Bull. Malays. Math. Sci. Soc. (2020) 43:609–618
https://doi.org/10.1007/s40840-018-00703-7

BULLETIN of the MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

ORIGINAL ARTICLE

Complementary Results on the Boundedness Problem of Factorizable Four-Dimensional Matrices

Gholamreza Talebi[1]

Received: 17 July 2018 / Revised: 18 October 2018 / Published online: 15 November 2018

Abstract
In this paper, the boundedness problem of factorizable four-dimensional matrices on the space of double sequences is investigated. As an application of our results, the lower bounds and the operator norms of four-dimensional Cesàro matrix and four-dimensional Copson matrix are obtained, which provide an extension of Hardy's discrete inequality and Copson's discrete inequality to double series, respectively. Finally, we present complementary results for the operator norm and lower bound of the four-dimensional Hausdorff matrices.

Keywords Operator norm and lower bound · Double sequences · Cesàro matrix

Mathematics Subject Classification 47A30 · 40G05

1 Introduction

By Ω, we denote the space of all real or complex valued double sequences which is the vector space with coordinatewise addition and scalar multiplication. The space $\mathcal{L}_p$ of double sequences [3] is defined by

$$\mathcal{L}_p = \left\{ \left(x_{n,m}\right) \in \Omega : \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left|x_{n,m}\right|^p < \infty \right\},$$

where $1 \leq p < \infty$, which is a complete space with the norm

Communicated by Fuad Kittaneh.

The paper is dedicated to the doyen of the martyrs, the chief of the youth of paradise, Imam Hossein ibn Ali (peace be upon him) in the memory of the 1379th occasion of his Arbaeen.

✉ Gholamreza Talebi
Gh.talebi@vru.ac.ir

1 Department of Mathematics, Faculty of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Islamic Republic of Iran

$$\|x\|_{\mathcal{L}_p} := \left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left|x_{n,m}\right|^p\right)^{1/p}.$$

For more information on the normed spaces of double sequences and domain of triangle matrices in normed/paranormed sequence spaces, and the matrix transformations and summability theory, we refer the readers to the textbook [1] and the recent papers [2,4,7–9,14,17] and [21–29].

Let X and Y be two double sequence spaces and $\mathsf{H} = (h_{nmjk})$ be a four-dimensional infinite matrix of real or complex numbers. Then, we say that H defines a matrix mapping from X into Y, and we denote it by writing $\mathsf{H} : X \to Y$, if for every double sequence $x = (x_{n,m}) \in X$ the double sequence $\mathsf{H}x = \{(\mathsf{H}x)_{n,m}\}$, the H-transform of x, exists and is in Y, where

$$(\mathsf{H}x)_{n,m} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} h_{nmjk}x_{j,k}, \quad (n, m = 0, 1, \ldots).$$

The purpose of this paper is to establish the lower bound and the operator norm of factorizable four-dimensional matrices as operators on the double sequence space $\mathcal{L}_p$. For $p \in \mathbb{R}\backslash\{0\}$, the lower bound involved here is the number $L_p(\mathsf{H})$, which is defined as the supremum of those ℓ, obeying the following inequality

$$\|\mathsf{H}x\|_{\mathcal{L}_p} \geq \ell\, \|x\|_{\mathcal{L}_p},$$

where $x \geq 0,\ x \in \mathcal{L}_p$ and $\mathsf{H} = (h_{nmjk})$ is a nonnegative four-dimensional matrix. Also, we consider the upper bound $k > 0$, of the form

$$\|\mathsf{H}x\|_{\mathcal{L}_p} \leq k\, \|x\|_{\mathcal{L}_p},$$

for all nonnegative sequences x. The constant k is not depending on x. We seek the smallest possible value of k and denote the best upper bound by $\|\mathsf{H}\|_{\mathcal{L}_p}$ as the operator norm of H on $\mathcal{L}_p$. When we deal with two-dimensional matrices, we use, respectively, the notation $L_p(\mathsf{A})$ and $\|\mathsf{A}\|_{\ell_p}$ for the lower bound and the operator norm of the matrix A on ℓ_p, where ℓ_p is the space of all real or complex $p-$absolutely summable sequences.

2 Main Results

Our main results are as follows.

Theorem 2.1 *Let $A = (a_{nj})$ and $B = (b_{mk})$ be two nonnegative infinite matrices with the lower bounds $L_p(A)$ and $L_p(B)$, respectively. Let $\mathsf{H} = (a_{nj}b_{mk})$ be the four-dimensional matrix constructed from A and B. Then,*

$$L_p(H) = L_p(A)L_p(B)$$

Proof Let $x = (x_{jk})$ be a nonnegative double sequence in $\mathcal{L}_p$. Then,

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_{nj}b_{mk}x_{jk}\right)^p &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{j=0}^{\infty}a_{nj}\sum_{k=0}^{\infty}b_{mk}x_{jk}\right)^p \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{j=0}^{\infty}a_{nj}y_{mj}\right)^p \quad \left(y_{mj} := \sum_{k=0}^{\infty}b_{mk}x_{jk}\right) \\ &\geq \left(L_p(\mathsf{A})\right)^p \sum_{m=0}^{\infty}\sum_{j=0}^{\infty}y_{mj}^p \\ &= \left(L_p(\mathsf{A})\right)^p \sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{k=0}^{\infty}b_{mk}x_{jk}\right)^p \\ &\geq \left(L_p(\mathsf{A})\right)^p\left(L_p(\mathsf{B})\right)^p \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}x_{jk}^p.\end{aligned}$$

This implies that

$$L_p(\mathsf{H}) \geq L_p(\mathsf{A})\, L_p(\mathsf{B})\,.$$

In order to see that one even has equality, look at double sequences of the form

$$x_{jk} = \varsigma_j \eta_k.$$

Then, one has that

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_{nj}b_{mk}x_{jk}\right)^p = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{\infty}a_{nj}\varsigma_j\right)^p \sum_{m=0}^{\infty}\left(\sum_{k=0}^{\infty}b_{mk}\eta_k\right)^p.$$

Now let $\alpha > L_p(\mathsf{A})$ and $\beta > L_p(\mathsf{B})$. Then, there exist nonnegative sequences (ς_j) and (η_k) such that

$$\sum_{n=0}^{\infty}\left(\sum_{j=0}^{\infty}a_{nj}\varsigma_j\right)^p < \alpha^p \sum_{j=0}^{\infty}\varsigma_j^p,$$

and

$$\sum_{m=0}^{\infty}\left(\sum_{k=0}^{\infty}b_{mk}\eta_k\right)^p < \beta^p \sum_{k=0}^{\infty}\eta_k^p.$$

Then,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_{nj}b_{mk}x_{jk}\right)^{p} < (\alpha\beta)^{p}\sum_{j=0}^{\infty}\varsigma_{j}^{p}\sum_{k=0}^{\infty}\eta_{k}^{p}.$$

This implies that

$$L_{p}\,(\mathsf{H}) < \alpha\beta.$$

Consequently,

$$L_{p}\,(\mathsf{H}) \leq L_{p}\,(\mathsf{A})\,L_{p}\,(\mathsf{B})\,. \qquad \square$$

Theorem 2.2 *Let $A = (a_{nj})$ and $B = (b_{mk})$ be two nonnegative infinite matrices with the norms $\|A\|_{\ell_p}$ and $\|B\|_{\ell_p}$, respectively. Let $H = (a_{nj}b_{mk})$ be the four-dimensional matrix constructed from A and B. Then,*

$$\|H\|_{\mathcal{L}_p} = \|A\|_{\ell_p}\|B\|_{\ell_p}$$

Proof The proof can be easily adapted from the one of Theorem 2.1 and so is omitted. □

To provide some applications of the above Theorems, we refer the readers to the next two sections.

3 Extension of Hardy's and Copson's Inequalities

In this section, using Theorems 2.1 and 2.2, we are going to provide an extension of Hardy's and Copson's inequalities to double series. First, consider the Hardy's inequality [13]

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{x_k}{n+1}\right)^{p} \leq \left(\frac{p}{p-1}\right)^{p}\sum_{k=0}^{\infty}x_{n}^{p} \quad (1 < p < \infty)\,, \tag{3.1}$$

where $x_k \geq 0$ for all $k \in \mathbb{N}$ and the constant $\left(\frac{p}{p-1}\right)^{p}$ is the best possible. Inequality (3.1) can be rewritten as $\|C(1)\|_{\ell_p} = \frac{p}{p-1}$, where $C(1) = (c_{nk})_{n,k\geq 0}$ is the Cesàro matrix of order 1, defined by

$$c_{nk} = \begin{cases} \frac{1}{n+1} & 0 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Now, consider the four-dimensional Cesàro matrix of order 1 and 1, $\mathsf{C}(1,1)=\left(h_{nmjk}\right)$ defined by

$$h_{nmjk}=\begin{cases}\frac{1}{(n+1)(m+1)} & (0\le j\le n,\ 0\le k\le m)\,,\\ 0 & o.w.\end{cases}$$

Obviously, this matrix can be factorized as $h_{nmjk}=c_{nj}c_{mk}$ where c_{nj} and c_{mk} are defined by (3.2). Applying Theorem 2.2, we have

$$\|\mathsf{C}(1,1)\|_{\mathcal{L}_p}=\|C(1)\|_{\ell_p}\|C(1)\|_{\ell_p}=\left(\frac{p}{p-1}\right)^2.$$

This leads us to the following generalization of Hardy's inequality.

Corollary 3.1 *Let $1<p<\infty$ and $x=(x_{nk})$ be nonnegative sequence of real numbers in $\mathcal{L}_p$. Then,*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{k=0}^{m}\frac{x_{jk}}{(n+1)\,(m+1)}\right)^p\le\left(\frac{p}{p-1}\right)^{2p}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}x_{nm}^p. \tag{3.3}$$

The constant $\left(\frac{p}{p-1}\right)^{2p}$ in (3.3) is the best possible.

Inequality (3.3) is an extension of Hardy's inequality to double series. It can be extended to multiple series [16].

Next, consider the Copson's inequality [11] [see also ([13], Theorem 344)]

$$\sum_{n=0}^{\infty}\left(\sum_{k=n}^{\infty}\frac{x_k}{k+1}\right)^p\ge p^p\sum_{k=0}^{\infty}x_k^p\qquad(0<p\le 1)\,, \tag{3.4}$$

where $x_k\ge 0$ for all $k\in\mathbb{N}$. The inequality switch order when $p>1$ and the constant p^p is the best possible. Again, inequality (3.4) can be rewritten as $L_p\left(C(1)^t\right)=p$, where $C(1)^t$ denotes the transpose of $C(1)$, and $C(1)$ is the Cesàro matrix of order 1, defined by (3.2). The transpose of the Cesàro matrix is called Copson matrix. Now, consider the four-dimensional Copson matrix $\mathsf{C}^t(1,1)=\left(h_{nmjk}\right)$ defined by

$$h_{nmjk}=\begin{cases}\frac{1}{(j+1)(k+1)} & (j\ge n\,,\ k\ge m\,)\,,\\ 0 & o.w.\end{cases}$$

Since the four-dimensional Copson matrix $\mathsf{C}^t(1,1)$ can be factorized as $h_{nmjk}=c_{nj}^tc_{mk}^t$, where c_{nj} and c_{mk} are defined by (3.2), applying Theorem 2.1 we have

$$L_p\left(\mathsf{C}^t(1,1)\right)=L_p\left(C(1)^t\right)L_p\left(C(1)^t\right)=p^2,$$

whenever $0 < p \leq 1$. Further, since $\left\| C(1)^t \right\|_{\ell_p} = p$, by Theorem 2.2, we deduce that

$$\left\| \mathsf{C}^{\mathsf{t}}(1,1) \right\|_{\mathcal{L}_p} = \left\| C(1)^t \right\|_{\ell_p} \left\| C(1)^t \right\|_{\ell_p} = p^2,$$

for $1 < p < \infty$. These lead us to the following generalization of Copson's inequality.

Corollary 3.2 *Let $x = (x_{nk})$ be nonnegative sequence of real numbers. Then*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left(\sum_{j=n}^{\infty} \sum_{k=m}^{\infty} \frac{x_{jk}}{(j+1)(k+1)} \right)^p \geq p^{2p} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x_{nm}^p \qquad (0 < p \leq 1). \quad (3.5)$$

The inequality switch order when $p > 1$ and the constant p^{2p} is the best possible.

Inequality (3.5) is an extension of Copson's inequality to double series. It can be extended to multiple series [18].

4 Complimentary Results for Four-Dimensional Hausdorff Matrices

Let $\mathrm{d}\mu$ and $\mathrm{d}\lambda$ be two Borel probability measures on [0,1] and $\mathsf{H}_{\mu\times\lambda} = (h_{nmjk})$ be the four-dimensional Hausdorff matrix defined by [15]

$$h_{nmjk} = \begin{cases} \int_0^1 \int_0^1 \binom{n}{j}\binom{m}{k} \alpha^j \beta^k (1-\alpha)^{n-j} (1-\beta)^{m-k} \mathrm{d}\mu(\alpha) \times \mathrm{d}\lambda(\beta), & 0 \leq j \leq n, 0 \leq k \leq m, \\ 0 & \text{otherwise,} \end{cases}$$

for all $n, m, j, k \in \mathbb{N}$. Clearly, we have

$$h_{nmjk} = \binom{n}{j} \binom{m}{k} \Delta_1^{n-j} \Delta_2^{m-k} \mu_{j,k},$$

where

$$\mu_{j,k} := \int_0^1 \int_0^1 \alpha^j \beta^k \mathrm{d}\mu(\alpha) \times \mathrm{d}\lambda(\beta), \qquad (j, k = 0, 1, \ldots)$$

and

$$\Delta_1^{n-j} \Delta_2^{m-k} \mu_{j,k} = \sum_{s=0}^{n-j} \sum_{t=0}^{m-k} (-1)^{s+t} \binom{n-j}{s} \binom{m-k}{t} \mu_{j+s,k+t}.$$

The four-dimensional Hausdorff matrix contains some famous classes of matrices. These classes are as follows:

1. The choices $\mathrm{d}\mu(\alpha) = \eta(1-\alpha)^{\eta-1}\mathrm{d}\alpha$ and $\mathrm{d}\lambda(\beta) = \gamma(1-\beta)^{\gamma-1}\mathrm{d}\beta$ give the four-dimensional Cesàro matrix of order η and γ which is denoted by $\mathsf{C}(\eta, \gamma)$.
2. The choices $\mathrm{d}\mu(\alpha) =$ point evaluation at $\alpha = \eta$ and $\mathrm{d}\lambda(\beta) =$ point evaluation at $\beta = \gamma$ give the four-dimensional Euler matrix of order η and γ which is denoted by $\mathsf{E}(\eta, \gamma)$,
3. The choices $\mathrm{d}\mu(\alpha) = |\log\alpha|^{\eta-1} / \Gamma(\eta)\mathrm{d}\alpha$ and $\mathrm{d}\lambda(\beta) = |\log\beta|^{\gamma-1} / \Gamma(\gamma)\mathrm{d}\beta$ give the four-dimensional Hölder matrix of order η and γ, which is denoted by $\mathsf{H}(\eta, \gamma)$.
4. The choices $\mathrm{d}\mu(\alpha) = \eta\alpha^{\eta-1}\mathrm{d}\alpha$ and $\mathrm{d}\lambda(\beta) = \gamma\beta^{\gamma-1}\mathrm{d}\beta$ give the four-dimensional Gamma matrix of order η and γ which is denoted by $\Gamma(\eta, \gamma)$.

The four-dimensional Cesàro, Hölder and Gamma matrices have nonnegative entries whenever $\eta > 0$ and $\gamma > 0$, and also the four-dimensional Euler matrices, when $0 < \eta < 1$ and $0 < \gamma < 1$.

The study of the boundedness problem of four-dimensional Hausdorff matrices goes back to the some recent works of the author. For example, it is proved in ([19], Theorem 3.1) that

$$\left\|\mathsf{H}_{\mu\times\lambda}\right\|_{\mathcal{L}_p} = \int_0^1\int_0^1 (\alpha\beta)^{-\frac{1}{p}}\mathrm{d}\mu(\alpha)\times\mathrm{d}\lambda(\beta) \quad (1 < p < \infty). \tag{4.1}$$

Further, it is proved in ([20], pp. 7–8) that

$$L_p\left(\mathsf{H}^t_{\mu\times\lambda}\right) = \int_0^1\int_0^1 (\alpha\beta)^{\frac{1-p}{p}}\mathrm{d}\mu(\alpha)\times\mathrm{d}\lambda(\beta) \quad (0 < p \leq 1), \tag{4.2}$$

and

$$L_p\left(\mathsf{H}_{\mu\times\lambda}\right) \geq \int_0^1\int_0^1 (\alpha\beta)^{-\frac{1}{p}}\mathrm{d}\mu(\alpha)\times\mathrm{d}\lambda(\beta) \quad (0 < p \leq 1). \tag{4.3}$$

According to the Hellinger–Toeplitz theorems ([6], Propositions 7.2 and 7.3), (4.1), (4.2) and (4.3), respectively, give

$$\left\|\mathsf{H}^t_{\mu\times\lambda}\right\|_{\mathcal{L}_p} = \int_0^1\int_0^1 (\alpha\beta)^{\frac{1-p}{p}}\mathrm{d}\mu(\alpha)\times\mathrm{d}\lambda(\beta) \quad (1 < p < \infty), \tag{4.4}$$

and

$$L_p\left(\mathsf{H}_{\mu\times\lambda}\right) = \int_0^1\int_0^1 (\alpha\beta)^{-\frac{1}{p}}\mathrm{d}\mu(\alpha)\times\mathrm{d}\lambda(\beta) \quad (-\infty < p < 0), \tag{4.5}$$

and

$$L_p\left(\mathsf{H}^t_{\mu\times\lambda}\right) \geq \int_0^1\int_0^1 (\alpha\beta)^{\frac{1-p}{p}}\mathrm{d}\mu(\alpha)\times\mathrm{d}\lambda(\beta) \quad (-\infty < p < 0). \tag{4.6}$$

The proof of (4.1), (4.2) and (4.3), in those papers, is all based on the special version of four-dimensional Euler matrix. On the other hand, the four-dimensional Hausdorff matrix can be factorized as

$$h_{nmjk} = h^{(\mu)}_{nj} h^{(\lambda)}_{mk}, \tag{4.7}$$

where $\mathsf{H}_\mu = (h^{(\mu)}_{nj})_{nj}$ is the classical (two-dimensional) Hausdorff matrix [5] corresponding to the Borel probability measure $\mathrm{d}\mu$ (and similarly for H_λ). A similar factorization holds for the transpose $\mathsf{H}^t_{\mu\times\lambda}$. Therefore, (4.1), (4.2) and (4.3), can be obtained by a different way. In fact they are all special cases of Theorems 2.1 and 2.2, using the classical results on lower bounds and norms of the classical (two-dimensional) Hausdorff matrices in [6,10,12]. For example, to achieve (4.4), consider the transpose of the four-dimensional Hausdorff matrices as operators selfmap of the space $\mathcal{L}_p$. Applying Theorem 2.2, we have the following result.

Theorem 4.1 *Let p be fixed,* $1 < p < \infty$ *and* $\mathsf{H}_{\mu\times\lambda}$ *be the four-dimensional Hausdorff matrix associated with the measures* $\mathrm{d}\mu$ *and* $\mathrm{d}\lambda$*. Then, the transpose* $\mathsf{H}^t_{\mu\times\lambda}$ *is bounded on* $\mathcal{L}_p$ *if and only if both* $\int_0^1 \alpha^{\frac{1-p}{p}} \mathrm{d}\mu(\alpha) < \infty$ *and* $\int_0^1 \beta^{\frac{1-p}{p}} \mathrm{d}\lambda(\beta) < \infty$, *and we have*

$$\left\| H^t_{\mu\times\lambda} \right\|_{\mathcal{L}_p} = \int_0^1 \int_0^1 (\alpha\beta)^{\frac{1-p}{p}} \mathrm{d}\mu(\alpha) \times \mathrm{d}\lambda(\beta).$$

Proof For the classical (two-dimensional) Hausdorff matrix H_μ, it is proved by Bennett [5] that $\mathsf{H}^{\mathsf{t}}_\mu$ is bounded on ℓ_p if and only if $\int_0^1 \alpha^{\frac{1-p}{p}} \mathrm{d}\mu(\alpha) < \infty$, and that $\left\|\mathsf{H}^{\mathsf{t}}_\mu\right\|_{\ell_p} = \int_0^1 \alpha^{\frac{1-p}{p}} \mathrm{d}\mu(\alpha)$. The result of the theorem is now a consequence of Theorem 2.2. □

Let $\mathsf{E}(\eta,\gamma)$, $\mathsf{C}(\eta,\gamma)$, $\mathsf{H}(\eta,\gamma)$ and $\Gamma(\eta,\gamma)$ be the four-dimensional Euler matrices, Cesàro matrices, Hölder matrices and Gamma matrices, respectively. Applying Theorem 4.1 to these matrices, we have the following corollary.

Corollary 4.2 *Let p be fixed,* $1 < p < \infty$ *and* $\eta, \gamma > 0$*. Then,*

1. $\left\|E^t(\eta,\gamma)\right\|_{\mathcal{L}_p} = (\eta\gamma)^{\frac{1-p}{p}}, \eta \le 1, \gamma \le 1.$
2. $\left\|C^t(\eta,\gamma)\right\|_{\mathcal{L}_p} = \frac{\Gamma(\eta+1)\Gamma^2\left(\frac{1}{p}\right)\Gamma(\gamma+1)}{\Gamma\left(\eta+\frac{1}{p}\right)\Gamma\left(\gamma+\frac{1}{p}\right)}.$
3. $\left\|H^t(\eta,\gamma)\right\|_{\mathcal{L}_p} = \frac{1}{\Gamma(\eta)\Gamma(\gamma)} \int_0^1\int_0^1 (\alpha\beta)^{\frac{p-1}{p}} |\log\alpha|^{\eta-1} |\log\beta|^{\gamma-1} \mathrm{d}\alpha \times \mathrm{d}\beta.$
4. $\left\|\Gamma^t(\eta,\gamma)\right\|_{\mathcal{L}_p} = \frac{p^2\eta\gamma}{(p\eta-p+1)(p\gamma-p+1)}, \quad \eta > \frac{p-1}{p}, \gamma > \frac{p-1}{p}.$

Putting $\eta = \gamma = 1$, the second part of the above corollary implies

$$\left\|\mathsf{C}^{\mathsf{t}}(1,1)\right\|_{\mathcal{L}_p} = \frac{\Gamma(2)\,\Gamma^2\left(\frac{1}{p}\right)\Gamma(2)}{\Gamma\left(1+\frac{1}{p}\right)\Gamma\left(1+\frac{1}{p}\right)} = p^2.$$

This says us that the second part of Corollary 4.2 is a generalization of Copson's inequality (see Corollary 3.2).

To the best of our knowledge the exact values of $L_p\left(\mathsf{H}_{\mu\times\lambda}\right)$ and $L_p\left(\mathsf{H}^t_{\mu\times\lambda}\right)$ for $1 < p < \infty$ have not been found yet. Keeping in mind factorization (4.7) and the results obtained by Chen and Wang in ([10], Theorem 2.2) for $L_p(\mathsf{H}_\mu)$ and $L_p(\mathsf{H}^{\mathsf{t}}_\mu)$ where $p > 1$ and H_μ is the classical (two-dimensional) Hausdorff matrix, we now enable to fill up this gap by the use of Theorem 2.1.

Theorem 4.3 *Let p be fixed, $1 < p < \infty$ and $\mathsf{H}_{\mu\times\lambda}$ be the four-dimensional Hausdorff matrix associated with the measures dμ and dλ. Then,*

$$L_p\left(H_{\mu\times\lambda}\right) = \mu\left(\{1\}\right) \times \lambda\left(\{1\}\right),$$

and

$$L_p\left(H^t_{\mu\times\lambda}\right) = \left((\mu\left(\{0\}\right))^p + (\mu\left(\{1\}\right))^p\right)^{\frac{1}{p}}\left((\lambda\left(\{0\}\right))^p + (\lambda\left(\{1\}\right))^p\right)^{\frac{1}{p}}.$$

We refer the readers to the paper [18] in which the operator norm and lower bound of some non-factorizable matrices are founded.

References

1. Başar, F.: Summability Theory and Its Applications, Bentham Science Publishers, e-books. Monographs, İstanbul (2012)
2. Başar, F., Çapan, H.: On the paranormed spaces of regularly convergent double sequences. Result. Math. **72**(1–2), 893–906 (2017)
3. Başar, F., Sever, Y.: The space $\mathcal{L}_p$ of double sequences. Math. J. Okayama Univ. **51**, 149–157 (2009)
4. Başar, F., Çapan, H.: On the paranormed space $M_u(t)$ of double sequences. Bol. Soc. Parana. Math. **37**(3), 97–109 (2018)
5. Bennett, G.: Lower bounds for matrices II. Can. J. Math. **44**, 54–74 (1991)
6. Bennett, G.: Factorizing the classical inequalities. Mem. Am. Math. Soc. **120**(576), 1–130 (1996)
7. Çapan, H., Başar, F.: Some paranormed difference spaces of double sequences. Indian J. Math. **58**(3), 405–427 (2016)
8. Çapan, H., Başar, F.: On some spaces isomorphic to the space of absolutely q-summable double sequences. Kyungpook Math. J. **58**(2), 271–289 (2018)
9. Çapan, H., Başar, F.: On the paranormed space $L(t)$ of double sequences. Filomat **32**(3), 1043–1053 (2018)
10. Chen, C.-P., Wang, K.-Z.: Lower bounds of Copson type for Hausdorff matrices II. Linear Algebra Appl. **422**, 563–573 (2007)
11. Copson, E.T.: Note on series of positive terms. J. Lond. Math. Soc. **3**, 49–51 (1928)
12. Hardy, G.H.: An inequality for Hausdorff means. J. Lond. Math. Soc. **18**, 46–50 (1943)
13. Hardy, G.H., Littlewood, J.E., Polya, G.: Inequalities, 2nd edn. Cambridge University Press, Cambridge (1967)
14. Mursaleen, M., Başar, F.: Domain of Cesáro mean of order one in some spaces of double sequences. Stud. Sci. Math. Hungar. **51**(3), 335–356 (2014)
15. Rhoades, B.E.: Some classes of doubly infinite matrices. Indian J. Pure Appl. Math. **23**(6), 419–427 (2003)
16. Salem, Sh, Zareen, A.A., Khan, R.Redheffer: Hardy's inequality in several variables. J. Math. Anal. Appl. **252**(2), 989–993 (2000)
17. Talebi, G.: On boundedness of four-dimensional Hausdorff matrices on the double Taylor sequence spaces. Indag. Math. **28**(5), 953–961 (2017)

18. Talebi, G.: Operator norm and lower bound of four-dimensional matrices. Indag. Math. **28**(6), 1134–1143 (2017)
19. Talebi, G.: Operator norms of four-dimensional Hausdorff matrices on the double Euler sequence spaces. Linear Multilinear Algebra **65**(11), 2257–2267 (2017)
20. Talebi, G.: Lower bound of four dimensional Hausdorff matrices (2018) **(preprint)**
21. Yeşilkayagil, M., Başar, F.: Domain of Riesz mean in some spaces of double sequences. Indag. Math. **29**(3), 1009–1029 (2018)
22. Yeşilkayagil, M., Başar, F.: A note on Abel summability of double series. Numer. Funct. Anal. Optim. **38**(8), 1069–1076 (2017)
23. Yeşilkayagil, M., Başar, F.: Domain of Riesz mean in the space L_s. Filomat **31**(4), 925–940 (2017)
24. Yeşilkayagil, M., Başar, F.: A note on Riesz summability of double series. Proc. Natl. Acad. Sci. India Sect. A Phys. Sci. **86**(3), 333–337 (2016)
25. Yeşilkayagil, M., Başar, F.: Some topological properties of the spaces of almost null and almost convergent double sequences. Turk. J. Math. **40**(3), 624–630 (2016)
26. Yeşilkayagil, M., Başar, F.: Mercerian theorem for four-dimensional matrices. Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **65**(1), 147–155 (2016)
27. Yeşilkayagil, M., Başar, F.: On the characterization of some classes of four-dimensional matrices and Steinhaus type theorems. Kragujev. J. Math. **40**(1), 35–45 (2016)
28. Yeşilkayagil, M., Başar, F.: Comparison of four-dimensional summability methods. Aligarh Bull. Math. **34**(1–2), 1–11 (2015)
29. Yeşilkayagil, M., Başar, F.: Four-dimensional dual and dual of the new sort summability methods. Contemp. Anal. Appl. Math. **3**(1), 13–29 (2015)