

Theorem 1 *Given positive constants $\gamma, N, \delta \geq 1$ unforced system* (1) ($u(k) = 0$) *is regular, causal and has unique solution if there exist symmetric positive-definite matrices P, Q, S_1, S_2 such that the following LMI holds* :

$$\Phi = \left[\, \Phi_{ij} \,\right]_{8\times 8} < 0. \tag{6}$$

Proof First, we prove that unforced system (1) is regular and causal. From (6), it follows that $\Phi_{11} < 0$. Since $Q > 0$, $S_1 > 0$, $F > 0$ and G is nonsingular, we have $G^\top(-\delta E^\top PE - P\bar{M}A - A\bar{M}^\top P)G < 0$ and hence

$$
\begin{aligned}
&-\delta G^\top E^\top M^\top M^{-\top} P M^{-1} M E G - G^\top P\bar{M}AG - G^\top A\bar{M}^\top PG < 0\\
&\Longleftrightarrow -\delta\begin{bmatrix} P_{11} & 0\\ 0 & 0\end{bmatrix} - \begin{bmatrix}(G^\top P)_{12}A_{21} & (G^\top P)_{12}A_{22}\\ (G^\top P)_{22}A_{21} & (G^\top P)_{22}A_{22}\end{bmatrix}\\
&\quad - \begin{bmatrix}(G^\top P)_{12}A_{21} & (G^\top P)_{12}A_{22}\\ (G^\top P)_{22}A_{21} & (G^\top P)_{22}A_{22}\end{bmatrix}^\top < 0\\
&\Longleftrightarrow \begin{bmatrix}\star & \star\\ \star & -(G^\top P)_{22}A_{22} - A_{22}^\top (G^\top P)_{22}^\top\end{bmatrix} < 0,
\end{aligned}
$$

where $\star$ represents matrices that are not relevant in the discussion. The last inequality shows that $(G^\top P)_{22}A_{22} + A_{22}^\top(G^\top P)_{22}^\top > 0$. Assume that A_{22} is singular, then there exists a vector $0 \neq \eta \in \mathbb{R}^{n-r}$ such that $A_{22}\eta = 0$. We have

$$
\eta^\top\left[(G^\top P)_{22}A_{22} + A_{22}^\top(G^\top P)_{22}^\top\right]\eta = \eta^\top(G^\top P)_{22}A_{22}\eta\eta^\top A_{22}^\top(G^\top P)_{22}^\top\eta = 0,
$$

i.e., the matrix $(G^\top P)_{22}A_{22} + A_{22}^\top(G^\top P)_{22}^\top$ is not positive definite. This contradiction enable us to confirm that A_{22} is nonsingular matrix. Hence, according to Definition 1 and [4], the system is regular and causal. We now are in position to prove that the system has a unique solution. By setting

$$
\begin{aligned}
\gamma_i &= \frac{\mathrm{d}}{\mathrm{d}(x_i(k))} f_i(x_i(k))\Big|_{x_i(k)=0},\\
\lambda_i &= \frac{\mathrm{d}}{\mathrm{d}(x_i(k-h(k)))} g_i(x_i(k-h(k)))\Big|_{x_i(k-h(k))=0},
\end{aligned}
$$

the functions $f_i(x_i(k))$ and $g_i(x_i(k-h(k)))$ can be presented in a neighborhood of the origin as

$$
\begin{aligned}
&f_i(x_i(k)) = \gamma_i x_i(k) + \alpha_i(x_i(k)),\\
&g_i(x_i(k-h(k))) = \lambda_i x_i(k-h(k)) + \beta_i(x_i(k-h(k))),
\end{aligned}
$$

where $\alpha_i(0) = 0,\ \beta_i(0) = 0$ and

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}(x_i(k))}\alpha_i(x_i(k))\Big|_{x_i(k)=0} = 0,\quad \lim_{x_i(k)\to 0}\frac{\alpha_i(x_i(k))}{x_i(k)} = 0,\\
&\frac{\mathrm{d}}{\mathrm{d}(x_i(k-h(k)))}\beta_i(x_i(k-h(k)))\Big|_{x_i(k-h(k))=0}\\
&\quad = 0,\quad \lim_{x_i(k-h(k))\to 0}\frac{\beta_i(x_i(k-h(k)))}{x_i(k-h(k))} = 0.
\end{aligned}
$$

We then have

$$\begin{aligned} f(x(k)) &= \Gamma x(k) + \alpha(x(k)), \\ g(x(k-h(k))) &= \Lambda x(k-h(k)) + \beta(x(k-h(k))), \end{aligned} \tag{7}$$

where $\Gamma = \mathrm{diag}\{\gamma_1, \ldots, \gamma_n\}$, $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$ and

$$\alpha(x(k)) = \left[\alpha_1(x_1(k)), \ldots, \alpha_n(x_n(k))\right]^\top, \ \beta(x(\cdot)) = \left[\beta_1(x_1(\cdot)), \ldots, \beta_n(x_n(\cdot))\right]^\top,$$
$$\lim_{x(k)\to 0} \frac{\alpha(x(k))}{\|x(k)\|} = 0, \quad \lim_{x(k-h(k))\to 0} \frac{\beta(x(k-h(k)))}{\|x(k-h(k))\|} = 0.$$

Therefore, unforced system (1) can be represented by

$$\begin{aligned} Ex(k+1) = (A + W\Gamma)x(k) + W_1\Lambda x(k-h(k)) + W\alpha(x(k)) \\ + W_1\beta(x(k-h(k))) + C\omega(k). \end{aligned}$$

Combining conditions (2) and (7) gives

$$\begin{aligned} &x^\top(k)\Gamma^2 x(k) + 2x^\top(k)\Gamma\alpha(x(k)) + \alpha^\top(x(k))\alpha(x(k)) \\ &= f^\top(x(k)) f(x(k)) \le x^\top(k) F^2 x(k). \end{aligned}$$

Letting $\|x(k)\| \to 0$, we can see that

$$\Gamma^2 \leqslant F^2. \tag{8}$$

Let $\Upsilon := \begin{bmatrix} I_{4\times 4} & 0 \\ \Gamma_{4,0} & I_{4\times 4} \end{bmatrix}$, $\Gamma_{4,0} = \begin{bmatrix} \Gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, then

$$\Phi < 0 \quad \Longleftrightarrow \quad \bar{\Phi} := \Upsilon^\top \Phi \Upsilon < 0$$

where $\bar{\Phi} = \left[\bar{\Phi}_{ij}\right]_{8\times 8}$ with

$$\begin{aligned} \bar{\Phi}_{11} = &-\delta E^\top PE + (h_2 - h_1 + 1)Q + S_1 + A_1^\top A_1 + F^2 - \Gamma^2 \\ &- P\bar{M}(A + W\Gamma) - (A + W\Gamma)^\top \bar{M}^\top P. \end{aligned}$$

From (8) and $\bar{\Phi}_{11} < 0$, we get

$$-\delta E^\top PE - P\bar{M}(A + W\Gamma) - (A + W\Gamma)^\top \bar{M}^\top P < 0.$$

Similar to the proof of (E, A) being regular and causal in the first step, it can be obtained that the pair $(E, A + W\Gamma)$ is regular and causal. That is the approximation system of unforced system (1) in a neighborhood of the origin:

$$Ex(k+1) = (A + W\Gamma)x(k) + W_1\Lambda x(k-h(k)) + C\omega(k) \tag{9}$$

is regular and causal. Setting

$$
\begin{aligned}
&MW\Gamma G = \begin{bmatrix} \widehat{\Gamma}_{11} & \widehat{\Gamma}_{12} \\ \widehat{\Gamma}_{21} & \widehat{\Gamma}_{22} \end{bmatrix}, \quad G^{-1}x(k) = \begin{bmatrix} x^1(k) \\ x^2(k) \end{bmatrix}, \\
&G^{-1}x(k-h(k)) = \begin{bmatrix} x^1(k-h(k)) \\ x^2(k-h(k)) \end{bmatrix}, \\
&G^{-1}\varphi(k) = \begin{bmatrix} \varphi^1(k) \\ \varphi^2(k) \end{bmatrix}, \quad MWf(x(k)) \\
&= \begin{bmatrix} f^1\left(x^1(k), x^2(k)\right) \\ f^2(x^1(k), x^2(k)) \end{bmatrix}, \quad MC\omega(k) = \begin{bmatrix} \omega^1(k) \\ \omega^2(k) \end{bmatrix} \\
&MW_1 g(x(k-h(k))) = \begin{bmatrix} g^1\left(x^1(k-h(k)), x^2(k-h(k))\right) \\ g^2\left(x^1(k-h(k)), x^2(k-h(k))\right) \end{bmatrix},
\end{aligned}
$$

unforced system (1) is restricted system equivalent to the following system

$$
\begin{aligned}
x^1(k+1) &= A_{11}x^1(k) + A_{12}x^2(k) + f^1(\cdot) + g^1(\cdot) + \omega^1(k) \\
0 &= A_{21}x^1(k) + A_{22}x^2(k) + f^2(\cdot) + g^2(\cdot) + \omega^2(k), \\
x^1(k) &= \varphi^1(k),\ x^2(k) = \varphi^2(k),\ k \in \{-h_2, \ldots, 0\}.
\end{aligned} \tag{10}
$$

Since system (9) is regular and causal, the matrix

$$
\frac{\partial F(x^1, x^2, f^2, g^2, w^2)}{\partial x^2(k)}\Bigg|_{\substack{x^1(k)=x^1(k-h(k))=0 \\ x^2(k)=x^2(k-h(k))=0 \\ \omega^2(k)=0}} = A_{22} + \widehat{\Gamma}_{22},
$$

where $F(x^1, x^2, f^2, g^2, w^2) := A_{21}x^1(k) + A_{22}x^2(k) + f^2(\cdot) + g^2(\cdot) + \omega^2(k)$, is nonsingular. From Proposition 2, it follows that in a neighborhood of $(0, 0, 0, 0, 0)$, there exists a unique continuous differentiable function $\hat{f}^2(x^1(k), x^1(k-h(k)), x^2(k-h(k)), \omega^2(k))$ on $x^1(k), x^1(k-h(k)), x^2(k-h(k)), \omega^2(k)$ such that

$$
0 = A_{21}x^1(k) + A_{22}\hat{f}^2(\cdot) + f^2(\cdot) + g^2(\cdot) + \omega^2(k),
$$

and $\hat{f}^2(0, 0, 0, 0) = 0$. That is in a neighborhood of $(0, 0, 0, 0, 0)$, the second equation of (10) has a unique solution:

$$
x^2(k) = \hat{f}^2(x^1(k), x^1(k-h(k)), x^2(k-h(k)), \omega^2(k)),\ \hat{f}^2(0, 0, 0, 0) = 0.
$$

Substituting the above solution to the first equation of (10), we obtain

$$
x^1(k+1) = A_{11}x^1(k) + A_{12}\hat{f}^2(\cdot) + f^1(\cdot) + g^1(\cdot) + \omega^1(k)
$$

So the system has a unique solution. This completes the proof of the theorem. □

Remark 1 It should be mentioned that the existence of a solution is a fundamental issue for nonlinear singular systems. The authors in [18] provided a sufficient condition for the existence and uniqueness of the solution of discrete systems with nonlinear perturbation by using the fixed point principle. In Theorem 1, using the implicit function theorem we propose a sufficient condition for not only the existence and uniqueness of the solution of system (1), but also the regularity and casualty of the system. The condition is given in terms of LMIs, which can be efficiently solved by using LMI control toolbox algorithm [6].

In the sequel, we give the solution to H_∞ finite-time stability of unforced system (1).

Theorem 2 *Given positive numbers* $\gamma, N, \delta \geq 1, c_1, c_2$ *and a symmetric positive-definite matrix* R. *Unforced system* (1) *is* H_∞ *finite-time stable w.r.t.* (c_1, c_2, R, N) *if there exist symmetric positive-definite matrices* P, Q, S_1, S_2, *positive scalars* $\lambda_i, i = \overline{1,5}$ *such that the following LMIs hold*:

$$\Psi = \left[\Psi_{ij}\right]_{11\times 11} < 0, \tag{11}$$

$$E^\top PE < \lambda_1 R,\ Q < \lambda_2 R,\ \lambda_3 R < S_1 < \lambda_4 R,\ S_2 < \lambda_5 R, \tag{12}$$

$$\left[v_{ij}\right]_{5\times 5} < 0, \tag{13}$$

where

$$\begin{aligned}
\Psi_{11} &= -\delta E^\top PE + (h_2 - h_1 + 1)Q + S_1 - P\bar{M}A - A\bar{M}^\top P,\ \Psi_{15} = -P\bar{M}W,\\
\Psi_{16} &= -P\bar{M}W_1,\ \Psi_{17} = -P\bar{M}C,\ \Psi_{19} = A_1^\top,\ \Psi_{1,10} = F,\ \Psi_{22} = \Phi_{22},\\
\Psi_{33} &= -\delta^{h_2} S_2,\ \Psi_{44} = -\delta^{h_1} Q,\ \Psi_{49} = D^\top,\ \Psi_{4,11} = H,\\
\Psi_{18} &= AP,\ \Psi_{55} = \Psi_{66} = \Psi_{99} = -I,\\
\Psi_{10,10} &= \Psi_{11,11} = -I,\ \Psi_{58} = W^\top P,\ \Psi_{68} = W_1^\top P,\ \Psi_{77} = -\frac{\gamma}{\delta^N} I,\\
\Psi_{78} &= C^\top P,\ \Psi_{88} = -P,\\
\Psi_{ij} &= 0 \text{ for any other } i, j\text{: } j > i,\ \Psi_{ij} = \Psi_{ji}^\top,\ i > j,\\
\rho &= c_1 \tfrac{h_2(h_2+1) - h_1(h_1-1)}{2} \delta^{N+h_2},\\
v_{11} &= \gamma d - c_2\lambda_3,\ v_{12} = c_1\delta^{N+1}\lambda_1,\ v_{13} = \rho\lambda_2,\ v_{33} = -\rho\lambda_2,\ v_{14}\\
&= c_1\delta^{N+h_1} h_1 \lambda_4,\\
v_{22} &= -c_1\delta^{N+1}\lambda_1,\ v_{15} = c_1\delta^{N+h_2}(h_2 - h_1)\lambda_5,\ v_{23} = v_{24} = v_{25} = v_{34}\\
&= v_{35} = v_{45} = 0,\ v_{44} = -c_1\delta^{N+h_1} h_1\lambda_4,\ v_{55} = -c_1\delta^{N+h_2}(h_2 - h_1)\lambda_5.
\end{aligned}$$

Proof Consider the following nonnegative quadratic functions: $V(k) = \sum_{i=1}^{3} V_i(k)$ where

$$\begin{aligned}
V_1(k) &= x^\top(k)E^\top PEx(k),\\
V_2(k) &= \sum_{s=-h_2+1}^{-h_1+1}\sum_{t=k-1+s}^{k-1} \delta^{k-1-t}x^\top(t)Qx(t),\\
V_3(k) &= \sum_{s=k-h_1}^{k-1} \delta^{k-1-s}x^\top(s)S_1x(s) + \sum_{s=k-h_2}^{k-h_1-1} \delta^{k-1-s}x^\top(s)S_2x(s).
\end{aligned}$$

Denoting $\eta(k) := [x^\top(k), f^\top(\cdot), g^\top(\cdot), \omega^\top(k)]^\top$, $\mathcal{M} := [A, W, W_1, C]$ and taking the difference variation of $V_i(k)$, $i = 1, 2, 3,$ we have

$$\begin{aligned}
V_1(k+1) - \delta V_1(k) &= x^\top(k+1)E^\top PEx(k+1) - \delta x^\top(k)E^\top PEx(k)\\
&= \eta^T(k)\mathcal{M}^\top P\mathcal{M}\eta(k) - \delta x^\top(k)E^\top PEx(k),\\
V_2(k+1) - \delta V_2(k) &= \sum_{s=-h_2+1}^{-h_1+1}\sum_{t=k+s}^{k} \delta^{k-t}x^\top(t)Qx(t)\\
&\quad - \sum_{s=-h_2+1}^{-h_1+1}\sum_{t=k-1+s}^{k-1} \delta^{k-t}x^\top(t)Qx(t)\\
&\leqslant (h_2 - h_1 + 1)x^\top(k)Qx(k) - \delta^{h_1}x^\top(k-h(k))Qx(k-h(k))\\
V_3(k+1) - \delta V_3(k) &= \sum_{s=k+1-h_1}^{k} \delta^{k-s}x^\top(s)S_1x(s) - \sum_{s=k-h_1}^{k-1} \delta^{k-s}x^\top(s)S_1x(s)\\
&\quad + \sum_{s=k+1-h_2}^{k-h_1} \delta^{k-s}x^\top(s)S_2x(s) - \sum_{s=k-h_2}^{k-h_1-1} \delta^{k-s}x^\top(s)S_2x(s)\\
&= x^\top(k)S_1x(k) + x^\top(k-h_1)\Big[\delta^{h_1}(-S_1 + S_2)\Big]x(k-h_1)\\
&\quad - \delta^{h_2}x^\top(k-h_2)S_2x(k-h_2).
\end{aligned}$$

Thus we get

$$\begin{aligned}V(k+1)-\delta V(k) \leq \eta^\top(k)\mathcal{M}^\top P\mathcal{M}\eta(k) &+ x^\top(k)[-\delta E^\top PE \\ &+ (h_2-h_1+1)Q+S_1 \\ &+ A_1^\top A_1]x(k)+2x^\top(k)A_1^\top Dx(k-h(k)) \\ &+ x^\top(k-h_1)\left[\delta^{h_1}(-S_1+S_2)\right]x(k-h_1) \\ &+ x^\top(k-h_2)\left[-\delta^{h_2}S_2\right]x(k-h_2) \\ &+ x^\top(k-h(k))\left[-\delta^{h_1}Q+D^\top D\right]x(k-h(k)) \\ &+ \omega^\top(k)\left[-\frac{\gamma}{\delta^N}I\right]\omega(k)+\frac{\gamma}{\delta^N}\omega^\top(k)\omega(k) \\ &- z^\top(k)z(k). \end{aligned} \tag{14}$$

The following estimations hold true by the assumption (2):

$$\begin{aligned}&0 \leq -f^\top(x(k))f(x(k))+x^\top(k)F^2x(k), \\ &0 \leq -g^\top(x(k-h(k)))g(x(k-h(k)))+x^\top(k-h(k))H^2x(k-h(k)).\end{aligned} \tag{15}$$

Multiplying by $-2x^\top(k)P\bar{M}$ the both side of Eq. (1) and note that $\bar{M}E=0$, we obtain

$$\begin{aligned}0 = &-2x^\top(k)P\bar{M}Ax(k)-2x^\top(k)P\bar{M}Wf(x(k)) \\ &- 2x^\top(k)P\bar{M}W_1g(x(k-h(k)))-2x^\top(k)P\bar{M}C\omega(k).\end{aligned} \tag{16}$$

By setting

$$\begin{aligned}\xi(k) := \Big[&x^\top(k), x^\top(k-h_1), x^\top(k-h_2), x^\top(k \\ &-h(k)), f^\top(x(k)), g^\top(x(k-h(k))), \omega^\top(k)\Big]^\top\end{aligned}$$

we see that

$$\eta^\top(k)\mathcal{M}^\top P\mathcal{M}\eta(k)=\xi^\top(k)\Upsilon^\top P^{-1}\Upsilon\xi(k),$$

where $\Upsilon := \begin{bmatrix}PA & 0 & 0 & 0 & PW & PW_1 & PC\end{bmatrix}$. Combining (14), (15), (16) gives

$$V(k+1)-\delta V(k) \leq \xi^\top(k)(\widetilde{\Phi}+\Upsilon^\top P^{-1}\Upsilon)\xi(k)+\frac{\gamma}{\delta^N}\omega^\top(k)\omega(k)-z^\top(k)z(k) \tag{17}$$

where

$$\widetilde{\Phi} = \begin{bmatrix} \Phi_{11} & 0 & 0 & \Phi_{14} & \Phi_{15} & \Phi_{16} & \Phi_{17} \\ * & \Phi_{22} & 0 & 0 & 0 & 0 & 0 \\ * & * & -\delta^{h_2} S_2 & 0 & 0 & 0 & 0 \\ * & * & * & \Phi_{44} & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & -\frac{\gamma}{\delta^N} I \end{bmatrix}.$$

Furthermore, if we set

$$\widehat{\Phi}_{11} = -\delta E^\top PE + (h_2 - h_1 + 1)Q + S_1 + F^2 - P\bar{M}A - A\bar{M}^\top P,$$

then the following relations holds

$$\widetilde{\Phi} + \Upsilon^\top P^{-1}\Upsilon < 0 \Longleftrightarrow \Phi < 0 \Longleftrightarrow \Psi < 0.$$

As a result, from (11) and (17) it follows that

$$V(k+1) \le \delta V(k) + \frac{\gamma}{\delta^N}\omega^\top(k)\omega(k) \quad \forall k \in \mathbb{Z}_+. \tag{18}$$

By iteration, and taking assumption (4) into account, the inequality (18) implies

$$\begin{aligned} V(k+1) &\le \delta^{k+1} V(0) + \frac{\gamma}{\delta^N} \sum_{s=0}^{k} \delta^{k-s}\omega^\top(s)\omega(s) \\ &< \delta^{N+1} V(0) + \gamma d, \quad \forall k = 0, 1, \dots, N. \end{aligned} \tag{19}$$

Using assumption (12) and $x(k) = \varphi(k)$, $k \in \{-h_2, -h_2+1, \dots, 0\}$, it is easily seen that

$$\begin{aligned} V(0) &= x^\top(0)E^\top PEx(0) + \sum_{s=-h_2+1}^{-h_1+1} \sum_{t=-1+s}^{-1} \delta^{-1-t} x^\top(t)Qx(t) \\ &+ \sum_{s=-h_1}^{-1} \frac{1}{\delta^{1+s}} x^\top(s)S_1 x(s) + \sum_{s=-h_2}^{-h_1-1} \frac{1}{\delta^{1+s}} x^\top(s)S_2 x(s) \\ &< \Big[\lambda_1 + \lambda_2 \frac{h_2(h_2+1) - h_1(h_1-1)}{2}\delta^{h_2-1} + \lambda_4 h_1 \delta^{h_1-1} \\ &+ \lambda_5(h_2 - h_1)\delta^{h_2-1}\Big] c_1. \end{aligned} \tag{20}$$

Associating (19) with (20), we get

$$V(k+1) < \delta^{N+1}\sigma + \gamma d \quad \forall k = 0, 1, \dots, N, \tag{21}$$

where

$$\sigma = \left[\lambda_1 + \lambda_2 \frac{h_2(h_2+1) - h_1(h_1-1)}{2\delta^{1-h_2}} + \lambda_4 h_1 \delta^{h_1-1} + \lambda_5 (h_2 - h_1)\delta^{h_2-1}\right] c_1.$$

On the other hand, according to (12) again, the following estimation holds

$$\begin{aligned} V(k+1) \geq V_3(k+1) \geq & \sum_{s=k+1-h_1}^{k} \delta^{k-s} x^\top(s) S_1 x(s) \\ & \geq x^\top(k) S_1 x(k) > \lambda_3 x^\top(k) R x(k). \end{aligned} \tag{22}$$

Moreover, by the Schur complement lemma ([3]) condition (13) is equivalent to

$$\begin{aligned} & \gamma d - c_2\lambda_3 + c_1\delta^{N+1}\lambda_1 + \rho\lambda_2 + c_1\delta^{N+h_1} h_1 \lambda_4 + c_1 \delta^{N+h_2}(h_2 - h_1)\lambda_5 < 0 \\ & \Longleftrightarrow \gamma d - c_2\lambda_3 + \delta^{N+1}\sigma < 0. \end{aligned} \tag{23}$$

Consequently, we get from (21), (22) and (23) that:

$$x^\top(k) R x(k) < \frac{1}{\lambda_3}[\delta^{N+1}\sigma + \gamma d] < c_2 \ \forall k = 1, 2, \ldots, N,$$

which implies that the unforced system is robustly finite-time stable w.r.t. (c_1, c_2, R, N). To complete the proof of the theorem, it remains to show the γ-level condition (5). For this, from (17) it follows that

$$V(k+1) \leqslant \delta V(k) + \frac{\gamma}{\delta^N}\omega^\top(k)\omega(k) - z^\top(k) z(k),$$

and hence by iteration it derives that

$$V(k) \leqslant \delta^k V(0) + \sum_{s=0}^{k-1} \frac{1}{\delta^{1+s-k}}\left[\frac{\gamma}{\delta^N}\|\omega(s)\|^2 - \|z(s)\|^2\right].$$

Since $V(0) = 0$, the above inequality implies

$$\sum_{s=0}^{k-1} \delta^{k-1-s} z^\top(s) z(s) \ \leqslant \ \sum_{s=0}^{k-1} \delta^{k-1-s} \frac{\gamma}{\delta^N}\omega^\top(s)\omega(s)$$

For $k = N+1$, we have

$$\sum_{s=0}^{N} \delta^{N-s} z^\top(s) z(s) \ \leqslant \ \gamma \sum_{s=0}^{N} \frac{\delta^{N-s}}{\delta^N}\omega^\top(s)\omega(s). \tag{24}$$

Since $1 \le \delta^{N-s} \le \delta^N \ \forall s \in \{0, 1, \ldots, N\}$, (24) immediately yields

$$\sum_{s=0}^{N} z^\top(s)z(s) \le \gamma \sum_{s=0}^{N} \omega^\top(s)\omega(s),$$

which implies that condition (5) holds. The proof of the theorem is completed. □

We are now in position to solve the problem of finite-time H_∞ control for system (1) by designing a state feedback controller $u(k) = Kx(k)$ such that the resulting closed-loop system

$$\begin{aligned} Ex(k+1) &= (A + BK)x(k) + Wf(x(k)) + W_1 g(x(k - h(k))) + C\omega(k),\ k \in \mathbb{Z}_+,\\ z(k) &= (A_1 + B_1 K)x(k) + Dx(k - h(k)),\\ x(k) &= \varphi(k),\ k \in \{-h_2, -h_2 + 1, \ldots, 0\}, \end{aligned} \tag{25}$$

is H_∞ finite-time stable.

Theorem 3 *Given positive constants $\gamma, N, \delta \ge 1, c_1, c_2$ and a symmetric positive-definite matrix R. The finite-time H_∞ control problem of system (1) has a solution if there exist symmetric positive-definite matrices U_i, V_j with $i = \overline{1,4}$, $j = \overline{1,5}$, a matrix Y such that the following LMIs hold:*

$$\Omega = \left[\Omega_{ij}\right]_{11\times 11} < 0, \tag{26}$$

$$\begin{bmatrix} -V_1 & U_1 E^\top \\ * & -U_1 \end{bmatrix} < 0, \tag{27}$$

$$U_2 < V_2, \quad U_3 < V_4, \quad U_4 < V_5, \tag{28}$$

$$[V_{ij}]_{5\times 5} < 0, \tag{29}$$

$$\begin{bmatrix} V_3 - c_2 U_3 & \gamma d U_1 R \\ * & -\gamma d R \end{bmatrix} < 0. \tag{30}$$

Moreover, the state feedback controller is given by

$$u(k) = Y U_1^{-1} x(k), \quad k \in \mathbb{Z}_+,$$

where $\rho = c_1 \frac{h_2(h_2+1) - h_1(h_1-1)}{2} \delta^{N+h_2}$ and

$$\begin{aligned} \Omega_{11} &= \delta U_1 + (h_2 - h_1 + 1)U_2 + U_3 + \delta(U_1 E^\top + EU_1) - \bar{M}(AU_1 + BY)\\ &\quad - (U_1 A + Y^\top B^\top)\bar{M}^\top,\\ \Omega_{15} &= -\bar{M}W,\ \Omega_{16} = -\bar{M}W_1,\ \Omega_{17} = -\bar{M}C,\ \Omega_{18} = U_1 A + Y^\top B^\top,\\ \Omega_{19} &= U_1 A_1^\top + Y^\top B_1^\top,\ \Omega_{1,10} = U_1 F,\ \Omega_{22} = \delta^{h_1}(-U_3 + U_4),\ \Omega_{33} = -\delta^{h_2} U_4,\\ \Omega_{44} &= -\delta^{h_1} U_2,\ \Omega_{49} = U_1 D^\top,\ \Omega_{4,11} = U_1 H,\ \Omega_{68} = W_1^\top,\ \Omega_{55} = \Omega_{66} = -I,\\ \Omega_{99} &= \Omega_{10,10} = \Omega_{11,11} = -I,\ \Omega_{77} = -\frac{\gamma}{\delta^N} I,\ \Omega_{78} = C^\top, \end{aligned}$$

$$\Omega_{88} = -U_1,\ \Omega_{58} = W^\top,$$
$$\Omega_{ij} = 0 \ \textit{for any other } i, j\text{: } j > i,\ \Omega_{ij} = \Omega_{ji}^\top,\ i > j,$$
$$V_{11} = -V_3,\ V_{12} = c_1\delta^{N+1}V_1,\ V_{13} = \rho V_2,\ V_{14} = c_1\delta^{N+h_1}h_1V_4,\ V_{15} = c_1\delta^{N+h_2}(h_2 - h_1)V_5,$$
$$V_{22} = -c_1\delta^{N+1}V_1,\ V_{23} = V_{24} = V_{25} = 0,\ V_{33} = -\rho V_2,\ V_{34} = V_{35} = V_{45} = 0,$$
$$V_{44} = -c_1\delta^{N+h_1}h_1V_4,\ V_{55} = -c_1\delta^{N+h_2}(h_2 - h_1)V_5.$$

Proof Using Theorem 2, closed-loop system (25) is H_∞ finite-time stable if there exist symmetric positive-definite matrices P, Q, S_1, S_2, positive scalars λ_i, $i = \overline{1, 5}$, such that conditions (11), (12) and (13), where matrices $A + BK$, $A_1 + B_1K$ will in place of the matrices A, A_1, hold. In other words, in proportion to (11), we have

$$\Theta = \left[\Theta_{ij}\right]_{11\times 11} < 0, \tag{31}$$

where

$$\Theta_{11} = -\delta E^\top PE + (h_2 - h_1 + 1)Q + S_1 - P\bar{M}(A + BK) - (A + BK)^\top \bar{M}^\top P,$$
$$\Theta_{18} = (A + BK)^\top P,\ \Theta_{19} = (A_1 + B_1K)^\top,\ \Theta_{ij} = \Psi_{ij} \ \text{ for any other } i, j\text{: } j \geqslant i,\ \Theta_{ij} = \Theta_{ji}^\top,\ i > j.$$

Pre- and post-multiplying (31) by the matrix:

$$\text{diag}\left\{P^{-1}, P^{-1}, P^{-1}, P^{-1}, I, I, I, P^{-1}, I, I, I\right\} > 0$$

and then define new matrix variables as follows:

$$U_1 = P^{-1},\quad U_2 = P^{-1}QP^{-1},\quad U_3 = P^{-1}S_1P^{-1},\quad U_4 = P^{-1}S_2P^{-1},$$

we easily obtain the following equivalent inequality

$$\bar{\Theta} < 0, \tag{32}$$

where $\bar{\Theta} = \left[\bar{\Theta}_{ij}\right]_{11\times 11}$ with

$$\begin{aligned}\bar{\Theta}_{11} &= -\delta U_1E^\top U_1^{-1}EU_1 + (h_2 - h_1 + 1)U_2 + U_3 - \bar{M}(A + BK)U_1\\ &\quad - U_1(A + BK)^\top \bar{M}^\top,\\ \bar{\Theta}_{18} &= U_1(A + BK)^\top,\ \bar{\Theta}_{19} = U_1(A_1 + B_1K)^\top,\\ \bar{\Theta}_{ij} &= \Omega_{ij} \ \text{ for any other } i, j:\ j \geqslant i,\ \bar{\Theta}_{ij} = \bar{\Theta}_{ji}^\top,\ i > j.\end{aligned}$$

Letting $Y^\top = U_1K^\top$, $K = YU_1^{-1}$, (32) becomes

$$\bar{\Omega} < 0, \tag{33}$$

where $\bar{\Omega} = \left[\bar{\Omega}_{ij}\right]_{11\times 11}$ with

$$\begin{aligned}
\bar{\Omega}_{11} &= -\delta U_1 E^\top U_1^{-1} E U_1 + (h_2 - h_1 + 1)U_2 + U_3 - \bar{M}(AU_1 + BY) \\
&\quad - (U_1 A + Y^\top B^\top)\bar{M}^\top, \\
\bar{\Omega}_{18} &= U_1 A + Y^\top B^\top, \quad \bar{\Omega}_{19} = U_1 A_1^\top + Y^\top B_1^\top, \\
\bar{\Omega}_{ij} &= \Omega_{ij}, \quad \text{for any other}, \quad i, j : j \geqslant i,\ \bar{\Omega}_{ij} = \bar{\Omega}_{ji}^\top,\ \forall i, j : i > j.
\end{aligned}$$

It is easy to see that

$$-\delta U_1 E^\top U_1^{-1} E U_1 \leqslant \delta(U_1 E^\top + E U_1 + U_1),$$

hence condition (33) holds if condition (26) holds. For getting (29), post-multiplying matrix (13): $[v_{ij} I]_{5\times 5}$ by the matrix $diag\{R, R, R, R, R\} > 0$ and then pre- and post-multiplying the derived matrix again by the matrix $\mathrm{diag}\{P^{-1}, P^{-1}, P^{-1}, P^{-1}, P^{-1}\} > 0$, and setting new variables

$$\begin{aligned}
V_1 &= P^{-1}(\lambda_1 R)P^{-1}, \quad V_2 = P^{-1}(\lambda_2 R)P^{-1}, \\
V_3 &= -\gamma d P^{-1} R P^{-1} + c_2 P^{-1}(\lambda_3 R)P^{-1}, \\
V_4 &= P^{-1}(\lambda_4 R)P^{-1}, \quad V_5 = P^{-1}(\lambda_5 R)P^{-1},
\end{aligned}$$

we reach (29) as expected. To obtain the inequalities (27) and (28), we just pre- and post-multiplying (12) by the matrix P^{-1}. Indeed, we prove (27) as illustrator

$$\begin{aligned}
E^\top P E < \lambda_1 R &\Longleftrightarrow P^{-1} E^\top P E P^{-1} < P^{-1}(\lambda_1 R)P^{-1} \\
&\Longleftrightarrow U_1 E^\top U_1^{-1} E U_1 < V_1 \\
&\Longleftrightarrow -V_1 + U_1 E^\top U_1^{-1} E U_1 < 0,
\end{aligned}$$

which is equivalent to (27) by Proposition 1. Finally, note that

$$\begin{aligned}
V_3 &= -\gamma \mathrm{d} P^{-1} R P^{-1} + c_2 P^{-1}(\lambda_3 R)P^{-1} < -\gamma \mathrm{d} P^{-1} R P^{-1} + c_2 P^{-1} S_1 P^{-1} \\
&= -\gamma \mathrm{d} U_1 R U_1 + c_2 U_3,
\end{aligned}$$

we get $V_3 - c_2 U_3 + \gamma \mathrm{d} U_1 R[\gamma \mathrm{d} R]^{-1} \gamma \mathrm{d} R U_1 < 0$, which is evidently equivalent to (30) by Proposition 1. The proof of the theorem is complete. □

Remark 2 The results obtained in Theorems 2 and 3 can be regarded as an extension of the results of [11,12,19] on H_∞ control for discrete-time neural network (1). To the best of our knowledge, this is the first time that the problem of H_∞ control of nonlinear singular discrete-time neural network systems with time-varying delays and disturbances. Note that Theorems 2 and 3 provide delay-dependent sufficient conditions for the H_∞ finite-time stability and control of the singular neural networks with time-varying delays. The obtained conditions are formulated in terms of LMIs, which can be efficiently solved by using various convex optimization algorithm.

4 Numerical Examples

In this section, we provide some numerical examples. It is worth noting that the finite-time stability and control problem for system (1) is first time studied and solved in our paper and there have not been any similar results obtained for system (1) such that the following examples are given to illustrate the validity and effectiveness of the derived conditions only. In the case, when the discrete-time neural networks (1) reduce to the nonsingular system ($E = I$), our result can be viewed as an extension of existing results [13,19,22,27].

Example 1 Consider unforced system (1) ($u(k) = 0$), where

$$\begin{aligned}
E &= \begin{bmatrix} 1 & -1.1 \\ 1 & -1.1 \end{bmatrix}, \quad A = \begin{bmatrix} 0.95 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad W = \begin{bmatrix} -0.025 & 0.02 \\ 0.015 & 0.025 \end{bmatrix}, \\
W_1 &= \begin{bmatrix} 0.02 & 0.01 \\ -0.025 & 0.02 \end{bmatrix}, \\
C &= \begin{bmatrix} 0.35 \\ 0.25 \end{bmatrix}, \quad F = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.35 \end{bmatrix}, \quad H = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \\
A_1 &= \begin{bmatrix} 0.7 & -0.3 \end{bmatrix}, \quad D = \begin{bmatrix} 0.2 & -0.1 \end{bmatrix}, \quad R = \begin{bmatrix} 1.7 & 0 \\ 0 & 1.3 \end{bmatrix}, \\
h(k) &= 2 + 13\cos^2 \frac{k\pi}{2}, k \in \mathbb{Z}_+.
\end{aligned}$$

By simple calculation, we can find

$$M = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 1.1 \\ 0 & 1 \end{bmatrix}, \quad MEG = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{M} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

For given $h_1 = 2,\ h_2 = 15,\ N = 60,\ d = 1,\ c_1 = 1,\ c_2 = 8$ and $\gamma = 1$, the LMIs (11)–(13) are feasible with $\delta = 1.0001$ and

$$\begin{aligned}
P &= \begin{bmatrix} 0.0078 & -0.1673 \\ -0.1673 & 12.0088 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.0899 & -0.0249 \\ -0.0249 & 0.0525 \end{bmatrix}, \\
S_1 &= \begin{bmatrix} 7.5618 & -0.0008 \\ -0.0008 & 5.7823 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0.0017 & 0 \\ 0 & 0.0013 \end{bmatrix}, \\
\lambda_1 &= 17.7476, \quad \lambda_2 = 0.0646, \quad \lambda_3 = 4.4471, \quad \lambda_4 = 4.4506, \quad \lambda_5 = 0.0015.
\end{aligned}$$

Since the inequalities (6) and (11) are equivalent, the system is regular, causal, and it has a unique solution and is robustly H_∞ finite-time stable w.r.t. $(1, 8, R, 60)$.

Example 2 Consider singular system (1), where

$$E=\begin{bmatrix}1 & -1.1\\ 1 & -1.1\end{bmatrix},\quad A=\begin{bmatrix}0.35 & 0\\ 0 & 0.15\end{bmatrix},\quad W=\begin{bmatrix}-0.02 & 0.015\\ 0.01 & 0.02\end{bmatrix},$$
$$W_1=\begin{bmatrix}0.01 & 0.015\\ -0.02 & 0.025\end{bmatrix},$$
$$B=\begin{bmatrix}0.25\\ 0.45\end{bmatrix},\quad C=\begin{bmatrix}0.15\\ 0.3\end{bmatrix},\quad F=\begin{bmatrix}0.25 & 0\\ 0 & 0.15\end{bmatrix},\quad H=\begin{bmatrix}0.15 & 0\\ 0 & 0.2\end{bmatrix},$$
$$A_1=\begin{bmatrix}0.75 & -0.15\end{bmatrix},$$
$$D=\begin{bmatrix}-0.15 & 0.1\end{bmatrix},\quad B_1=0.2,\quad R=\begin{bmatrix}1.2 & 1\\ 1 & 1.4\end{bmatrix},$$
$$h(k)=2+12\sin^2\frac{k\pi}{2}.$$

We can find that

$$M=\begin{bmatrix}2 & 0\\ 2 & -2\end{bmatrix},\quad G=\begin{bmatrix}0.5 & 0.55\\ 0 & 0.5\end{bmatrix},\quad MEG=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\quad \bar{M}=\begin{bmatrix}0 & 0\\ 2 & -2\end{bmatrix}.$$

For given $h_1=2,\ h_2=14,\ N=40,\ d=1,\ c_1=2, c_2=25$ and $\gamma=1$, the LMIs (26)–(30) are feasible with $\delta=1.0001$ and

$$U_1=\begin{bmatrix}0.1054 & 0.2102\\ 0.2102 & 0.4589\end{bmatrix},\quad U_2=\begin{bmatrix}0.0022 & 0.0048\\ 0.0048 & 0.0107\end{bmatrix},$$
$$U_3=\begin{bmatrix}0.0844 & 0.1942\\ 0.1942 & 0.4585\end{bmatrix},\quad U_4=\begin{bmatrix}0.0012 & 0.0032\\ 0.0032 & 0.0102\end{bmatrix},$$
$$V_1=\begin{bmatrix}0.5522 & 1.2965\\ 1.2965 & 3.0661\end{bmatrix},\quad V_2=\begin{bmatrix}0.0023 & 0.0049\\ 0.0049 & 0.0110\end{bmatrix},$$
$$V_3=\begin{bmatrix}1.9826 & 4.5768\\ 4.5768 & 10.8473\end{bmatrix},\quad V_4=\begin{bmatrix}0.0869 & 0.2012\\ 0.2012 & 0.4841\end{bmatrix},$$
$$V_5=\begin{bmatrix}0.0018 & 0.0048\\ 0.0048 & 0.0156\end{bmatrix},\quad Y=\begin{bmatrix}-0.2115 & -1.0007\end{bmatrix}.$$

The H_∞ finite-time control problem of system (1), by Theorem 3, has a solution, and the state feedback controller is given by

$$u(k)=\begin{bmatrix}27.1655 & -14.6259\end{bmatrix}x(k),\quad k\in\mathbb{Z}_+.$$

Figure 1 shows the response solution with the initial condition

$$\varphi(k)=\begin{bmatrix}0.4\\ 0.8\end{bmatrix},\quad k\in\{-14,-13,\ldots,0\}.$$

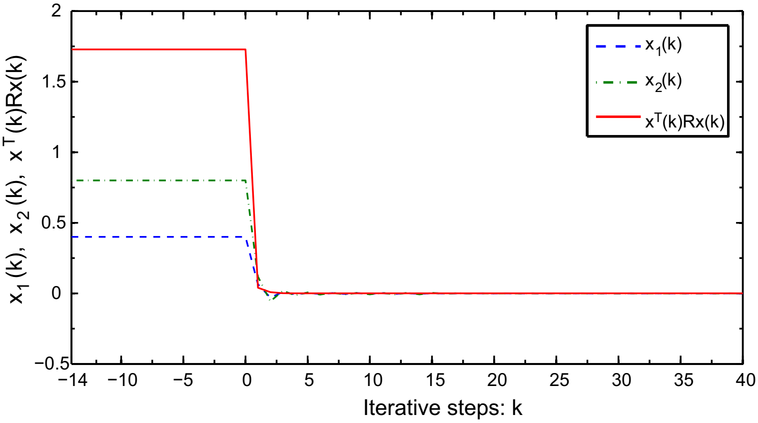

Fig. 1 Response solution of the closed-loop system

5 Conclusion

The problem of H_∞ finite-time stability and control of nonlinear singular discrete-time neural networks with time-varying delays and disturbances has been studied in this paper. Based on the singular systems theory and Lyapunov functional method, we have provided new delay-dependent sufficient conditions for the existence and uniqueness of solutions and the H_∞ finite-time control for such systems. The conditions for the existence of state feedback controllers are easy to check by using MATLAB LMI control toolbox.

Acknowledgements This work was supported by the National Foundation for Science and Technology Development, Vietnam, Grant 101.01.2017.300. The authors wish to thank anonymous reviewers for valuable comments and suggestions, which allowed us to improve the paper.

References

1. Amato, F., Ambrosino, R., Ariola, M., Cosentino, C., Tommasi, G.D.: Finite-Time Stability and Control. Springer, Berlin (2014)
2. Bao, H., Cao, J.: Finite-time generalized synchronization of nonidentical delayed chaotic systems. Nonlinear Anal. Model. Control **21**(2), 306–324 (2016)
3. Boyd, S., Ghaoui, L.El, Feron, E., Balakrishnan, V.: Linear Matrix Inequalities in System and Control Theory. SIAM, Philadelphia (1994)
4. Dai, L.: Singular control systems. In: Lecture Notes in Control and Information Sciences. Springer, Heidelberg (1989)
5. Dorato, P.: Short time stability in linear time-varying systems. Proc. IRE Int. Conv. Rec. **4**, 83–87 (1961)
6. Gahinet, P., Nemirovskii, A., Laub, A.J., Chilali, M.: LMI Control Toolbox for Use with MATLAB. The MathWorks Inc. Natick (1995)
7. Feng, Z., Shi, P.: Two equivalent sets: application to singular systems. Automatica **77**, 198–205 (2017)
8. Feng, Z., Shi, P.: Sliding mode control of singular stochastic Markov jump systems. IEEE Trans. Autom. Control **62**(8), 4266–4273 (2017)
9. Heaton, J.: Introduction to the Math of Neural Networks. Heaton Research Inc. Chesterfield (2012)

10. Kang, D., Li, J., Xu, L.: Existence and nonexistence of ground state solutions to singular elliptic systems. Bull. Malays. Math. Sci. Soc. (2017). https://doi.org/10.1007/s40840-017-0518-4
11. Kanjilal, P., Dey, P., Banerjee, D.: Reduced-size neural networks through singular value decomposition and subset selection. Electron. Lett. **29**, 1516–1518 (1993)
12. Kumaresan, N., Balasubramaniam, P.: Optimal control for stochastic linear quadratic singular system using neural networks. J. Process Control **19**, 482–488 (2009)
13. Kwon, O.M., Park, M.J., Park, J.H., Lee, S.M., Cha, E.J.: New criteria on delay-dependent stability for discrete-time neural networks with time-varying delays. Neurocomputing **121**, 185–194 (2013)
14. Lee, T.H., Park, M.J., Park, J.H., Kwon, O.M., Lee, S.M.: Extended dissipative analysis for neural networks with time-varying delays. IEEE Trans. Neural Netw. Learn. Syst. **25**(10), 1936–1941 (2014)
15. Liang, S.: Positive solutions for singular boundary value problem with fractional q-differences. Bull. Malays. Math. Sci. Soc. **38**(2), 647–666 (2015)
16. Liu, X., Park, J.H., Jiang, N., Cao, J.: Nonsmooth finite-time stabilization of neural networks with discontinuous activations. Neural Netw. **52**, 25–32 (2015)
17. Li, Z., Wang, S.: Robust optimal H_∞ control for irregular buildings with AMD via LMI approach. Nonlinear Anal. Modell. Control **19**(2), 256–271 (2014)
18. Lu, G.P., Ho, D.W.C.: Generalized quadratic stability for continuous-time singular systems with nonlinear perturbation. IEEE Trans. Autom. Control **51**(5), 818–823 (2006)
19. Ma, Y., Zheng, Y.: Delay-dependent stochastic stability for discrete singular neural networks with Markovian jump and mixed time-delays. Neural Comput. Appl. (2016). https://doi.org/10.1007/s00521-016-2414-5
20. Phat, V.N., Trinh, H.: Exponential stabilization of neural networks with various activation functions and mixed time-varying delays. IEEE Trans. Neural. Netw. **21**, 1180–1185 (2010)
21. Ratnavelu, K., Kalpana, M., Balasubramaniam, P.: Stability analysis of fuzzy genetic regulatory networks with various time delays. Bull. Malays. Math. Sci. Soc. (2016). https://doi.org/10.1007/s40840-016-0427-y
22. Sakthivel, R., Mathiyalagan, K., Anthoni, S.: Robust H_∞ control for uncertain discrete-time stochastic neural networks with time-varying delays. IET Control Theory Appl. **6**(9), 1220–1228 (2012)
23. Shen, H., Park, J.H., Wu, Z.G.: Finite-time synchronization control for uncertain Markov jump neural networks with input constraints. Nonlinear Dyn. **77**(4), 1709–1720 (2014)
24. Shen, H., Su, L., Park, J.H.: Extended passive filtering for discrete-time singular Markov jump systems with time-varying delays. Signal Process. **128**, 68–77 (2016)
25. Song, S., Ma, S., Zhang, C.: Stability and robust stabilisation for a class of nonlinear uncertain discrete-time descriptor Markov jump systems. IET Control Theory Appl. **6**, 2518–2527 (2012)
26. Tuan, L.A., Nam, P.T., Phat, V.N.: New H_∞ controller design for neural networks with interval time-varying delays in state and observation. Neural Process. Lett. **37**, 235–249 (2013)
27. Tuan, L.A., Phat, V.N.: Robust finite-time stability and H_∞ control of linear discrete-time delay systems with norm-bounded disturbances. Acta Math. Vietnam. **41**, 481–493 (2016)
28. Wade, W.R.: An Introduction to Analysis, 4th edn. Prentice Hall, New Jersey (2009)
29. Wu, Z.G., Su, H., Shi, P., Chu, J.: Analysis and Synthesis of Singular Systems with Time-Delays. Springer, Heidelberg (2013)
30. Wu, Y., Cao, J., Alofi, A., AL-Mazrooei, A., Elaiw, A.: Finite-time boundedness and stabilization of uncertain switched neural networks with time-varying delay. Neural Netw. **69**, 135–143 (2015)
31. Xu, C., Li, P.: p-th moment exponential stability of stochastic fuzzy Cohen Grossberg neural networks with discrete and distributed delays. Nonlinear Anal. Modell. Control **22**(2), 531–544 (2017)
32. Zhang, Y., Shi, P., Nguang, S.K., Song, Y.: Robust finite-time H_∞ control for uncertain discrete-time singular systems with Markovian jumps. IET Control Theory Appl. **8**, 1105–1111 (2014)