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Positive Solutions for Elliptic Problems Involving Hardy–Sobolev–Maz'ya Terms

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Abstract In the present paper, we study the semilinear elliptic problem $-\Delta u - \mu \frac{u}{|y|^2} = \frac{|u|^{2^*(s)-2}u}{|y|^s} + f(x,u)$ in bounded domain. Replacing the Ambrosetti–Rabinowitz condition by general superquadratic assumptions and the nonquadratic assumption, we establish the existence results of positive solutions.

Keywords Positive solutions · Hardy–Sobolev–Maz'ya terms · Hardy–Sobolev critical exponents · Mountain Pass Lemma · Local Palais–Smale condition

1 Introduction and Main Results

In this paper, we deal with the following semilinear elliptic problem with Dirichlet boundary value conditions

$$\begin{cases} -\Delta u - \mu \frac{u}{|y|^2} = \frac{|u|^{2^*(s)-2}u}{|y|^s} + f(x,u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, Ω is a smooth bounded domain in $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$ with $2 \le k < N$, a point $x \in \mathbb{R}^N$ is denoted as $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ and $(0, z^0) \in \Omega$, $0 \le \mu < \bar{\mu} = \frac{(k-2)^2}{4}$ for $k > 2$, and $\mu = 0$ for $k = 2$. The so-called Hardy–Sobolev critical exponent is denoted as $2^*(s) = \frac{2(N-s)}{N-2}$, where $0 \le s < 2$. Clearly, $2^* = 2^*(0) = \frac{2N}{N-2}$ is the Sobolev critical exponent. $F(x,t)$ is the primitive function of $f(x,t)$ defined as $F(x,t) = \displaystyle\int_0^t f(x,s)\mathrm{d}s$. $H_0^1(\Omega)$ is the Sobolev space with its equivalent norm

$$\|u\| = \left(\int_\Omega \left(|\nabla u|^2 - \mu \frac{u^2}{|y|^2}\right) \mathrm{d}x\right)^{\frac{1}{2}}$$

due to the Hardy inequality

$$C_k \int_{\mathbb{R}^N} \frac{u^2}{|y|^2} \mathrm{d}x \le \int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x, \quad \forall u \in D^{1,2}(\mathbb{R}^N),$$

where $C_k = (\frac{k-2}{2})^2$ is the best constant and is not attained. Let S_μ be the best Hardy–Sobolev constant defined as

$$S_\mu = \inf_{u \in D^{1,2}(\mathbb{R}^N \setminus (0,z^0)), u \neq 0} \frac{\int_{\mathbb{R}^N} \left(|\nabla u|^2 - \mu \frac{u^2}{|y|^2}\right) \mathrm{d}x}{\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*(s)}}{|y|^s} \mathrm{d}x\right)^{\frac{2}{2^*(s)}}}. \tag{1.2}$$

When $k = N$, (1.1) becomes

$$\begin{cases} -\Delta u - \mu \frac{u}{|x|^2} = \frac{|u|^{2^*(s)-2}u}{|x|^s} + f(x,u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

after the work of Brezis and Nirenberg [3], there are many papers concerning the Dirichlet problem with critical exponents (see [1,6–9,11,17,19,26]). When $\mu = 0$ and $s = 0$, problem (1.3) becomes the well-known Brezis–Nirenberg problem and is studied extensively; for example, Nguyen and Lu [23] established the existence of nontrivial nonnegative solutions in dimension two involving exponential nonlinearities which had subcritical or critical exponential growth and did not satisfy the (AR) condition. When $\mu \neq 0$, the problem has its singularity at 0 and attracts much attention. For instance, Ding and Tang [12] studied the existence of positive solutions for $N \ge 3$ and $0 \le s < 2$. Kang and Peng [18] showed the existence of positive solutions replacing $f(x,u)$ by $\lambda |u|^{q-2}u$ with $q > 2$ for $0 \le s < 2$.

When $2 \le k < N$, the singularity of the problem is more complicated. Very recently, it attracts more attention. Bhakta and Sandeep [2] studied the regularity, Palais–Smale characterization and existence of solutions in some special bounded domain and proved nonexistence of nontrivial solution with $f(x, u) = 0$. Ganguly and Sandeep [14] researched the existence and nonexistence of sign-changing solutions for the Brezis–Nirenberg type problem in the hyperbolic space, which is closely related to Hardy–Sobolev–Maz'ya equations. Yang [28] showed the existence of positive solutions for $N \geq 3$ with Neumann boundary condition and $f(x, u)$ satisfying some conditions. Wang and Wang [27] showed that the existence of infinitely many solutions replacing $f(x, u)$ by λu for $N > 6+s$. More details about Hardy–Sobolev–Maz'ya equations and elliptic equations in $\mathbb{H}^n$ (n-dimensional Hyperbolic space) can be seen in [4,5,13,22] and their references.

In order to get a nontrivial solution, the Mountain Pass Lemma [25] is generally exploited, when the equation involves superlinearity. To use this lemma, the authors assume that $f(x, t)$ satisfies the well-known Ambrosetti–Rabinowitz (AR) condition, that is, for some $\rho > 2$, $M > 0$, for a.e. $x \in \overline{\Omega}$ and all $|t| \geq M$, there holds

$$0 < \rho F(x, t) \leq f(x, t)t.$$

It is known that the (AR) condition plays an important role in ensuring that any *Cerami* (*Ce*) sequence of the functional is bounded. But this condition is very restrictive, and there are many functions which do not satisfy the (AR) condition, for example

$$f(x, t) = 2t \ln(1 + |t|).$$

The main purpose of this present paper is to establish the existence of positive solutions for problem (1.1) with $2 \leq k < N$, $0 \leq \mu < \bar{\mu}$ under the case $s = 2 - \dfrac{N-2}{N-k+\sqrt{(k-2)^2 - 4\mu}}$ and $f(x, t)$ satisfying different conditions which are weaker than the (AR) condition but play the same role as the (AR) condition. Here are the main results of this paper:

Theorem 1 *Suppose* $N \geq 2k - 2 - 2\sqrt{(k-2)^2 - 4\mu}$, $2 \leq k < N$, $0 \leq \mu < \bar{\mu}$, *and* $s = 2 - \dfrac{N-2}{N-k+\sqrt{(k-2)^2 - 4\mu}}$. $f \in C(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R})$ *satisfies*

(f_1) $f(x, t) \geq 0$ *for* $t \geq 0$ *and* $f(x, t) = 0$ *for* $t \leq 0$. $\limsup_{t \to 0^+} \dfrac{f(x, t)}{t} < \lambda$ *uniformly for* $x \in \overline{\Omega}$, *where* $0 < \lambda < \lambda_1$ *and* λ_1 *is the first eigenvalue of* $-\Delta - \mu|y|^{-2}$,

(f_2) $\lim_{t \to +\infty} \dfrac{f(x, t)}{t^{2^*(s)-1}} = 0$ *uniformly for* $x \in \overline{\Omega}$,

(f_3) *there exist a positive constant* σ, *a nonempty open subset* ω *with* $(0, z^0) \in \omega \subset \Omega$, *and a nonempty open interval* $I \subset (0, +\infty)$, *so that* $f(x, t) \geq \sigma > 0$ *for almost everywhere* $x \in \omega$ *and for all* $t \in I$.

Then, problem (1.1) *admits at least one positive solution.*

Remark 1 Firstly, when $k = N$, problem (1.1) has been researched in [20], in which $f(x, t)$ did not satisfy the (AR) condition. Secondly, there are many examples satisfying the assumptions of Theorem 1. For instance, we may take $f(x, t) = \lambda t$ with $0 < \lambda < \lambda_1$, or $f(x, t) = \lambda t^q$ with $\lambda > 0$ and $1 < q < 2^*(s) - 1$.

Theorem 2 *Suppose $N \geq 2k-2-2\sqrt{(k-2)^2-4\mu}$, $2 \leq k < N$, $0 \leq \mu < \bar{\mu}$ and $s = 2 - \dfrac{N-2}{N-k+\sqrt{(k-2)^2-4\mu}}$. $f \in C(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R}^+)$ satisfies (f_3) and*

(f_4) $\lim_{t\to 0^+} \dfrac{f(x,t)}{t} = 0$ *uniformly for $x \in \overline{\Omega}$,*
(f_5) $\lim_{t\to +\infty} \dfrac{f(x,t)}{t^{2^*-1}} = 0$ *uniformly for $x \in \overline{\Omega}$,*
(f_6) $\lim_{t\to +\infty} \dfrac{f(x,t)}{t} = +\infty$ *uniformly for $x \in \overline{\Omega}$,*
(f_7) $|f(x,t)|^\tau \leq a_1 \widetilde{F}(x,t)|t|^\tau$ *for some $a_1 > 0$, $\tau > 1$ and $(x,t) \in \overline{\Omega} \times \mathbb{R}^+$ with t large enough, where $\widetilde{F}(x,t) = \frac{1}{2}f(x,t)t - F(x,t)$.*

Then, problem (1.1) possesses at least a positive solution.

Remark 2 Firstly, (f_6) and (f_7) can lead to $\widetilde{F}(x,t) = \frac{1}{2}f(x,t)t - F(x,t) \to +\infty$ uniformly in $x \in \overline{\Omega}$ as $t \to +\infty$. Secondly, there are also many functions satisfying the conditions of Theorem 2. For example, one may take $f(x,t) = \lambda t^q$ with $\lambda > 0$ and $1 < q < 2^*-1$. Thirdly, when $k = N$, $\mu \neq 0$, and $s \neq 0$, Ding and Tang [12] obtained the existence of positive solutions with $f(x,u)$ satisfying a global (AR) condition. Here, we obtain the similar results as those in [12] when $2 \leq k < N$. Thus, our results complete the existence of positive solutions for elliptic problem with Hardy–Sobolev critical exponents.

Theorem 3 *Suppose $N \geq 2k-2-2\sqrt{(k-2)^2-4\mu}$, $2 \leq k < N$, $0 \leq \mu < \bar{\mu}$, and $s = 2 - \dfrac{N-2}{N-k+\sqrt{(k-2)^2-4\mu}}$. $f \in C(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R})$ satisfies (f_1), (f_3), (f_5), (f_6) and*

(f_8) *there exist two constants $\theta \geq 1$, $\theta_0 > 0$ such that $\theta H(x,t) \geq H(x,st) - \theta_0$ for all $x \in \overline{\Omega}$, $t \geq 0$ and $s \in [0,1]$, where $H(x,t) = f(x,t)t - 2F(x,t)$ and $F(x,t) = \int_0^t f(x,s)ds$.*

Then, problem (1.1) admits at least one positive solution.

Remark 3 A condition similar to (f_8) was introduced by Jeanjean [16]. We can easily verify that when $\theta = 1$, (f_8) means that $\frac{f(x,t)}{t}$ is nondecreasing with respect to $t \geq 0$, which leads to the (AR) condition. Thus, (f_8) gives a more general monotonicity when $\theta > 1$. Moreover, one can find some examples that satisfy (f_8) but $\frac{f(x,t)}{t}$ is not monotone. For example, let

$$F(x,t) = t^2 \ln(1+t^2) + t \sin t,$$

it follows that

$$f(x,t) = 2t\ln(1+t^2) + \frac{2t^3}{1+t^2} + \sin t + t\cos t,$$

then

$$H(x,t) = 2(t^2-1) + \frac{2}{1+t^2} + t^2\cos t - t\sin t.$$

Let $\theta = 1000$, we can prove by some simple computation that $f(x,t)$ satisfies (f_8) but $\frac{f(x,t)}{t}$ is not monotone any more.

Theorem 4 *Suppose* $N \geq 2k-2-2\sqrt{(k-2)^2-4\mu}$, $2 \leq k < N$, $0 \leq \mu < \bar{\mu}$, *and* $s = 2 - \dfrac{N-2}{N-k+\sqrt{(k-2)^2-4\mu}}$. $f \in C(\overline{\Omega}\times\mathbb{R}^+,\mathbb{R})$ *satisfies* (f_1), (f_3) *and*

(f_9) *there exists* $q \in [2, 2^*)$ *such that* $\lim\limits_{t\to+\infty}\dfrac{f(x,t)}{t^{q-1}} = 0$ *uniformly for* $x \in \overline{\Omega}$,

(f_{10}) *there exist constants* $D > 0$, $L > 0$, *and* $\delta > \frac{N(q-2)}{2}$, *such that*

$$\frac{f(x,t)t - 2F(x,t)}{|t|^\delta} \geq D,$$

for $t \geq L$ *and a.e.* $x \in \overline{\Omega}$.

Then, problem (1.1) *admits at least one positive solution.*

Remark 4 A nonquadratic condition similar to (f_{10}) was introduced in [10]. Although (f_{10}) is weaker than the (AR) condition, it can guarantee the boundedness of the (Ce) sequence. There are also many functions that satisfy (f_{10}) but do not satisfy the (AR) condition. For example, $f(x,t) = 2t\ln(1+t^2) + \frac{2t^3}{1+t^2}$, $t \in \mathbb{R}$.

2 Proof of Theorems

To verify our main results, we make use of the following notations.

- The dual space of a Banach space E will be denoted by E'.
- $L^p(\Omega, |y|^{-s}\mathrm{d}x)$ denotes the weighted Sobolev space.
- $\to$(resp. $\rightharpoonup$) denotes the strong (resp. weak) convergence.
- C, C_i (i=0, 1, 2 …) will denote various positive constants, and their values can vary from line to line.

In order to study the positive solutions of problem (1.1), we first consider the existence of nontrivial solutions to the problem

$$\begin{cases} -\Delta u - \mu\frac{u}{|y|^2} = \frac{(u^+)^{2^*(s)-1}}{|y|^s} + f(x,u^+), & \text{in} \quad \Omega, \\ u > 0, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where $u^+ = \max\{u, 0\}$. The energy functional corresponding to problem (2.1) is given by

$$I(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \mu\frac{u^2}{|y|^2}\right)\mathrm{d}x - \frac{1}{2^*(s)}\int_\Omega \frac{(u^+)^{2^*(s)}}{|y|^s}\mathrm{d}x - \int_\Omega F(x, u^+)\mathrm{d}x, \tag{2.2}$$

for $u \in H_0^1(\Omega)$. Clearly, I is well defined and is C^1 smooth thanks to the Hardy–Sobolev–Maz'ya inequality [21]

$$\left(\int_{\mathbb{R}^N} \frac{|u|^{2^*(s)}}{|y|^s}\mathrm{d}x\right)^{\frac{2}{2^*(s)}} \le S_\mu^{-1}\int_{\mathbb{R}^N}\left(|\nabla u|^2 - \mu\frac{u^2}{|y|^2}\right)\mathrm{d}x, \tag{2.3}$$

where $S_\mu = S(\mu, N, k, s)$ is the best constant defined in (1.2). By the existence of the one-to-one correspondence between the critical points of I and the weak solutions of problem (2.1), we know that if u is a weak solution of problem (2.1), there holds

$$\begin{aligned}\langle I'(u), v\rangle = \int_\Omega \left((\nabla u, \nabla v) - \mu\frac{uv}{|y|^2}\right)\mathrm{d}x - \int_\Omega \frac{(u^+)^{2^*(s)-1}v}{|y|^s}\mathrm{d}x \\ - \int_\Omega f(x, u^+)v\mathrm{d}x = 0,\end{aligned}$$

for any $v \in H_0^1(\Omega)$.

Before proving our main results, we need the following lemmas. First, it is necessary to give the estimates below. From [2,4,5,22], when $s = 2 - \frac{N-2}{N-k+\sqrt{(k-2)^2-4\mu}}$, the best constant S_μ given in (1.2) can be achieved by the following form of the extremal function

$$U(y, z) = c(\mu, k, N)\frac{|y|^{\frac{\sqrt{(k-2)^2-4\mu}-(k-2)}{2}}}{((1+|y|)^2 + |z|^2)^{\frac{1}{2^*(s)-2}}},$$

where $c(\mu, k, N)$ is a constant. In order to guarantee $s \ge 0$, we suppose $N \ge 2k - 2 - 2\sqrt{(k-2)^2 - 4\mu}$ in this article. For $(0, z^0) \in \Omega$, we can choose $\rho, R > 0$, satisfying $B_\rho(0, z^0) \subset \Omega \subset B_R(0, z^0)$. Let $\varphi \in C_0^\infty(\Omega)$ be a cutoff function such that $0 \le \varphi(x) \le 1$ and

$$\varphi(x) = \begin{cases} 1, & x \in B_{\frac{\rho}{2}}(0, z^0), \\ 0, & x \notin B_\rho(0, z^0). \end{cases}$$

Denote $T = \frac{\sqrt{(k-2)^2-4\mu}-(k-2)}{2}$. Set $u_\varepsilon^* = \varepsilon^{\frac{2-N}{2}}U(\frac{y}{\varepsilon}, \frac{z-z^0}{\varepsilon})$ for $\varepsilon > 0$. Then,

$$u_\varepsilon^*(x) = c(\mu, k, N)\varepsilon^{\frac{1}{2^*(s)-2}}\frac{|y|^T}{((\varepsilon+|y|)^2 + |z-z^0|^2)^{\frac{1}{2^*(s)-2}}}$$

is also an extremal function of S_μ and solves the equation

$$-\Delta u-\mu\frac{u}{|y|^2}=\frac{|u|^{2^*(s)-2}u}{|y|^s},\quad \text{in}\quad \mathbb{R}^N.$$

For $\varepsilon>0$, we define $u_\varepsilon=\varphi(x)u_\varepsilon^*(x)$. We have the following estimates for u_ε.

Lemma 2.1 *Suppose* $N\geq 2k-2-2\sqrt{(k-2)^2-4\mu}$, $2\leq k<N$, $0\leq\mu<\bar{\mu}$, *and* $s=2-\dfrac{N-2}{N-k+\sqrt{(k-2)^2-4\mu}}$, *then the following estimates hold*

$$\int_\Omega|\nabla u_\varepsilon|^2dx=\int_{\mathbb{R}^N}|\nabla u_\varepsilon^*|^2dx+O\left(\varepsilon^{\frac{2}{2^*(s)-2}}\right),\tag{2.4}$$

$$\int_\Omega\frac{u_\varepsilon^2}{|y|^2}dx=\int_{\mathbb{R}^N}\frac{(u_\varepsilon^*)^2}{|y|^2}dx+O\left(\varepsilon^{\frac{2}{2^*(s)-2}}\right),\tag{2.5}$$

$$\int_\Omega\frac{|u_\varepsilon|^{2^*(s)}}{|y|^s}dx=\int_{\mathbb{R}^N}\frac{|u_\varepsilon^*|^{2^*(s)}}{|y|^s}dx+O\left(\varepsilon^{\frac{2^*(s)}{2^*(s)-2}}\right).\tag{2.6}$$

Proof First, we estimate (2.5). There holds

$$\begin{aligned}\int_\Omega\frac{u_\varepsilon^2}{|y|^2}\mathrm{d}x&=\int_\Omega\frac{(\varphi u_\varepsilon^*)^2}{|y|^2}\mathrm{d}x=\int_\Omega\frac{(u_\varepsilon^*)^2}{|y|^2}\mathrm{d}x-\int_\Omega(1-\varphi^2)\frac{(u_\varepsilon^*)^2}{|y|^2}\mathrm{d}x\\&=\int_{\mathbb{R}^N}\frac{(u_\varepsilon^*)^2}{|y|^2}\mathrm{d}x-\int_{\mathbb{R}^N\setminus\Omega}\frac{(u_\varepsilon^*)^2}{|y|^2}\mathrm{d}x-\int_\Omega(1-\varphi^2)\frac{(u_\varepsilon^*)^2}{|y|^2}\mathrm{d}x\\&=\int_{\mathbb{R}^N}\frac{(u_\varepsilon^*)^2}{|y|^2}\mathrm{d}x-\int_{\mathbb{R}^N\setminus\Omega}\frac{(u_\varepsilon^*)^2}{|y|^2}\mathrm{d}x-\int_{\Omega\setminus B_{\frac{\rho}{2}}(0,z^0)}(1-\varphi^2)\frac{(u_\varepsilon^*)^2}{|y|^2}\mathrm{d}x.\end{aligned}$$

Then, we have

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_\rho(0,z^0)}\frac{(u_\varepsilon^*)^2}{|y|^2}\mathrm{d}x&=\varepsilon^{\frac{2}{2^*(s)-2}}\int_{\mathbb{R}^N\setminus B_\rho(0,z^0)}\frac{|y|^{2T-2}}{((\varepsilon+|y|)^2+|z-z^0|^2)^{\frac{2}{2^*(s)-2}}}\mathrm{d}x\\&=\varepsilon^{\frac{2}{2^*(s)-2}}\int_{\mathbb{R}^N\setminus B_\rho(0)}\frac{|y|^{2T-2}}{((\varepsilon+|y|)^2+|z|^2)^{\frac{2}{2^*(s)-2}}}\mathrm{d}x\\&\leq\varepsilon^{\frac{2}{2^*(s)-2}}\Bigg(\int_{\frac{\rho}{2}}^{+\infty}\int_{\frac{\rho}{2}}^{+\infty}\frac{r^{2T+k-3}t^{N-k-1}}{((\varepsilon+r)^2+t^2)^{\frac{2}{2^*(s)-2}}}\mathrm{d}r\mathrm{d}t\\&\quad+\int_0^{\frac{\rho}{2}}\int_{\frac{\rho}{2}}^{+\infty}\frac{r^{2T+k-3}t^{N-k-1}}{((\varepsilon+r)^2+t^2)^{\frac{2}{2^*(s)-2}}}\mathrm{d}r\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&+\int_{\frac{\rho}{2}}^{+\infty}\int_0^{\frac{\rho}{2}}\frac{r^{2T+k-3}t^{N-k-1}}{((\varepsilon+r)^2+t^2)^{\frac{2}{2^*(s)-2}}}\mathrm{d}r\,\mathrm{d}t\Bigg)\\
&=\varepsilon^{2T+N-2-\frac{2}{2^*(s)-2}}\Bigg(\int_{\frac{\rho}{2\varepsilon}}^{+\infty}r^{2T+k-3}\mathrm{d}r\int_{\frac{\rho}{2\varepsilon}}^{+\infty}\frac{t^{N-k-1}}{((1+r)^2+t^2)^{\frac{2}{2^*(s)-2}}}\mathrm{d}t\\
&+\int_0^{\frac{\rho}{2\varepsilon}}r^{2T+k-3}\mathrm{d}r\int_{\frac{\rho}{2\varepsilon}}^{+\infty}\frac{t^{N-k-1}}{((1+r)^2+t^2)^{\frac{2}{2^*(s)-2}}}\mathrm{d}t\\
&+\int_{\frac{\rho}{2\varepsilon}}^{+\infty}r^{2T+k-3}\mathrm{d}r\int_0^{\frac{\rho}{2\varepsilon}}\frac{t^{N-k-1}}{((1+r)^2+t^2)^{\frac{2}{2^*(s)-2}}}\mathrm{d}t\Bigg)\\
&\le\varepsilon^{2T+N-2-\frac{2}{2^*(s)-2}}\Bigg(\int_{\frac{\rho}{2\varepsilon}}^{+\infty}\frac{r^{2T+k-3}}{(1+r)^{\frac{4}{2^*(s)-2}-N+k}}\mathrm{d}r\int_0^{+\infty}\frac{t^{N-k-1}}{(1+t^2)^{\frac{2}{2^*(s)-2}}}\mathrm{d}t\\
&+\int_0^{\frac{\rho}{2\varepsilon}}\frac{r^{2T+k-3}}{(1+r)^{\frac{4}{2^*(s)-2}-N+k}}\mathrm{d}r\int_0^{+\infty}\frac{t^{N-k-1}}{(1+t^2)^{\frac{2}{2^*(s)-2}}}\mathrm{d}t\\
&+\int_{\frac{\rho}{2\varepsilon}}^{+\infty}\frac{r^{2T+k-3}}{(1+r)^{\frac{4}{2^*(s)-2}-N+k}}\mathrm{d}r\int_0^{+\infty}\frac{t^{N-k-1}}{(1+t^2)^{\frac{2}{2^*(s)-2}}}\mathrm{d}t\Bigg)\\
&\le C\varepsilon^{2T+N-2-\frac{2}{2^*(s)-2}}\Bigg(\int_{\frac{\rho}{2\varepsilon}}^{+\infty}\frac{r^{2T+k-3}}{(1+r)^{\frac{4}{2^*(s)-2}-N+k}}\mathrm{d}r\\
&+\int_0^{\frac{\rho}{2\varepsilon}}\frac{r^{2T+k-3}}{(1+r)^{\frac{4}{2^*(s)-2}-N+k}}\mathrm{d}r+\int_{\frac{\rho}{2\varepsilon}}^{+\infty}\frac{r^{2T+k-3}}{(1+r)^{\frac{4}{2^*(s)-2}-N+k}}\mathrm{d}r\Bigg)\\
&\le C\varepsilon^{\frac{2}{2^*(s)-2}}.
\end{aligned}
$$

Then from above, one has

$$
\int_{\mathbb{R}^N\setminus B_\rho(0,z^0)}\frac{(u_\varepsilon^*)^2}{|y|^2}\mathrm{d}x=O\left(\varepsilon^{\frac{2}{2^*(s)-2}}\right). \tag{2.7}
$$

By the same method, we get

$$
\int_{\mathbb{R}^N\setminus B_R(0,z^0)}\frac{(u_\varepsilon^*)^2}{|y|^2}\mathrm{d}x=O\left(\varepsilon^{\frac{2}{2^*(s)-2}}\right).
$$

Since

$$
\int_{\mathbb{R}^N\setminus B_R(0,z^0)}\frac{(u_\varepsilon^*)^2}{|y|^2}\mathrm{d}x\le\int_{\mathbb{R}^N\setminus\Omega}\frac{(u_\varepsilon^*)^2}{|y|^2}\mathrm{d}x\le\int_{\mathbb{R}^N\setminus B_\rho(0,z^0)}\frac{(u_\varepsilon^*)^2}{|y|^2}\mathrm{d}x,
$$

we obtain

$$
\int_\Omega\frac{u_\varepsilon^2}{|y|^2}\mathrm{d}x=\int_{\mathbb{R}^N}\frac{(u_\varepsilon^*)^2}{|y|^2}\mathrm{d}x+O\left(\varepsilon^{\frac{2}{2^*(s)-2}}\right).
$$

Next, we estimate (2.6). In fact, there holds

$$\begin{aligned}\int_\Omega \frac{|u_\varepsilon|^{2^*(s)}}{|y|^s}\mathrm{d}x &= \int_\Omega \frac{|\varphi u_\varepsilon^*|^{2^*(s)}}{|y|^s}\mathrm{d}x\\ &= \int_\Omega \frac{|u_\varepsilon^*|^{2^*(s)}}{|y|^s}\mathrm{d}x - \int_\Omega (1-\varphi^{2^*(s)})\frac{|u_\varepsilon^*|^{2^*(s)}}{|y|^s}\mathrm{d}x\\ &= \int_{\mathbb{R}^N} \frac{|u_\varepsilon^*|^{2^*(s)}}{|y|^s}\mathrm{d}x - \int_{\mathbb{R}^N\setminus\Omega} \frac{|u_\varepsilon^*|^{2^*(s)}}{|y|^s}\mathrm{d}x - \int_\Omega (1-\varphi^{2^*(s)})\frac{|u_\varepsilon^*|^{2^*(s)}}{|y|^s}\mathrm{d}x\\ &= \int_{\mathbb{R}^N} \frac{|u_\varepsilon^*|^{2^*(s)}}{|y|^s}\mathrm{d}x - \int_{\mathbb{R}^N\setminus\Omega} \frac{|u_\varepsilon^*|^{2^*(s)}}{|y|^s}\mathrm{d}x - \int_{\Omega\setminus B_{\frac{\rho}{2}}(0,z^0)} (1-\varphi^{2^*(s)})\frac{|u_\varepsilon^*|^{2^*(s)}}{|y|^s}\mathrm{d}x.\end{aligned}$$

Since

$$\int_{\mathbb{R}^N\setminus B_R(0,z^0)} \frac{|u_\varepsilon^*|^{2^*(s)}}{|y|^s}\mathrm{d}x \le \int_{\mathbb{R}^N\setminus\Omega} \frac{|u_\varepsilon^*|^{2^*(s)}}{|y|^s}\mathrm{d}x \le \int_{\mathbb{R}^N\setminus B_\rho(0,z^0)} \frac{|u_\varepsilon^*|^{2^*(s)}}{|y|^s}\mathrm{d}x,$$

using the method similar to (2.7), one gets

$$\int_{\mathbb{R}^N\setminus B_R(0,z^0)} \frac{|u_\varepsilon^*|^{2^*(s)}}{|y|^s}\mathrm{d}x = O\left(\varepsilon^{\frac{2^*(s)}{2^*(s)-2}}\right).$$

Thus, we deduce

$$\int_\Omega \frac{|u_\varepsilon|^{2^*(s)}}{|y|^s}\mathrm{d}x = \int_{\mathbb{R}^N} \frac{|u_\varepsilon^*|^{2^*(s)}}{|y|^s}\mathrm{d}x + O\left(\varepsilon^{\frac{2^*(s)}{2^*(s)-2}}\right).$$

Now, we estimate (2.4). Observe that

$$\int_\Omega |\nabla u_\varepsilon|^2\mathrm{d}x = \int_\Omega |\nabla(\varphi u_\varepsilon^*)|^2\mathrm{d}x = \int_\Omega |u_\varepsilon^*|^2|\nabla\varphi|^2\mathrm{d}x + \int_\Omega \left(\nabla u_\varepsilon^*, \nabla(\varphi^2 u_\varepsilon^*)\right)\mathrm{d}x,$$

and $-\Delta u_\varepsilon^* - \mu\frac{u_\varepsilon^*}{|y|^2} = \frac{|u_\varepsilon^*|^{2^*(s)-2}u_\varepsilon^*}{|y|^s}$, one has

$$\int_\Omega \left(\nabla u_\varepsilon^*, \nabla(\varphi^2 u_\varepsilon^*)\right)\mathrm{d}x = \mu\int_\Omega \frac{(\varphi u_\varepsilon^*)^2}{|y|^2}\mathrm{d}x + \int_\Omega \varphi^2\frac{|u_\varepsilon^*|^{2^*(s)}}{|y|^s}\mathrm{d}x.$$

When $x \in B_{\frac{\rho}{2}}(0, z^0)$, one has $\nabla\varphi = 0$, then

$$\int_\Omega |u_\varepsilon^*|^2|\nabla\varphi|^2\mathrm{d}x = O\left(\varepsilon^{\frac{2}{2^*(s)-2}}\right).$$

Therefore, one has

$$\begin{aligned}
&\int_{\Omega} |\nabla u_{\varepsilon}|^2 \mathrm{d}x \\
&= \int_{\Omega} |u_{\varepsilon}^*|^2 |\nabla \varphi|^2 \mathrm{d}x + \int_{\Omega} \varphi^2 \frac{|u_{\varepsilon}^*|^{2^*(s)}}{|y|^s} \mathrm{d}x + \mu \int_{\Omega} \varphi^2 \frac{|u_{\varepsilon}^*|^2}{|y|^2} \mathrm{d}x \\
&\leq C\varepsilon^{\frac{2}{2^*(s)-2}} + \int_{\mathbb{R}^N} \frac{|u_{\varepsilon}^*|^{2^*(s)}}{|y|^s} \mathrm{d}x + C\varepsilon^{\frac{2^*(s)}{2^*(s)-2}} + \mu \int_{\mathbb{R}^N} \frac{|u_{\varepsilon}^*|^2}{|y|^2} \mathrm{d}x + C\varepsilon^{\frac{2}{2^*(s)-2}} \\
&= \int_{\mathbb{R}^N} \frac{|u_{\varepsilon}^*|^{2^*(s)}}{|y|^s} \mathrm{d}x + \mu \int_{\mathbb{R}^N} \frac{|u_{\varepsilon}^*|^2}{|y|^2} \mathrm{d}x + C\varepsilon^{\frac{2}{2^*(s)-2}} + C\varepsilon^{\frac{2^*(s)}{2^*(s)-2}} \\
&= \int_{\mathbb{R}^N} |\nabla u_{\varepsilon}^*|^2 \mathrm{d}x + C_2 \varepsilon^{\frac{2}{2^*(s)-2}}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Omega} |\nabla u_{\varepsilon}|^2 \mathrm{d}x \\
&= \int_{\Omega} |u_{\varepsilon}^*|^2 |\nabla \varphi|^2 \mathrm{d}x + \int_{\Omega} \varphi^2 \frac{|u_{\varepsilon}^*|^{2^*(s)}}{|y|^s} \mathrm{d}x + \mu \int_{\Omega} \varphi^2 \frac{|u_{\varepsilon}^*|^2}{|y|^2} \mathrm{d}x \\
&\geq C\varepsilon^{\frac{2}{2^*(s)-2}} + \int_{\mathbb{R}^N} \frac{|u_{\varepsilon}^*|^{2^*(s)}}{|y|^s} \mathrm{d}x + C\varepsilon^{\frac{2^*(s)}{2^*(s)-2}} + \mu \int_{\mathbb{R}^N} \frac{|u_{\varepsilon}^*|^2}{|y|^2} \mathrm{d}x + C\varepsilon^{\frac{2}{2^*(s)-2}} \\
&= \int_{\mathbb{R}^N} \frac{|u_{\varepsilon}^*|^{2^*(s)}}{|y|^s} \mathrm{d}x + \mu \int_{\mathbb{R}^N} \frac{|u_{\varepsilon}^*|^2}{|y|^2} \mathrm{d}x + C\varepsilon^{\frac{2}{2^*(s)-2}} + C\varepsilon^{\frac{2^*(s)}{2^*(s)-2}} \\
&= \int_{\mathbb{R}^N} |\nabla u_{\varepsilon}^*|^2 \mathrm{d}x + C_1 \varepsilon^{\frac{2}{2^*(s)-2}}.
\end{aligned}$$

Thus, we obtain

$$\int_{\mathbb{R}^N} |\nabla u_{\varepsilon}^*|^2 \mathrm{d}x + C_1 \varepsilon^{\frac{2}{2^*(s)-2}} \leq \int_{\Omega} |\nabla u_{\varepsilon}|^2 \mathrm{d}x \leq \int_{\mathbb{R}^N} |\nabla u_{\varepsilon}^*|^2 \mathrm{d}x + C_2 \varepsilon^{\frac{2}{2^*(s)-2}}.$$

The proof is completed. □

For convenience, it is necessary to get the following estimates. Set

$$v_{\varepsilon} = u_{\varepsilon} \Big/ \left(\int_{\Omega} \frac{u_{\varepsilon}^{2^*(s)}}{|y|^s} \mathrm{d}x \right)^{\frac{1}{2^*(s)}}.$$

Clearly,

$$\int_{\Omega} \frac{v_{\varepsilon}^{2^*(s)}}{|y|^s} \mathrm{d}x = 1.$$

Then, the following results can be obtained by the methods used in [15],

$$S_{\mu} + C_7 \varepsilon^{\frac{2}{2^*(s)-2}} \leq \|v_{\varepsilon}\|^2 \leq S_{\mu} + C_8 \varepsilon^{\frac{2}{2^*(s)-2}}, \tag{2.8}$$

and

$$\int_\Omega v_\varepsilon^q \mathrm{d}x = \begin{cases} O(\varepsilon^{\frac{q}{2^*(s)-2}}), & 1 < q < \frac{2N}{2N+\sqrt{(k-2)^2-4\mu}-(k-2)}, \\ O(\varepsilon^{\frac{q}{2^*(s)-2}}|\ln\varepsilon|), & q = \frac{2N}{2N+\sqrt{(k-2)^2-4\mu}-(k-2)}, \\ O(\varepsilon^{Tq+N-\frac{q}{2^*(s)-2}}), & \frac{2N}{2N+\sqrt{(k-2)^2-4\mu}-(k-2)} < q < 2^*. \end{cases} \tag{2.9}$$

We will use the function v_ε as a test function to estimate $I(u)$ below.

Lemma 2.2 *Suppose* $N \geq 2k-2-2\sqrt{(k-2)^2-4\mu}$, $0 \leq \mu < \bar{\mu}$, *and* $s = 2 - \dfrac{N-2}{N-k+\sqrt{(k-2)^2-4\mu}}$. *Assume that* (f_1), (f_3), *and* (f_5) *hold. There exists* $u' \in H_0^1(\Omega)$ *with* $u' \not\equiv 0$, *such that*

$$\sup_{t\geq 0} I(tu') < \frac{2-s}{2(N-s)} S_\mu^{\frac{2^*(s)}{2^*(s)-2}}.$$

Proof Considering the functions

$$g(t) = I(tv_\varepsilon) = \frac{1}{2}t^2\|v_\varepsilon\|^2 - \frac{t^{2^*(s)}}{2^*(s)} - \int_\Omega F(x, tv_\varepsilon)\mathrm{d}x,$$
$$\tilde{g}(t) = \frac{1}{2}t^2\|v_\varepsilon\|^2 - \frac{t^{2^*(s)}}{2^*(s)}.$$

Note that $g(0) = 0$, $g(t) > 0$ for $t > 0$ small enough, and $\lim_{t\to+\infty} g(t) = -\infty$. It follows that $\sup_{t\geq 0} g(t)$ can be achieved by some $t_\varepsilon > 0$.

First, we claim that t_ε is bounded. By

$$0 = g'(t_\varepsilon) = t_\varepsilon\left(\|v_\varepsilon\|^2 - t_\varepsilon^{2^*(s)-2} - \frac{1}{t_\varepsilon}\int_\Omega f(x, tv_\varepsilon)v_\varepsilon \mathrm{d}x\right),$$

we have $\|v_\varepsilon\|^2 = t_\varepsilon^{2^*(s)-2} + \dfrac{1}{t_\varepsilon}\displaystyle\int_\Omega f(x, tv_\varepsilon)v_\varepsilon \mathrm{d}x \geq t_\varepsilon^{2^*(s)-2}$; therefore, one gets

$$t_\varepsilon \leq \|v_\varepsilon\|^{\frac{2}{2^*(s)-2}} \triangleq t_\varepsilon^0.$$

By (2.8), we get

$$t_\varepsilon \leq C_{12}. \tag{2.10}$$

Now, we prove that t_ε is bounded below under (f_1) and (f_5). Obviously, one has $|f(x,t)t| \leq \varepsilon|t|^{2^*} + Ct^2$, then $\|v_\varepsilon\|^2 \leq t_\varepsilon^{2^*(s)-2} + \varepsilon\displaystyle\int_\Omega |t_\varepsilon|^{2^*-2}|v_\varepsilon|^{2^*}\mathrm{d}x + C\int_\Omega |v_\varepsilon|^2\mathrm{d}x$. Due to $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ and (2.8), we can obtain $\displaystyle\int_\Omega |v_\varepsilon|^{2^*}\mathrm{d}x \leq$

$C\|v_\varepsilon\|^{2^*} \le C_{13}(2S_\mu)^{\frac{2^*}{2}}$ for $\varepsilon > 0$ small enough. From (2.8), one has $\int_\Omega |v_\varepsilon|^2 \mathrm{d}x \to 0$, as $\varepsilon \to 0$. From above, combining with (2.9), we deduce that

$$t_\varepsilon^{2^*(s)-2} \ge C. \tag{2.11}$$

From (2.10) and (2.11), we obtain that t_ε is bounded for $\varepsilon > 0$ small enough.

Secondly, we compute $\sup\limits_{t\ge 0} g(t)$. Now we claim that

$$\|v_\varepsilon\|^{\frac{2\cdot 2^*(s)}{2^*(s)-2}} \le S_\mu^{\frac{2^*(s)}{2^*(s)-2}} + C_{14}\varepsilon^{\frac{2}{2^*(s)-2}}. \tag{2.12}$$

In order to prove, we first verify the following inequality

$$(a+b)^r \le a^r + r(a+1)^{r-1}b, \;\; a > 0, \; r \ge 1, \; 0 \le b \le 1.$$

Indeed, set $\phi(x) = (a+x)^r - a^r - r(a+1)^{r-1}x$, for $0 \le x \le 1$. Obviously, $\phi'(x) \le 0$ for all $0 \le x \le 1$, so $\phi(b) \le \phi(0) = 0$; then, the inequality above holds. Then, let $a = S_\mu$, $b = C\varepsilon^{\frac{2}{2^*(s)-2}}$, $r = \frac{2^*(s)}{2^*(s)-2}$, and combining with (2.8), we get (2.12).

It is easy to get that $\tilde{g}(t)$ attains its maximum at t_ε^0 and is increasing in the interval $[0, t_\varepsilon^0]$, and combining with (2.12) we conclude that

$$\begin{aligned} I(t_\varepsilon v_\varepsilon) = g(t_\varepsilon) &\le \tilde{g}(t_\varepsilon^0) - \int_\Omega F(x, t_\varepsilon v_\varepsilon)\mathrm{d}x \\ &= \frac{2-s}{2(N-s)}\|v_\varepsilon\|^{\frac{2\cdot 2^*(s)}{2^*(s)-2}} - \int_\Omega F(x, t_\varepsilon v_\varepsilon)\mathrm{d}x \\ &\le \frac{2-s}{2(N-s)} S_\mu^{\frac{2^*(s)}{2^*(s)-2}} + C_{14}\varepsilon^{\frac{2}{2^*(s)-2}} - \int_\Omega F(x, t_\varepsilon v_\varepsilon)\mathrm{d}x. \end{aligned}$$

Therefore, in order to verify that $\sup\limits_{t\ge 0} I(tu') < \frac{2-s}{2(N-s)} S_\mu^{\frac{2^*(s)}{2^*(s)-2}}$, it is sufficient to show that

$$C_{14}\varepsilon^{\frac{2}{2^*(s)-2}} - \int_\Omega F(x, t_\varepsilon v_\varepsilon)\mathrm{d}x < 0,$$

for $\varepsilon > 0$ small enough. To this purpose, we prove

$$\lim_{\varepsilon\to 0^+} \varepsilon^{-\frac{2}{2^*(s)-2}} \int_\Omega F(x, t_\varepsilon v_\varepsilon)\mathrm{d}x = +\infty. \tag{2.13}$$

In fact, if there exists $m(t)$ such that $f(x,t) \ge m(t) > 0$, combining with the definition of v_ε, (2.6) and the boundedness of t_ε, we only need to verify

$$\lim_{\varepsilon\to 0^+} \varepsilon^{-\frac{2}{2^*(s)-2}} \int_{|x|<R} M\left(\frac{C_\varepsilon \varepsilon^{\frac{1}{2^*(s)-2}}|y|^T}{\left((\varepsilon+|y|)^2+|z|^2\right)^{\frac{1}{2^*(s)-2}}}\right)\mathrm{d}x = +\infty, \tag{2.14}$$

where $B_R(0,z^0)\subset\Omega$, without loss of generality, we assume $R>2$ and $M(t)=\int_0^t m(s)\mathrm{d}s$ is the primitive function of $m(t)$ and

$$C_\varepsilon=t_\varepsilon\left(\int_\Omega \frac{u_\varepsilon^{2^*(s)}}{|y|^s}\mathrm{d}x\right)^{-\frac{1}{2^*(s)}}$$

By (2.6), (2.10), and (2.11), we obtain that C_ε is bounded. Through computation, one has

$$\begin{aligned}
&\varepsilon^{-\frac{2}{2^*(s)-2}}\int_{|x|<R} M\left(\frac{C_\varepsilon \varepsilon^{\frac{1}{2^*(s)-2}}|y|^T}{\left((\varepsilon+|y|)^2+|z|^2\right)^{\frac{1}{2^*(s)-2}}}\right)\mathrm{d}x\\
&\geq \varepsilon^{-\frac{2}{2^*(s)-2}}\int_0^{\frac{R}{2}}\int_0^{\frac{R}{2}} M\left(\frac{C_\varepsilon \varepsilon^{\frac{1}{2^*(s)-2}}r^T}{\left((\varepsilon+r)^2+s^2\right)^{\frac{1}{2^*(s)-2}}}\right) r^{k-1}s^{N-k-1}\mathrm{d}r\mathrm{d}s\\
&= \varepsilon^{N-\frac{2}{2^*(s)-2}}\int_0^{\frac{R}{2\varepsilon}}\int_0^{\frac{R}{2\varepsilon}} M\left(\frac{C_\varepsilon \varepsilon^{T-\frac{1}{2^*(s)-2}}r^T}{\left((1+r)^2+s^2\right)^{\frac{1}{2^*(s)-2}}}\right) r^{k-1}s^{N-k-1}\mathrm{d}r\mathrm{d}s\\
&= \varepsilon^{N-\frac{2}{2^*(s)-2}}\int_0^{\frac{R}{2\varepsilon}} r^{k-1}(1+r)^{N-k}\mathrm{d}r\\
&\quad\int_0^{\frac{R}{2\varepsilon(1+r)}} M\left(\frac{C_\varepsilon \varepsilon^{T-\frac{1}{2^*(s)-2}}r^T}{(1+r)^{\frac{2}{2^*(s)-2}}(1+\rho^2)^{\frac{1}{2^*(s)-2}}}\right)\rho^{N-k-1}\mathrm{d}\rho.
\end{aligned}$$

Thus, we can deduce that for $R'=\frac{R}{2}>1$, (2.14) is equivalent to

$$\begin{aligned}
&\frac{\varepsilon^N}{\varepsilon^{\frac{2}{2^*(s)-2}}}\int_0^{\frac{R'}{\varepsilon}} r^{k-1}(1+r)^{N-k}\mathrm{d}r\\
&\int_0^{\frac{R'}{\varepsilon(1+r)}} M\left(\frac{C_\varepsilon \varepsilon^{T-\frac{1}{2^*(s)-2}}r^T}{(1+r)^{\frac{2}{2^*(s)-2}}(1+\rho^2)^{\frac{1}{2^*(s)-2}}}\right)\rho^{N-k-1}\mathrm{d}\rho\to+\infty,
\end{aligned}\tag{2.15}$$

as $\varepsilon\to0^+$. Then, if we prove that

$$\begin{aligned}
&\frac{\varepsilon^N}{\varepsilon^{\frac{2}{2^*(s)-2}}}\int_0^{\frac{1}{\varepsilon}} r^{k-1}(1+r)^{N-k}\mathrm{d}r\\
&\int_0^{\frac{1}{\varepsilon(1+r)}} M\left(\frac{C_\varepsilon \varepsilon^{T-\frac{1}{2^*(s)-2}}r^T}{(1+r)^{\frac{2}{2^*(s)-2}}(1+\rho^2)^{\frac{1}{2^*(s)-2}}}\right)\rho^{N-k-1}\mathrm{d}\rho\to+\infty.
\end{aligned}\tag{2.16}$$

as $\varepsilon\to0^+$, then it is easy to check that (2.15) is established.

Last, we will prove that (2.16) holds under (f_3). By (f_3), one gets

$$f(x,t) \geq \sigma \chi_I(t) \triangleq m(t),$$

for almost everywhere $x \in \omega$ and for all $t \geq 0$, where χ_I is the characteristic function of $I(I \subset (0,+\infty))$. Thus, for some constants $\eta > 0$ and $B > 0$, it follows that

$$M(t) \geq \eta > 0,$$

for all $t \geq B$. Then, we obtain

$$M\left(\frac{C_\varepsilon \varepsilon^{T-\frac{1}{2^*(s)-2}} r^T}{(1+r)^{\frac{2}{2^*(s)-2}}(1+\rho^2)^{\frac{1}{2^*(s)-2}}}\right) \geq \eta,$$

for all ρ satisfying $\rho \leq C'_\varepsilon \dfrac{r^{\frac{T(2^*(s)-2)}{2}}}{1+r} \varepsilon^{\frac{T(2^*(s)-2)-1}{2}}$, where $0 < r < \varepsilon^{-1}$ and C'_ε is bounded and related to B and C_ε. Then, it leads to

$$\begin{aligned}
&\varepsilon^{N-\frac{2}{2^*(s)-2}} \int_0^{\frac{1}{\varepsilon}} r^{k-1}(1+r)^{N-k} \mathrm{d}r \int_0^{\frac{1}{\varepsilon(1+r)}} M\left(\frac{C_\varepsilon \varepsilon^{T-\frac{1}{2^*(s)-2}} r^T}{(1+r)^{\frac{2}{2^*(s)-2}}(1+\rho^2)^{\frac{1}{2^*(s)-2}}}\right) \rho^{N-k-1} \mathrm{d}\rho \\
&\geq \eta \varepsilon^{N-\frac{2}{2^*(s)-2}} \int_0^{\frac{1}{\varepsilon}} r^{k-1}(1+r)^{N-k} \mathrm{d}r \int_0^{C'_\varepsilon \frac{r^{\frac{T(2^*(s)-2)}{2}}}{1+r} \varepsilon^{\frac{T(2^*(s)-2)-1}{2}}} \rho^{N-k-1} \mathrm{d}\rho \\
&\geq C\eta \varepsilon^{N-\frac{2}{2^*(s)-2}+\frac{T(2^*(s)-2)-1}{2}(N-k)} \int_0^{\frac{1}{\varepsilon}} r^{k-1+\frac{T(2^*(s)-2)(N-k)}{2}} \mathrm{d}r \\
&= C_{18} \varepsilon^{\frac{N-k}{2}-\frac{2}{2^*(s)-2}},
\end{aligned}$$

for $N \geq 2k-2-\sqrt{(k-2)^2-4\mu}$ and $s = 2 - \dfrac{N-2}{N-k+\sqrt{(k-2)^2-4\mu}}$; then, one has $\frac{N-k}{2} - \frac{2}{2^*(s)-2} < 0$. Therefore, (2.16) holds. Then, we complete the proof of Lemma 2.2. □

Proof of Theorem 1 From the continuity of embeddings

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega)(1 \leq q \leq 2^*) \text{ and } H_0^1(\Omega) \hookrightarrow L^{2^*(s)}(\Omega, |y|^{-s}\mathrm{d}x),$$

there exist $C_{19}, C_{20} > 0$ such that

$$\int_\Omega |u|^q \mathrm{d}x \leq C_{19}\|u\|^q, \quad \int_\Omega \frac{|u|^{2^*(s)}}{|y|^s} \mathrm{d}x \leq C_{20}\|u\|^{2^*(s)}. \tag{2.17}$$

It follows from (f_1) and (f_2) that

$$|F(x,t)| \leq \frac{1}{2}\lambda |t|^2 + C_{21}|t|^{2^*(s)}, \tag{2.18}$$

for all $t \in \mathbb{R}^+$ and $x \in \overline{\Omega}$. By (2.17) and (2.18), one has

$$
\begin{aligned}
I(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{2^*(s)}\int_\Omega \frac{(u^+)^{2^*(s)}}{|y|^s}\mathrm{d}x - \int_\Omega F(x,u^+)\mathrm{d}x \\
&\geq \frac{1}{2}\|u\|^2 - \frac{1}{2^*(s)}\int_\Omega \frac{(u^+)^{2^*(s)}}{|y|^s}\mathrm{d}x - \frac{\lambda}{2}\int_\Omega |u|^2\mathrm{d}x - C_{22}\int_\Omega |u|^{2^*(s)}\mathrm{d}x \\
&\geq \frac{1}{2}\|u\|^2 - \frac{C_{23}}{2^*(s)}\|u\|^{2^*(s)} - \frac{\lambda}{2\lambda_1}\|u\|^2 - C_{24}\|u\|^{2^*(s)},
\end{aligned}
$$

for $0 < \lambda < \lambda_1$; therefore, there exists $\alpha > 0$ such that $I(u) \geq \alpha > 0$ for all $\|u\| = r$, where $r > 0$ small enough. For any $u \in H_0^1(\Omega)$ with $u^+ \not\equiv 0$, together with the nonnegativity of $F(x,t)$, one has

$$
\begin{aligned}
I(tu) &= \frac{1}{2}t^2\|u\|^2 - \frac{1}{2^*(s)}t^{2^*(s)}\int_\Omega \frac{(u^+)^{2^*(s)}}{|y|^s}\mathrm{d}x - \int_\Omega F(x,tu^+)\mathrm{d}x \\
&\leq \frac{1}{2}t^2\|u\|^2 - \frac{1}{2^*(s)}t^{2^*(s)}\int_\Omega \frac{(u^+)^{2^*(s)}}{|y|^s}\mathrm{d}x,
\end{aligned}
$$

then $\lim\limits_{t\to+\infty} I(tu) \to -\infty$. Thus, we can find $t' > 0$ such that $I(t'u) < 0$ when $\|t'u\| > r$. According to the Mountain Pass Lemma (see [25]), there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$, such that as $n \to \infty$,

$$
I(u_n) \to c > \alpha \ \text{ and } \ (1 + \|u_n\|)\|I'(u_n)\| \to 0 \text{ in } \left(H_0^1(\Omega)\right)',
$$

where

$$
\begin{aligned}
&c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)),\\
&\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) | \gamma(0) = 0, \gamma(1) = t'u\}.
\end{aligned}
$$

It is easy to obtain that (f_2) leads to (f_5), then Lemma 2.2 holds if we replace (f_5) by (f_2). By the definition of c and Lemma 2.2, we obtain

$$
0 < \alpha < c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)) \leq \max_{t\in[0,1]} I(tt'u) \leq \sup_{t\geq 0} I(tu) < \frac{2-s}{2(N-s)} S_\mu^{\frac{2^*(s)}{2^*(s)-2}}.
$$

First, we claim that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Indeed, by (f_2), for any $\varepsilon > 0$, there exists $M > 0$, such that

$$
\begin{aligned}
&|F(x,t)| \leq \varepsilon|t|^{2^*(s)},\ x \in \Omega,\ t \geq M;\ \ |F(x,t)| \leq C_1(\varepsilon),\ t \in (0, M];\\
&|f(x,t)t| \leq \varepsilon|t|^{2^*(s)},\ x \in \Omega,\ t \geq M;\ \ |f(x,t)t| \leq C_2(\varepsilon),\ t \in (0, M].
\end{aligned}
$$

Thus, we have

$$
|F(x,t)| \leq C_1(\varepsilon) + \varepsilon|t|^{2^*(s)},\ \ |f(x,t)t| \leq C_2(\varepsilon) + \varepsilon|t|^{2^*(s)}, \tag{2.19}
$$

for any $(x,t)\in\overline{\Omega}\times\mathbb{R}^+$. Then, for $\xi\in(2,2^*(s))$, one has

$$F(x,t)-\frac{1}{2}f(x,t)t\le F(x,t)-\frac{1}{\xi}f(x,t)t\le C_3(\varepsilon)+\varepsilon|t|^{2^*(s)},\qquad(2.20)$$

for any $(x,t)\in\overline{\Omega}\times\mathbb{R}^+$. Set $l(x,t)\triangleq|y|^{-s}|t|^{2^*(s)-1}+f(x,t)$, we claim that $l(x,t)$ satisfies the (AR) condition. By (2.20), one easily gets

$$\begin{aligned}\xi L(x,t)-l(x,t)t&=\left(\frac{\xi}{2^*(s)}-1\right)|y|^{-s}|t|^{2^*(s)}+\big(\xi F(x,t)-f(x,t)t\big)\\&\le\left(\frac{\xi}{2^*(s)}-1\right)|y|^{-s}|t|^{2^*(s)}+\xi C_4(\varepsilon)+\xi\varepsilon|t|^{2^*(s)}\\&=\left(\left(\frac{\xi}{2^*(s)}-1\right)|y|^{-s}+\xi\varepsilon\right)|t|^{2^*(s)}+\xi C_4(\varepsilon),\end{aligned}$$

so for $\varepsilon>0$ sufficiently small, there exists $M'>0$, such that

$$0\le\xi L(x,t)\le l(x,t)t,\quad x\in\overline{\Omega}\backslash\{(0,z^0)\},\ t\ge M',$$

where $L(x,t)=\int_0^t l(x,s)\mathrm{d}s$. Moreover, by (f_2), we obtain

$$L(x,t)-\frac{1}{\xi}l(x,t)t\le\max_{x\in\overline{\Omega}\backslash\{(0,z^0)\},0\le t\le M'}\left(F(x,t)-\frac{1}{\xi}f(x,t)t\right)\triangleq M_0.$$

It follows from the inequalities above that

$$L(x,t)-\frac{1}{\xi}l(x,t)t\le M_0,\quad\text{for all}\quad x\in\overline{\Omega}\backslash\{(0,z^0)\},\ t\ge0.\qquad(2.21)$$

Then, one has

$$\begin{aligned}c+1+o(1)&\ge I(u_n)-\frac{1}{\xi}\langle I'(u_n),u_n\rangle\\&=\left(\frac{1}{2}-\frac{1}{\xi}\right)\|u_n\|^2+\left(\frac{1}{\xi}-\frac{1}{2^*(s)}\right)\int_\Omega\frac{(u_n^+)^{2^*(s)}}{|y|^s}\mathrm{d}x\\&\quad-\int_\Omega\left(F(x,u_n^+)-\frac{1}{\xi}f(x,u_n^+)u_n^+\right)\mathrm{d}x\\&\ge\left(\frac{1}{2}-\frac{1}{\xi}\right)\|u_n\|^2-\int_\Omega\left(L(x,u_n^+)-\frac{1}{\xi}l(x,u_n^+)u_n^+\right)\mathrm{d}x\\&\ge\left(\frac{1}{2}-\frac{1}{\xi}\right)\|u_n\|^2-M_0|\Omega|.\end{aligned}$$

Thus, $\{u_n\}$ is bounded. Due to the continuity of embedding $H_0^1(\Omega)\hookrightarrow L^{2^*(s)}(\Omega)$, we have $\int_\Omega|u_n|^{2^*(s)}\mathrm{d}x\le C<\infty$. Up to a subsequence, still denoted by $\{u_n\}$, there exists $u_0\in H_0^1(\Omega)$ satisfying

$$\begin{cases} u_n \rightharpoonup u_0, & \text{weakly in } H_0^1(\Omega), \\ u_n \to u_0, & \text{strongly in } L^p(\Omega),\ 1 \le p < 2^*, \\ u_n(x) \to u_0(x), & \text{a.e. in } \Omega, \\ u_n^{2^*(s)-1} \rightharpoonup u_0^{2^*(s)-1}, & \text{weakly in } \left(L^{2^*(s)}(\Omega, |y|^{-s}\mathrm{d}x)\right)', \end{cases} \tag{2.22}$$

as $n \to \infty$. By (f_2), for any $\varepsilon > 0$, there exists $a(\varepsilon) > 0$ such that

$$|F(x,t)| \le \frac{1}{2C_{24}}\varepsilon|t|^{2^*(s)} + a(\varepsilon) \quad \text{for } (x,t) \in \Omega \times \mathbb{R}^+.$$

Set $\delta = \frac{\varepsilon}{2a(\varepsilon)} > 0$. When $E \subset \Omega$, meas $E < \delta$, one gets

$$\begin{aligned} \left|\int_E F(x,u_n^+)\mathrm{d}x\right| &\le \int_E |F(x,u_n^+)|\mathrm{d}x \\ &\le \int_E a(\varepsilon)\mathrm{d}x + \frac{1}{2C_{24}}\varepsilon\int_E |u_n|^{2^*(s)}\mathrm{d}x \\ &\le a(\varepsilon)\mathrm{meas}E + \frac{1}{2C_{24}}\varepsilon C_{24} \\ &\le \varepsilon. \end{aligned}$$

Hence, $\left\{\int_\Omega F(x,u_n^+)\mathrm{d}x, n \in N\right\}$ is equi-absolutely continuous. It follows from the Vitali Convergence Theorem that

$$\int_\Omega F(x,u_n^+)\mathrm{d}x \to \int_\Omega F(x,u_0^+)\mathrm{d}x, \tag{2.23}$$

as $n \to \infty$. Applying the same method, one has

$$\int_\Omega f(x,u_n^+)u_n\mathrm{d}x \to \int_\Omega f(x,u_0^+)u_0\mathrm{d}x \tag{2.24}$$

By (2.22) and (2.24), we have

$$\begin{aligned} \lim_{n\to+\infty}\langle I'(u_n), v\rangle = &\int_\Omega \left((\nabla u_0, \nabla v) - \mu\frac{u_0 v}{|y|^2}\right)\mathrm{d}x - \int_\Omega \frac{(u_0^+)^{2^*(s)-1}v}{|y|^s}\mathrm{d}x \\ &- \int_\Omega f(x,u_0^+)v\mathrm{d}x = 0, \end{aligned} \tag{2.25}$$

for all $v \in H_0^1(\Omega)$. Thus, u_0 is a critical point of I, that is, u_0 is a solution of problem (1.1). Now we verify that $u_0 \not\equiv 0$. Let $v = u_0$ in (2.25), we get

$$\|u_0\|^2 - \int_\Omega \frac{(u_0^+)^{2^*(s)}}{|y|^s}\mathrm{d}x - \int_\Omega f(x,u_0^+)u_0\mathrm{d}x = 0. \tag{2.26}$$

Set $w_n = u_n - u_0$, then we have

$$\int_\Omega |\nabla u_n|^2 \mathrm{d}x = \int_\Omega |\nabla u_0|^2 \mathrm{d}x + \int_\Omega |\nabla w_n|^2 \mathrm{d}x + o(1). \tag{2.27}$$

From Brézis–Lieb's lemma (see [3]), it follows that

$$\int_\Omega \frac{u_n^2}{|y|^2} \mathrm{d}x = \int_\Omega \frac{u_0^2}{|y|^2} \mathrm{d}x + \int_\Omega \frac{w_n^2}{|y|^2} \mathrm{d}x + o(1). \tag{2.28}$$

$$\int_\Omega \frac{(u_n^+)^{2^*(s)}}{|y|^s} \mathrm{d}x = \int_\Omega \frac{(u_0^+)^{2^*(s)}}{|y|^s} \mathrm{d}x + \int_\Omega \frac{(w_n^+)^{2^*(s)}}{|y|^s} \mathrm{d}x + o(1). \tag{2.29}$$

By (2.23) and (2.27)–(2.29), one has

$$I(u_n) = I(u_0) + \frac{1}{2}\|w_n\|^2 - \frac{1}{2^*(s)} \int_\Omega \frac{(w_n^+)^{2^*(s)}}{|y|^s} \mathrm{d}x = c + o(1). \tag{2.30}$$

Since $\langle I'(u_n), u_n \rangle = o(1)$, combining with (2.23), (2.26), one has

$$\|w_n\|^2 - \int_\Omega \frac{(w_n^+)^{2^*(s)}}{|y|^s} \mathrm{d}x = o(1).$$

We may assume that as $n \to \infty$,

$$\|w_n\|^2 \to b, \qquad \int_\Omega \frac{(w_n^+)^{2^*(s)}}{|y|^s} \mathrm{d}x \to b.$$

Clearly, $b \geq 0$. We now suppose that $u_0 \equiv 0$. If $b = 0$, then from (2.30), $c = I(0) = 0$, which contradicts with $c > 0$. If $b \neq 0$, we have from the definition of S_μ that

$$\|w_n\|^2 = \int_\Omega \left(|\nabla w_n|^2 - \mu \frac{w_n^2}{|y|^2} \right) \mathrm{d}x \geq S_\mu \left(\int_\Omega \frac{(w_n^+)^{2^*(s)}}{|y|^s} \mathrm{d}x \right)^{\frac{2}{2^*(s)}},$$

and $b \geq S_\mu b^{\frac{2}{2^*(s)}}$, together with (2.30), we deduce

$$\begin{aligned} c + o(1) &= I(u_0) + \frac{1}{2}\|w_n\|^2 - \frac{1}{2^*(s)} \int_\Omega \frac{(w_n^+)^{2^*(s)}}{|y|^s} \mathrm{d}x + o(1) \\ &= I(u_0) + o(1) + \left(\frac{1}{2} - \frac{1}{2^*(s)} \right) b \\ &\geq \frac{2-s}{2(N-s)} S_\mu^{\frac{2^*(s)}{2^*(s)-2}} + o(1), \end{aligned}$$

which contradicts $c < \dfrac{2-s}{2(N-s)} S_\mu^{\frac{2^*(s)}{2^*(s)-2}}$. Therefore, $u_0 \not\equiv 0$ and u_0 is a nontrivial solution of problem (2.1). Then, by $\langle I'(u_0), u_0^- \rangle = 0$ where $u_0^- = \min\{u_0, 0\}$, one has $\|u_0^-\| = 0$, which implies that $u_0 \geq 0$. From (2.25), we get $\int_\Omega (\nabla u_0, \nabla v)\mathrm{d}x \geq 0$ for any $v \in H_0^1(\Omega)$, which means $-\triangle u_0 \geq 0$ in Ω. By the strong maximum principle, we know u_0 is a positive solution of problem (1.1). Therefore, Theorem 1 holds. □

Proof of Theorem 2 Obviously, (f_4) and (f_5) can ensure that I has a mountain pass geometry and then there exists a $(Ce)_c$ sequence $\{u_n\}$, that is,

$$I(u_n) \to c \;\text{ and }\; (1 + \|u_n\|)\|I'(u_n)\| \to 0 \quad \text{as } n \to \infty.$$

We claim that $\{u_n\}$ is bounded. In fact, there exists $n_0 > 0$, such that for $n \geq n_0$, one has

$$\begin{aligned} c + 1 &\geq I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{2^*(s)}\right) \int_\Omega \frac{(u_n^+)^{2^*(s)}}{|y|^s}\mathrm{d}x + \int_\Omega \widetilde{F}(x, u_n^+)\mathrm{d}x \\ &\geq \int_\Omega \widetilde{F}(x, u_n^+)\mathrm{d}x. \end{aligned} \tag{2.31}$$

Set

$$g(r) := \inf\left\{\widetilde{F}(x, u) : x \in \overline{\Omega},\ |u| \geq r\right\}.$$

By Remark 2, one deduces $g(r) \to +\infty$ as $r \to +\infty$. For $0 \leq a < b$, let

$$\Omega_n(a, b) := \left\{x \in \overline{\Omega},\ a \leq u_n^+(x) < b\right\}$$

and

$$C_a^b := \inf\left\{\frac{\widetilde{F}(x, u)}{u^2} : x \in \overline{\Omega},\ a \leq |u(x)| < b\right\};$$

thus for all $x \in \Omega_n(a, b)$, one obtains

$$\widetilde{F}\big(x, u_n^+(x)\big) \geq C_a^b (u_n^+(x))^2.$$

It follows from (2.31) that

$$\begin{aligned} c + 1 &\geq \int_{\Omega_n(0,b)} \widetilde{F}(x, u_n^+)\mathrm{d}x + \int_{\Omega_n(b,+\infty)} \widetilde{F}(x, u_n^+)\mathrm{d}x \\ &\geq C_0^b \int_{\Omega_n(a,b)} |u_n^+|^2\mathrm{d}x + g(b)|\Omega_n(b, +\infty)|. \end{aligned} \tag{2.32}$$

Arguing directly, assume $\|u_n\| \to +\infty$. Set $v_n = \dfrac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$ and $\int_\Omega |v_n|^s \mathrm{d}x \le M_s$ for all $s \in [2, 2^*]$. Using (2.32),

$$|\Omega_n(b, +\infty)| \le \frac{c+1}{g(b)} \to 0, \tag{2.33}$$

uniformly in $n \ge n_0$ as $b \to +\infty$. And for any fixed $0 \le a < b$,

$$\int_{\Omega_n(a,b)} (v_n^+)^2 \mathrm{d}x = \frac{1}{\|u_n\|^2} \int_{\Omega_n(a,b)} (u_n^+)^2 \mathrm{d}x \le \frac{C}{\|u_n\|^2} \to 0, \tag{2.34}$$

as $n \to \infty$. It follows from (2.33) and the Hölder inequality that for $r \in [2, 2^*]$, $p \in (r, 2^*)$, and a suitable constant C_*,

$$\int_{\Omega_n(b,+\infty)} (v_n^+)^r \mathrm{d}x \le C_* |\Omega_n(b, \infty)|^{\frac{p-r}{p}} \to 0, \tag{2.35}$$

uniformly in $n \ge n_0$ as $b \to +\infty$. Let $\varepsilon > 0$, by (f_4), there exists $a_\varepsilon > 0$ such that $|f(x,u)| \le \dfrac{\varepsilon}{M_2}|u|$ for all $|u| < a_\varepsilon$. Consequently,

$$\int_{\Omega_n(0,a_\varepsilon)} \frac{f(x,u_n^+)}{u_n^+}(v_n^+)^2 \mathrm{d}x \le \int_{\Omega_n(0,a_\varepsilon)} \frac{\varepsilon}{M_2}(v_n^+)^2 \mathrm{d}x \le \varepsilon, \tag{2.36}$$

for all n. Combining (f_7), (2.31) with (2.35), we can take b_ε so large that

$$\begin{aligned} &\int_{\Omega_n(b_\varepsilon,+\infty)} \frac{f(x,u_n^+)}{u_n^+}(v_n^+)^2 \mathrm{d}x \\ &\le \left(\int_{\Omega_n(b_\varepsilon,+\infty)} \left|\frac{f(x,u_n^+)}{u_n^+}\right|^\tau \mathrm{d}x\right)^{\frac{1}{\tau}} \left(\int_{\Omega_n(b_\varepsilon,+\infty)} (v_n^+)^\alpha \mathrm{d}x\right)^{\frac{2}{\alpha}} \\ &\le \left(\int_{\Omega_n(b_\varepsilon,+\infty)} a_1 \tilde{F}(x,u_n^+) \mathrm{d}x\right)^{\frac{1}{\tau}} \left(\int_{\Omega_n(b_\varepsilon,+\infty)} (v_n^+)^\alpha \mathrm{d}x\right)^{\frac{2}{\alpha}} \\ &\le \varepsilon, \end{aligned} \tag{2.37}$$

for all $n \ge n_0$, where $\alpha = \dfrac{2\tau}{\tau - 1}$. Note that there is $\gamma = \gamma(\varepsilon) > 0$ independent of n such that $|f(x,u_n^+)| \le \gamma u_n^+$ for $x \in \Omega_n(a_\varepsilon, b_\varepsilon)$. By (2.34), there exists $n_1 > 0$ such that

$$\int_{\Omega_n(a_\varepsilon,b_\varepsilon)} \frac{f(x,u_n^+)}{u_n^+}(v_n^+)^2 \mathrm{d}x \le \gamma \int_{\Omega_n(a_\varepsilon,b_\varepsilon)} (v_n^+)^2 \mathrm{d}x \le \varepsilon, \tag{2.38}$$

for all $n \ge n_1$. Now (2.36), (2.37), and (2.38) imply that

$$\int_\Omega \frac{f(x,u_n^+)u_n^+}{\|u_n\|^2} \mathrm{d}x = \int_\Omega \frac{f(x,u_n^+)}{u_n^+}(v_n^+)^2 \mathrm{d}x < 3\varepsilon, \tag{2.39}$$

for $n \geq \max\{n_0, n_1\}$. Since $\|v_n\| = 1$, passing to a subsequence, there exists $v \in H_0^1(\Omega)$, such that

$$\begin{cases} v_n \rightharpoonup v, & \text{weakly in } H_0^1(\Omega), \\ v_n \to v, & \text{strongly in } L^p(\Omega),\ 1 \leq p < 2^*, \\ v_n(x) \to v(x), & \text{a.e. in } \Omega. \end{cases}$$

Set $\Omega' = \{x \in \overline{\Omega} : v^+(x) \neq 0\}$, if meas $\Omega' > 0$, then $u_n^+(x) \to +\infty$ for a.e. $x \in \Omega'$. By (f_6), it is easy to get that for any $M > 0$, there exists $C(M) > 0$ such that

$$f(x, u_n^+)u_n^+ \geq M(u_n^+)^2 - C(M)$$

for all $x \in \overline{\Omega}$ and n large enough. Hence,

$$\int_\Omega \frac{f(x, u_n^+)u_n^+}{\|u_n\|^2} \mathrm{d}x \geq M \int_\Omega (v_n^+)^2 \mathrm{d}x - \frac{C(M)}{\|u_n\|^2},$$

then

$$0 = \lim_{n\to\infty} \int_\Omega \frac{f(x, u_n^+)u_n^+}{\|u_n\|^2} \mathrm{d}x \geq M \int_\Omega (v^+)^2 \mathrm{d}x > 0,$$

which is a contradiction. Hence, meas Ω'=0. Therefore, $v^+(x) = 0$ a.e. $x \in \overline{\Omega}$. By Remark 2, for any $m > 0$, there exists $L_0 > 0$ such that

$$tf(x, t) - 2F(x, t) \geq m > 0,$$

for $t > L_0$. It follows from (f_4) and (f_5) that $|F(x, t)| \leq C_{25}(t^2 + t^{2^*})$ for all $(x, t) \in \overline{\Omega} \times \mathbb{R}^+$. Hence, we have, for $x \in \overline{\Omega}$ and $|t| \leq L_0$,

$$|tf(x, t) - 2F(x, t)| \leq C_{26}t^2,$$

where $C_{26} = 2C_{25}(1 + L_0^{2^*-2})$. The two inequalities above show that

$$tf(x, t) - 2F(x, t) \geq -C_{27}t^2, \tag{2.40}$$

for all $(x, t) \in \overline{\Omega} \times \mathbb{R}^+$. From $(1 + \|u_n\|)\|I'(u_n)\| \to 0$, one has $\langle I'(u_n), u_n\rangle \to 0$, that is,

$$\|u_n\|^2 - \int_\Omega \frac{(u_n^+)^{2^*(s)}}{|y|^s} \mathrm{d}x - \int_\Omega f(x, u_n^+)u_n^+ \mathrm{d}x = o(1),$$

and combining with (2.31), one deduces

$$\begin{aligned} c+1+o(1) \geq\, & I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle \\ =\, & \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\|u_n\|^2 - \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\int_\Omega f(x, u_n^+)u_n^+ \mathrm{d}x \\ & + \frac{1}{2}\int_\Omega \big(f(x, u_n^+)u_n^+ - 2F(x, u_n^+)\big)\mathrm{d}x. \end{aligned}$$

Consequently, together with (2.39) and (2.40), we deduce

$$\begin{aligned} \frac{1}{\|u_n\|^2}\Big(c+1+o(1)\Big) \geq\, & \left(\frac{1}{2} - \frac{1}{2^*(s)}\right) - \frac{1}{\|u_n\|^2}\left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\int_\Omega f(x, u_n^+)u_n^+ \mathrm{d}x \\ & + \frac{1}{2\|u_n\|^2}\int_\Omega \big(f(x, u_n^+)u_n^+ - 2F(x, u_n^+)\big)\mathrm{d}x \\ \geq\, & \left(\frac{1}{2} - \frac{1}{2^*(s)}\right) - 3\left(\frac{1}{2} - \frac{1}{2^*(s)}\right)\varepsilon - C_{27}\int_\Omega (v_n^+)^2 \mathrm{d}x, \end{aligned}$$

which implies $0 \geq \frac{1}{2} - \frac{1}{2^*(s)}$ as $n \to \infty$, a contradiction. Thus, $\{u_n\}$ is bounded.

It is obvious that (f_4) leads to (f_1) and then Lemma 2.2 is also true if we replace (f_1) by (f_4). Thanks to (f_4), (f_5), and Lemma 2.2, similar to the proof of Theorem 1, we obtain a positive solution of problem (1.1). □

Proof of Theorem 3 Due to (f_1), (f_5), and Lemma 2.2, one obtains that the proof of Theorem 3 is similar to the proof of Theorem 2 and we only need to prove that the $(Ce)_c$ sequence is bounded. In fact, let $\{u_n\} \subset H_0^1(\Omega)$ be a $(Ce)_c$ sequence, that is,

$$I(u_n) \to c \;\text{ and }\; (1+\|u_n\|)\|I'(u_n)\| \to 0 \quad \text{as } n \to \infty. \tag{2.41}$$

Assume that $\{u_n\}$ is unbounded, there is a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) satisfying $\|u_n\| \to +\infty$. Set $\omega_n = \frac{u_n}{\|u_n\|}$, then $\|\omega_n\| = 1$. Then, there exists $\omega \in H_0^1(\Omega)$ such that

$$\begin{cases} \omega_n \rightharpoonup \omega, & \text{weakly in } H_0^1(\Omega), \\ \omega_n \to \omega, & \text{strongly in } L^p(\Omega),\ 1 \leq p < 2^*, \\ \omega_n(x) \to \omega(x), & \text{a.e. in } \Omega, \end{cases} \tag{2.42}$$

as $n \to \infty$. We claim that $\omega^+ = 0$. It follows from (2.41) that

$$\|u_n\|^2 - \int_\Omega \frac{(u_n^+)^{2^*(s)}}{|y|^s}\mathrm{d}x - \int_\Omega f(x, u_n^+)u_n^+ \mathrm{d}x = o(1),$$

which implies

$$\int_\Omega f(x,u_n^+)u_n^+\mathrm{d}x = \|u_n\|^2 - \int_\Omega \frac{(u_n^+)^{2^*(s)}}{|y|^s}\mathrm{d}x + o(1) \le \|u_n\|^2 + o(1).$$

Then,

$$\int_\Omega \frac{f(x,u_n^+)u_n^+}{\|u_n\|^2}\mathrm{d}x \le 1 + o(1). \tag{2.43}$$

For $x \in \Omega^+ := \{x \in \overline{\Omega} : \omega^+(x) > 0\}$, $u_n^+(x) \to +\infty$ as $n \to \infty$. Combining with (f_6), we have

$$\lim_{n\to\infty} \frac{f(x,u_n^+)}{u_n^+}(\omega_n^+)^2 = +\infty, \quad \text{a.e. } x \in \Omega^+. \tag{2.44}$$

If meas $\Omega^+ > 0$, using the Fatou's lemma, (2.44) implies that as $n \to \infty$,

$$\int_{\omega>0} \frac{f(x,u_n^+)}{u_n^+}(\omega_n^+)^2\mathrm{d}x \to +\infty.$$

From (2.43), one has

$$1 + o(1) \ge \int_\Omega \frac{f(x,u_n^+)u_n^+}{\|u_n\|^2}\mathrm{d}x \ge \int_{\Omega^+} \frac{f(x,u_n^+)}{u_n^+}(\omega_n^+)^2\mathrm{d}x \to +\infty,$$

which is a contradiction; then, one has meas $\Omega^+ = 0$, that is, $\omega^+ = 0$. Set a sequence $\{t_n\}$ of real numbers such that $I(t_n u_n^+) = \max_{t\in[0,1]} I(tu_n^+)$. Let $v_n = S_\mu^{\frac{2^*(s)}{2(2^*(s)-2)}}\omega_n$, due to the continuity of embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, we have $\int_\Omega |v_n|^{2^*}\mathrm{d}x \le C_{23} < \infty$. By (f_5) and the same method as the proof of Theorem 1, one has

$$\int_\Omega F(x,v_n^+)\mathrm{d}x \to \int_\Omega F(x,0)\mathrm{d}x = 0,$$

as $n \to \infty$. Because $\|u_n\| \to +\infty$ as $n \to \infty$, one has $\dfrac{S_\mu^{\frac{2^*(s)}{2(2^*(s)-2)}}}{\|u_n\|} \in [0,1]$ for n large enough. By the definition of t_n and $v_n = \dfrac{S_\mu^{\frac{2^*(s)}{2(2^*(s)-2)}}}{\|u_n\|}u_n$, one has

$$\begin{aligned}
I(t_n u_n) \ge I(v_n) &= \frac{1}{2}\|v_n\|^2 - \frac{1}{2^*(s)}\int_\Omega \frac{(v_n^+)^{2^*(s)}}{|y|^s}\mathrm{d}x - \int_\Omega F(x,v_n^+)\mathrm{d}x \\
&\ge \frac{1}{2}\|v_n\|^2 - \frac{1}{2^*(s)}S_\mu^{-\frac{2^*(s)}{2}}\|v_n\|^{2^*(s)} - \int_\Omega F(x,v_n^+)\mathrm{d}x \\
&= \left(\frac{1}{2} - \frac{1}{2^*(s)}\right)S_\mu^{\frac{2^*(s)}{2^*(s)-2}} - \int_\Omega F(x,v_n^+)\mathrm{d}x,
\end{aligned}$$

which implies that $I(t_nu_n) \to +\infty$ as $n \to \infty$. Noting that $I(0) = 0$, $I(u_n) \to c$, thus $0 < t_n < 1$ when n is large enough. It follows that

$$\int_\Omega \left(|\nabla(t_nu_n)|^2 - \mu\frac{(t_nu_n)^2}{|y|^2} \right) \mathrm{d}x - \int_\Omega \frac{(t_nu_n^+)^{2^*(s)}}{|y|^s}\mathrm{d}x - \int_\Omega f(x, t_nu_n^+)t_nu_n^+\mathrm{d}x$$
$$= \langle I'(t_nu_n), t_nu_n\rangle = t_n\frac{dI(t_nu_n)}{\mathrm{d}t} \;|_{t=t_n}= 0.$$

By (f_8), for $0 \le t_n \le 1$, we have $\theta H(x, u_n^+) \ge H(x, t_nu_n^+) - \theta_0$, then

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2} f(x, u_n^+)u_n^+ - F(x, u_n^+)\right) \mathrm{d}x \\
&= \frac{1}{2} \int_\Omega H(x, u_n^+)\mathrm{d}x \\
&\ge \frac{1}{2\theta} \int_\Omega \left(H(x, t_nu_n^+) - \theta_0\right)\mathrm{d}x \\
&= \frac{1}{\theta} \int_\Omega \left(\frac{1}{2} f(x, t_nu_n^+)t_nu_n^+ - F(x, t_nu_n^+)\right) \mathrm{d}x - \frac{\theta_0}{2\theta}|\Omega| \\
&= \frac{1}{\theta} \left(\frac{1}{2}\|t_nu_n\|^2 - \frac{1}{2} \int_\Omega \frac{(t_nu_n^+)^{2^*(s)}}{|y|^s}\mathrm{d}x - \int_\Omega F(x, t_nu_n^+)\mathrm{d}x\right) - \frac{\theta_0}{2\theta}|\Omega| \\
&= \frac{1}{\theta}I(t_nu_n) + \frac{1}{\theta} \left(\frac{1}{2^*(s)} - \frac{1}{2}\right) \int_\Omega \frac{(t_nu_n^+)^{2^*(s)}}{|y|^s}\mathrm{d}x - \frac{\theta_0}{2\theta}|\Omega|,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\int_\Omega \left(\frac{1}{2} f(x, u_n^+)u_n^+ - F(x, u_n^+)\right) \mathrm{d}x + \frac{1}{\theta} \left(\frac{1}{2} - \frac{1}{2^*(s)}\right) \int_\Omega \frac{(t_nu_n^+)^{2^*(s)}}{|y|^s}\mathrm{d}x \\
&\ge \frac{1}{\theta}I(t_nu_n) - \frac{\theta_0}{2\theta}|\Omega| \\
&\to +\infty,
\end{aligned} \tag{2.45}$$

as $n \to \infty$. But by $\theta \ge 1$ and $0 \le t_n \le 1$, we have $\frac{t_n^{2^*(s)}}{\theta} \le 1$; then, it follows from (2.41) that

$$
\begin{aligned}
c+1+o(1) &\geq I(u_n) - \frac{1}{2}\langle I'(u_n), u_n\rangle \\
&\geq \left(\frac{1}{2}-\frac{1}{2^*(s)}\right)\int_\Omega \frac{(u_n^+)^{2^*(s)}}{|y|^s}\mathrm{d}x \\
&\quad + \int_\Omega \left(\tfrac{1}{2} f(x,u_n^+)u_n^+ - F(x,u_n^+)\right)\mathrm{d}x \\
&\geq \frac{1}{\theta}\left(\frac{1}{2}-\frac{1}{2^*(s)}\right)\int_\Omega \frac{(t_n u_n^+)^{2^*(s)}}{|y|^s}\mathrm{d}x \\
&\quad + \int_\Omega \left(\frac{1}{2} f(x,u_n^+)u_n^+ - F(x,u_n^+)\right)\mathrm{d}x,
\end{aligned}
$$

which contradicts (2.45). Therefore, $\{u_n\}$ is bounded. □

Proof of Theorem 4 It is obvious that (f_9) leads to (f_5) and then Lemma 2.2 is also true if we replace (f_5) by (f_9). Due to (f_1), (f_9), and Lemma 2.2, one obtains that the proof of Theorem 4 is similar to the proof of Theorem 2, and we only need to prove that the $(Ce)_c$ sequence is bounded. In fact, let $\{u_n\} \subset H_0^1(\Omega)$ be a $(Ce)_c$ sequence, that is,

$$
I(u_n) \to c \ \text{ and } \ (1+\|u_n\|)\|I'(u_n)\| \to 0 \quad \text{as } n \to \infty. \tag{2.46}
$$

By (f_{10}), there exist positive constants $D, C_{28} > 0$, such that

$$
f(x,t)t - 2F(x,t) \geq D|t|^\sigma - C_{28}, \tag{2.47}
$$

for all $t \in \mathbb{R}^+$ and a.e. $x \in \overline{\Omega}$. Together with (2.46), one has

$$
\begin{aligned}
2c+1+o(1) &\geq 2I(u_n) - \langle I'(u_n), u_n\rangle \\
&= \left(1-\frac{2}{2^*(s)}\right)\int_\Omega \frac{(u_n^+)^{2^*(s)}}{|y|^s}\mathrm{d}x + \int_\Omega \Big(f(x,u_n^+)u_n - 2F(x,u_n^+)\Big)\mathrm{d}x \\
&\geq \left(1-\frac{2}{2^*(s)}\right)\int_\Omega \frac{(u_n^+)^{2^*(s)}}{|y|^s}\mathrm{d}x + D\int_\Omega |u_n|^\sigma \mathrm{d}x - C_{28}|\Omega|.
\end{aligned}
$$

From above, we easily obtain that there exist constants $C_{29}, C_{30} > 0$ such that

$$
\int_\Omega \frac{(u_n^+)^{2^*(s)}}{|y|^s}\mathrm{d}x \leq C_{29}, \qquad \int_\Omega |u_n|^\sigma \mathrm{d}x \leq C_{30}.
$$

By (f_9), for any $\varepsilon > 0$, there exists $a(\varepsilon) > 0$ such that

$$
|F(x,t)| \leq \varepsilon t^q + a(\varepsilon) \quad \text{for } (x,t) \in \Omega \times \mathbb{R}^+,
$$

then it follows that

$$
\begin{aligned}
\frac{1}{2}\|u_n\|^2 - I(u_n) &= \frac{1}{2^*(s)}\int_\Omega \frac{(u_n^+)^{2^*(s)}}{|y|^s}\mathrm{d}x + \int_\Omega F(x,u_n^+)\mathrm{d}x \\
&\leq \frac{C_{29}}{2^*(s)} + C_{31}\int_\Omega |u_n|^q \mathrm{d}x + a(\varepsilon)|\Omega|.
\end{aligned}
$$

By the Gagliardo–Nirenberg interpolation inequality, one has

$$\int_\Omega |u_n|^q \mathrm{d}x \le \left(\int_\Omega |u_n|^\sigma \mathrm{d}x\right)^{\frac{qt}{\sigma}} \left(\int_\Omega |u_n|^{2^*} \mathrm{d}x\right)^{\frac{q(1-t)}{2^*}},$$

where $0 < \sigma \le q < 2^*$, $\frac{1}{q} = \frac{t}{\sigma} + \frac{1-t}{2^*}$, and $t \in (0, 1]$. Then, we deduce from above inequality and Sobolev inequality

$$\begin{aligned}\frac{1}{2}\|u_n\|^2 &\le \frac{C_{29}}{2^*(s)} + a(\varepsilon)|\Omega| + C_{31}\int_\Omega |u_n|^q \mathrm{d}x + I(u_n)\\ &\le \frac{C_{29}}{2^*(s)} + C_{31}\left(\int_\Omega |u_n|^\sigma \mathrm{d}x\right)^{\frac{qt}{\sigma}} \left(\int_\Omega |u_n|^{2^*} \mathrm{d}x\right)^{\frac{q(1-t)}{2^*}} + a(\varepsilon)|\Omega| + c + 1\\ &\le \frac{C_{29}}{2^*(s)} + C_{32}\|u_n\|^{q(1-t)} + a(\varepsilon)|\Omega| + c + 1.\end{aligned}$$

Since by definition of q, we have $q(1-t) = \frac{2^*(q-\sigma)}{2^*-\sigma}$ with $\sigma > \frac{N(q-2)}{2}$, it follows that $q(1-t) < 2$. Thus, $\{u_n\}$ is bounded. □

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