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# Directional Time–Frequency Analysis and Directional Regularity

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**Abstract** We improve some of the results related to the directional short-time Fourier transform by fixing the direction and extend them to the spaces $\mathscr{K}_1(\mathbb{R}^n)$ and $\mathscr{K}_1(\mathbb{R})\widehat{\otimes}\mathscr{U}(\mathbb{C}^n)$ and their duals. Then, we define multidimensional short-time Fourier transform in the direction of $u^k$ for tempered distributions, directional regular sets and their complements, directional wave fronts. Different windows with mild conditions on their support show the invariance of these notions related to window functions. Smoothness of $f$ follows from the assumptions of the directional regularity in any direction.

**Keywords** Short-time Fourier transform · Distributions · Directional wave front

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## 1 Introduction

In multidimensional time–frequency analysis, wave fronts are useful concepts when analyzing where, how and why one distribution is singular, and when observing the direction in which the singularity occurs. Also, wave fronts are one of the crucial elements in the recent studies of the theory of distributions because of their ability to control the product of distributions.

The motivation of this paper is coming from [2], where Grafakos and Sansing developed a theory that merges the Radon transform and time–frequency theory, and introduced the concept of directionally sensitive time–frequency analysis. Let $g \in \mathscr{S}(\mathbb{R})$ be a nonzero window function, $(u, x, \xi) \in \mathbb{S}^{n-1} \times \mathbb{R} \times \mathbb{R}^n$, where $\mathbb{S}^{n-1}$ is the unit sphere and $f \in L^1(\mathbb{R}^n)$. Then,

$$g_{u,x,\xi}(t) = \mathrm{e}^{2\pi i \xi(u\cdot t - x)} g(u \cdot t - x), \quad t \in \mathbb{R}^n, \tag{1}$$

called Gabor ridge functions, can be viewed as time–frequency analysis elements in the Radon domain. By pairing the function $f$ with $g_{u,x,\xi}$, Grafakos and Sansing provided an idea to localize information in time, frequency and direction defining a directionally sensitive variant of the short-time Fourier transform (STFT). They have shown that it is not possible to obtain an exact reconstruction of a signal using the Gabor ridge functions [2, Thrm. 1], and therefore they have modified their class of functions to the weighted Gabor ridge functions (see [2] for details). Their results for directionally sensitive time–frequency decompositions in $L^2(\mathbb{R}^n)$ based on Gabor systems in $L^2(\mathbb{R})$ are generalized in [8], by showing similar results for discrete and continuous frames.

Giv [1] introduced another transform which is also a directionally sensitive variant of the STFT, letting

$$g_{u,x,\xi}\,(t) = \mathrm{e}^{2\pi i t\cdot\xi} g\,(t \cdot u - x)\,, \quad t \in \mathbb{R}^n.$$

Using these functions he defined the directional short-time Fourier transform (DSTFT) and proved several orthogonality results and reconstruction formulas for it [1].

The aim of this paper is twofold. In the first part (Sect. 3), we study the DSTFT by fixing the direction $u$. This new transform will be called short-time Fourier transform in the direction of $u$, and the appropriate synthesis operator will be introduced. We defined them on the exponential-type distributions, as an extension of the results of two of us (cf. [10]) for tempered distributions to distributions of exponential-type $\mathscr{K}_1'(\mathbb{R}^n)$. In this part we improve some results of [1,10] by observing that the original function can be recovered from the STFT in any specified direction.

In the second part of the paper we give an extension, introducing the multidimensional STFT in the direction of $u^k = (u_1, \ldots, u_k)$, where $u_i, i = 1, \ldots, k$ are independent vectors of $\mathbb{S}^{n-1}$ (Sect. 4). Moreover, by a simple transformation of coordinates, we simplify our exposition considering directions of orthonormal basis $e_1, \ldots, e_k$ of $\mathbb{R}^k$ in the framework of $\mathbb{R}^n$. In this way we present our main aim, namely the analysis of the regularity properties of a signal $f(t), t \in \mathbb{R}^n$, being a tempered distribution, through the knowledge of the short-time Fourier transform in direction of selected coordinates. In other words, we introduce and analyze the directional wave fronts which can be applied in the time–frequency analysis.

## 2 Preliminaries

### 2.1 Notation

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is defined as $\mathscr{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(x)dx$, $\xi \in \mathbb{R}^n$. The translation and modulation operators are given by $T_x f(\,\cdot\,) = f(\,\cdot\, - x)$ and $M_\xi f(\,\cdot\,) = e^{2\pi i \xi\,\cdot} f(\,\cdot\,)$, $x, \xi \in \mathbb{R}^n$, respectively. The operators $M_\xi T_x$ and $T_x M_\xi$ are called time–frequency shifts. The notation $\langle f, \varphi\rangle$ means dual pairing, whereas $(f, \varphi)$ stands for the $L^2$ inner product. The set of all nonnegative integers is denoted by $\mathbb{N}_0$.

### 2.2 Spaces

The Schwartz space of rapidly decreasing smooth functions and its dual, the space of tempered distributions, are denoted by $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$, respectively, [12]. Recall [5] that the space of exponentially rapidly decreasing smooth functions $\mathscr{K}_1(\mathbb{R}^n)$ is the space that consists of $\varphi \in C^\infty(\mathbb{R}^n)$ for which all the norms

$$\rho_k(\varphi) := \sup_{t\in\mathbb{R}^n,\ |\alpha|\le k} e^{k|t|}|\varphi^{(\alpha)}(t)|, \quad k \in \mathbb{N}_0,$$

are finite. It is an FS-space and therefore Montel and reflexive. Moreover, the space $\mathscr{K}_1(\mathbb{R}^n)$ is nuclear. The dual space $\mathscr{K}_1'(\mathbb{R}^n)$ consists of all distributions of the exponential form $f = \sum_{|\alpha|\le l}(e^{s|\,\cdot\,|} f_\alpha)^{(\alpha)}$, where $f_\alpha \in L^\infty(\mathbb{R}^n)$ [5]. Next, recall [5] that $\mathscr{U}(\mathbb{C}^n)$ is the space of entire functions such that $\varphi \in \mathscr{U}(\mathbb{C}^n)$ if and only if

$$\theta_k(\varphi) := \sup_{z\in\Pi_k} (1 + |z|^2)^{k/2}|\varphi(z)| < \infty, \quad \forall k \in \mathbb{N}_0,$$

where $\Pi_k$ is the tube $\Pi_k = \mathbb{R}^n + i[-k, k]^n$. The dual space $\mathscr{U}'(\mathbb{C}^n)$, known as the space of Silva tempered ultradistributions (see [6,7,11,14]), contains the space of analytic functionals.

As it turns out, the Fourier transform is a topological isomorphism from $\mathscr{K}_1(\mathbb{R}^n)$ onto $\mathscr{U}(\mathbb{C}^n)$ and extends to a topological isomorphism (with respect to strong topologies) $\mathscr{F} : \mathscr{K}_1'(\mathbb{R}^n) \to \mathscr{U}'(\mathbb{C}^n)$ [5,14].

Next, we introduce the topological tensor product space $\mathscr{K}_1(\mathbb{R})\widehat{\otimes}\mathscr{U}(\mathbb{C}^n)$ derived as the completion of the tensor product $\mathscr{K}_1(\mathbb{R}) \otimes \mathscr{U}(\mathbb{C}^n)$ in the $\pi$- topology, the same as the completion in the $\varepsilon$-topology [13]. The topology of $\mathscr{K}_1(\mathbb{R})\widehat{\otimes}\mathscr{U}(\mathbb{C}^n)$ is given by the family of the norms

$$\rho_k^l(\Phi) := \sup_{(x,z)\in\mathbb{R}\times\Pi_k} e^{k|x|}(1 + |z|^2)^{k/2}\left|\frac{\partial^l}{\partial x^l}\Phi(x, z)\right|, \quad k, l \in \mathbb{N}_0.$$

Its dual $(\mathscr{K}_1(\mathbb{R})\widehat{\otimes}\mathscr{U}(\mathbb{C}^n))' = \mathscr{K}_1'(\mathbb{R})\widehat{\otimes}\mathscr{U}'(\mathbb{C}^n)$ will be used in our definition of the DSTFT of exponential distributions as it contains the range of this transform (cf. Sect. 3.2). If a measurable function $F$ satisfies

$$|F(x,z)| \leq C\mathrm{e}^{s|x|}(1+|z|)^s, \quad (x,z) \in \mathbb{R}\times\mathbb{C}^n,$$

for some $s,\ C > 0$, then we shall identify $F$ with an element of $\mathscr{K}_1'(\mathbb{R})\widehat{\otimes}\mathscr{U}'(\mathbb{C}^n)$ via

$$\langle F, \Phi\rangle := \int_{\mathbb{R}}\int_{\mathbb{R}^n} F(x, \xi+i\eta)\,\Phi(x,\xi+i\eta)\,\mathrm{d}\xi\mathrm{d}x, \tag{2}$$

$z=\xi+i\eta,\ \xi,\eta\in\mathbb{R}^n,\ \Phi\in\mathscr{K}_1(\mathbb{R})\widehat{\otimes}\mathscr{U}(\mathbb{C}^n)$. (2) holds due to the Cauchy integral theorem.

### 2.3 The Short-Time Fourier Transform

Let $f\in L^2(\mathbb{R}^n)$. Recall that the short-time Fourier transform (STFT) of $f$ with respect to a window function $g\in L^2(\mathbb{R}^n)$ is given by

$$V_g f(x,\xi) := \langle f(t), \overline{M_\xi T_x g(t)}\rangle_t = \int_{\mathbb{R}^n} f(t)\overline{g(t-x)}\mathrm{e}^{-2\pi i\xi\cdot t}\mathrm{d}t, \quad x,\xi\in\mathbb{R}^n. \tag{3}$$

The adjoint of $V_g$, over $L^2(\mathbb{R}^{2n})$, is given by

$$V_g^* F(t) = \iint_{\mathbb{R}^{2n}} F(x,\xi)g(t-x)\mathrm{e}^{2\pi i\xi\cdot t}\mathrm{d}x\mathrm{d}\xi.$$

If $g\neq 0$ and $\psi\in L^2(\mathbb{R}^n)$ is a synthesis window for $g$, that is, one for which $(g,\psi)\neq 0$, then any $f\in L^2(\mathbb{R}^n)$ can be recovered from its STFT via the inversion formula

$$f(t) = \frac{1}{(g,\psi)}\iint_{\mathbb{R}^{2n}} V_g f(x,\xi) M_\xi T_x\psi(t)\mathrm{d}x\mathrm{d}\xi. \tag{4}$$

Whenever the generalized inner product in (3) is well defined, the definition of $V_g f$ can be viewed in a larger class than $L^2(\mathbb{R}^n)$. It is easy to show that if $g\in\mathscr{S}(\mathbb{R}^n)\backslash\{0\}$ is a fixed window, then $V_g:\mathscr{S}(\mathbb{R}^n)\to\mathscr{S}(\mathbb{R}^{2n})$ and $V_g^*:\mathscr{S}(\mathbb{R}^{2n})\to\mathscr{S}(\mathbb{R}^n)$ are continuous mappings. We refer to [3,4] for the basic STFT theory.

Moreover, in [9] the authors have shown that if $g\in\mathscr{K}_1(\mathbb{R}^n)\backslash\{0\}$, then $V_g:\mathscr{K}_1(\mathbb{R}^n)\to\mathscr{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathscr{U}(\mathbb{C}^n)$ and $V_g^*:\mathscr{K}_1(\mathbb{R}^n)\widehat{\otimes}\mathscr{U}(\mathbb{C}^n)\to\mathscr{K}_1(\mathbb{R}^n)$ are continuous mappings.

One can define the STFT of a distribution $f\in\mathscr{K}_1'(\mathbb{R}^n)$ (resp. $\mathscr{S}'(\mathbb{R}^n)$) with respect to a window $g\in\mathscr{K}_1(\mathbb{R}^n)$ (resp. $g\in\mathscr{S}(\mathbb{R}^n)$) as

$$V_g f(x,\xi) = \langle f, \overline{M_\xi T_x g}\rangle. \tag{5}$$

### 2.4 The Short-Time Fourier Transform in the Direction of *u*

The directional short-time Fourier transform (DSTFT) of an integrable function $f \in L^1(\mathbb{R}^n)$ (or $f \in \mathscr{D}'_{L^1}(\mathbb{R}^n)$) with respect to the window $g \in \mathscr{S}(\mathbb{R})$ is given by

$$\int_{\mathbb{R}^n} f(t)\overline{g(u \cdot t - x)}\mathrm{e}^{-2\pi i t\cdot\xi}\mathrm{d}t = \left\langle f(t), \overline{g}_{u,x,\xi}(t)\right\rangle_t, \tag{6}$$

where $(u, x, \xi) \in \mathbb{S}^{n-1} \times \mathbb{R} \times \mathbb{R}^n$, [1]. For a given $u \in \mathbb{S}^{n-1}$, we will call this transform *STFT in the direction of* $u$ and denote by $\mathrm{DS}_{g,u} f(x, \xi)$.

One can show, by the use of results of Gröchenig [3] (we will demonstrate this in the proof of Proposition 4 of Sect. 4), that for a nontrivial $g \in \mathscr{S}(\mathbb{R})$, with synthesis window $\psi \in \mathscr{S}(\mathbb{R})$ and $f \in L^1(\mathbb{R}^n)$, the following reconstruction formula holds pointwisely,

$$f(t) = \frac{1}{(g, \psi)} \int_{\mathbb{R}^n} \int_{\mathbb{R}} \mathrm{DS}_{g,u} f(x, \xi)\psi_{u,x,\xi}(t)\mathrm{d}x\mathrm{d}\xi. \tag{7}$$

Reconstruction formula (7) allows us define an operator that maps functions on $\mathbb{R} \times \mathbb{R}^n$ to functions on $\mathbb{R}^n$ as superposition of functions $g_{u,x,\xi}$. Given $g \in \mathscr{S}(\mathbb{R})$, we introduce the appropriate *synthesis operator* as

$$\mathrm{DS}^*_{g,u}\Phi(t) := \int_{\mathbb{R}^n} \int_{\mathbb{R}} \Phi(x, \xi)g_{u,x,\xi}(t)\mathrm{d}x\mathrm{d}\xi, \quad t \in \mathbb{R}^n. \tag{8}$$

Thus, relation (7) takes the form $(\mathrm{DS}^*_{\psi,u} \circ \mathrm{DS}_{g,u}) f = (g, \psi) f$.

In [10] the authors have discussed the problem of extending the definition of DSTFT to the space of tempered distributions. Here, we study the STFT in the direction of $u$ in the context of the space $\mathscr{K}_1'(\mathbb{R}^n)$ of distributions of exponential type.

If $f \in \mathscr{K}_1(\mathbb{R}^n)$ and $g \in \mathscr{K}_1(\mathbb{R})$, then we immediately get that $\mathrm{DS}_{g,u} f(x, \xi)$ extends to a holomorphic function in the second variable. This means that $\mathrm{DS}_{g,u} f(x, z)$ is entire in $z \in \mathbb{C}^n$. We write in the sequel $z = \xi + i\eta$ with $\xi, \eta \in \mathbb{R}^n$. Note also that if $\Phi \in \mathscr{K}_1(\mathbb{R})\widehat{\otimes}\mathscr{U}(\mathbb{C}^n)$ and $g \in \mathscr{K}_1(\mathbb{R})$, then, using the Cauchy theorem, we may write $\mathrm{DS}^*_{g,u}\Phi$ as

$$\mathrm{DS}^*_{g,u}\Phi(t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}} \Phi(x, \xi + i\eta)g(u \cdot t - x)\mathrm{e}^{2\pi i(\xi+i\eta)\cdot t}\mathrm{d}x\mathrm{d}\xi, \tag{9}$$

for arbitrary $\eta \in \mathbb{R}^n$. In the next section, we will show that if $g \in \mathscr{K}_1(\mathbb{R})$, then $\mathrm{DS}^*_{g,u}$ maps continuously $\mathscr{K}_1(\mathbb{R})\widehat{\otimes}\mathscr{U}(\mathbb{C}^n) \to \mathscr{K}_1(\mathbb{R}^n)$. It will then be shown that $\mathrm{DS}^*_{g,u}$ can be even extended to act on the distribution space $\mathscr{K}_1'(\mathbb{R})\widehat{\otimes}\mathscr{U}'(\mathbb{C}^n)$.

As a simple consequence of Fubini's theorem, if $g \in \mathscr{K}_1(\mathbb{R})$, $f \in L^1(\mathbb{R}^n)$ and $\Phi \in \mathscr{K}_1(\mathbb{R})\widehat{\otimes}\mathscr{U}(\mathbb{C}^n)$, then one can easily prove

$$\int_{\mathbb{R}^n} f(t)\mathrm{DS}^*_{g,u}\Phi(t)\mathrm{d}t = \int_{\mathbb{R}^n} \int_{\mathbb{R}} \mathrm{DS}_{\overline{g},u} f(x, \xi + i\eta)\Phi(x, \xi + i\eta)\mathrm{d}x\mathrm{d}\xi, \tag{10}$$

$\xi, \eta \in \mathbb{R}^n$, and this can be written as $\left\langle f, \mathrm{DS}^*_{\overline{g},u}\Phi\right\rangle = \left\langle \mathrm{DS}_{g,u} f, \Phi\right\rangle$ using (2). As in [10], we use this dual relation when defining STFT in the direction of $u$ of exponential distributions.

# 3 The STFT in the Direction of $u$ of Exponential Distributions

## 3.1 Continuity of the STFT in the Direction of $u$ on $\mathscr{K}_1(\mathbb{R}^n)$

Let $g \in \mathscr{K}_1(\mathbb{R}) \setminus \{0\}$. Then $\mathrm{DS}_{g,u}$ is injective and $\mathrm{DS}^*_{g,u}$ is surjective, due to reconstruction formula (7).

Notice that we can extend the definition of the STFT in the direction of $u$ as a sesquilinear mapping $\mathrm{DS} : (\varphi, g) \mapsto \mathrm{DS}_{g,u}\varphi$, $\varphi \in \mathscr{K}_1(\mathbb{R}^n)$, $g \in \mathscr{K}_1(\mathbb{R})$, whereas the synthesis operator extends to the bilinear form $\mathrm{DS}^* : (\Phi, g) \mapsto \mathrm{DS}^*_{g,u}\Phi$, $\Phi \in \mathscr{K}_1(\mathbb{R})\hat{\otimes}\mathscr{U}(\mathbb{C}^n)$.

**Theorem 1** *The mapping DS* : $\mathscr{K}_1(\mathbb{R}^n) \times \mathscr{K}_1(\mathbb{R}) \to \mathscr{K}_1(\mathbb{R})\hat{\otimes}\mathscr{U}(\mathbb{C}^n)$ *is continuous.*

*Proof* We will show that for given $k, l \in \mathbb{N}_0$, there exist $\nu, \tau \in \mathbb{N}_0$ and $C > 0$ such that

$$\rho_k^l(\mathrm{DS}_{g,u}\varphi) \le C\rho_\nu(\varphi)\rho_\tau(g), \quad \varphi \in \mathscr{K}_1(\mathbb{R}^n),\ g \in \mathscr{K}_1(\mathbb{R}). \tag{11}$$

Indeed, we have

$$\begin{aligned}
&\mathrm{e}^{k|x|}(1+|z|^2)^{k/2}\left|\frac{\partial^l}{\partial x^l}\mathrm{DS}_{g,u}\varphi\,(x,z)\right| \\
&= \mathrm{e}^{k|x|}(1+|\xi+i\eta|^2)^{k/2}\left|\frac{\partial^l}{\partial x^l}\int_{\mathbb{R}^n}\varphi(t)\overline{g(u\cdot t-x)}\mathrm{e}^{-2\pi i t\cdot(\xi+i\eta)}\mathrm{d}t\right| \\
&\le \mathrm{e}^{k|x|}(1+|\xi|^2)^{k/2}(1+|\eta|^2)^{k/2}\left|\int_{\mathbb{R}^n}\varphi(t)\overline{g^{(l)}(u\cdot t-x)}(-1)^l\mathrm{e}^{-2\pi i t\cdot\xi}\mathrm{e}^{2\pi t\cdot\eta}\mathrm{d}t\right| \\
&\le C\mathrm{e}^{k|x|}(1+nk^2)^{k/2}\left|\int_{\mathbb{R}^n}\varphi(t)\overline{g^{(l)}(u\cdot t-x)}(-1)^l(1-\triangle_t)^{k/2}(\mathrm{e}^{-2\pi i t\cdot\xi})\mathrm{e}^{2\pi t\cdot\eta}\mathrm{d}t\right| \\
&\le C\mathrm{e}^{k|x|}(1+nk^2)^{k/2}\int_{\mathbb{R}^n}\left|(1-\triangle_t)^{k/2}\left(\varphi(t)\overline{g^{(l)}(u\cdot t-x)}\right)\right|\left|\mathrm{e}^{2\pi t\cdot\eta}\right|\mathrm{d}t \\
&\le C\mathrm{e}^{k|x|}(1+nk^2)^{k/2}\sum_{|k_1|+|k_2|=k}\binom{k}{k_1,k_2}\int_{\mathbb{R}^n}\left|\varphi^{(k_1)}(t)\overline{g^{(l+k_2)}(u\cdot t-x)}\right|\mathrm{e}^{2\pi k|t|}\mathrm{d}t \\
&\le \widetilde{C}_k\sum_{|k_1|+|k_2|=k}\binom{k}{k_1,k_2}\int_{\mathbb{R}^n}\mathrm{e}^{k|x-u\cdot t+u\cdot t|}\left|\varphi^{(k_1)}(t)\overline{g^{(l+k_2)}(u\cdot t-x)}\right|\mathrm{e}^{2\pi k|t|}\mathrm{d}t \\
&\le \widetilde{C}_k\sum_{|k_1|+|k_2|=k}\binom{k}{k_1,k_2}\int_{\mathbb{R}^n}\mathrm{e}^{k|u\cdot t-x|}\mathrm{e}^{k|t|}\left|\varphi^{(k_1)}(t)\overline{g^{(l+k_2)}(u\cdot t-x)}\right|\mathrm{e}^{2\pi k|t|}\mathrm{d}t \\
&= \widetilde{C}_k\sum_{|k_1|+|k_2|=k}\binom{k}{k_1,k_2}\int_{\mathbb{R}^n}\mathrm{e}^{k|u\cdot t-x|}\mathrm{e}^{(1+2\pi)k|t|}\left|\varphi^{(k_1)}(t)\right|\left|\overline{g^{(l+k_2)}(u\cdot t-x)}\right|\mathrm{d}t.
\end{aligned}$$

We now analyze the synthesis operator. □

**Theorem 2** *The bilinear mapping* $DS^* : (\mathscr{K}_1(\mathbb{R})\hat{\otimes}\mathscr{U}(\mathbb{C}^n)) \times \mathscr{K}_1(\mathbb{R}) \to \mathscr{K}_1(\mathbb{R}^n)$ *is continuous.*

*Proof* Let $g \in \mathscr{K}_1(\mathbb{R})$, $\Phi \in \mathscr{K}_1(\mathbb{R})\hat{\otimes}\mathscr{U}(\mathbb{C}^n)$ and $\varphi = \mathrm{DS}^*_{g,u}\Phi$. Let $\widehat{\Phi}_1(z_1, z)$ denote the Fourier transform of $\Phi(x, z)$ with respect to the first variable and $\mathscr{F}_2^{-1}(\Phi)(x, t)$ denote the inverse Fourier transform of $\Phi(x, z)$ with respect to the second variable. We remark that $\widehat{\Phi}_1(z_1, z)$ is an entire function in $z_1 = \omega + i\mu$, $\omega,\ \mu \in \mathbb{R}$. An application of the Cauchy theorem and the Parseval formula gives

$$\int_{\mathbb{R}} \Phi(x, z)g(u \cdot t - x)\mathrm{d}x = \int_{\mathbb{R}} \widehat{\Phi}_1(\omega + i\mu, z)\mathrm{e}^{-2\pi i(\omega+i\mu)u\cdot t}\widehat{g}(\omega + i\mu)\mathrm{d}\omega.$$

Observe that

$$\begin{aligned}\varphi(t) = \mathrm{DS}^*_{g,u}\Phi(t) &= \int_{\mathbb{R}}\int_{\mathbb{R}^n} \Phi(x, \xi + i\eta)g(u \cdot t - x)\mathrm{e}^{2\pi i(\xi+i\eta)\cdot t}\mathrm{d}\xi\,\mathrm{d}x \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^n} \Big(\mathrm{e}^{-2\pi i(\omega+i\mu)u\cdot t}\widehat{\Phi}_1(\omega + i\mu, \xi + i\eta)\widehat{g}(\omega + i\mu)\Big) \\ &\quad \mathrm{e}^{2\pi i t\cdot(\xi+i\eta)}\mathrm{d}\xi\,\mathrm{d}\omega \\ &= \int_{\mathbb{R}} \mathrm{e}^{-2\pi i(\omega+i\mu)u\cdot t}\widehat{g}(\omega + i\mu) \\ &\qquad \left(\int_{\mathbb{R}^n} \widehat{\Phi}_1(\omega + i\mu, \xi + i\eta)\mathrm{e}^{2\pi i t\cdot(\xi+i\eta)}\mathrm{d}\xi\right)\mathrm{d}\omega \\ &= \int_{\mathbb{R}} \mathrm{e}^{-2\pi i(\omega+i\mu)u\cdot t}\widehat{g}(\omega + i\mu)\mathscr{F}_2^{-1}(\hat{\Phi}_1)(\omega + i\mu, t)\mathrm{d}\omega.\end{aligned}$$

Hence,

$$\begin{aligned}\widehat{\varphi}(z) &= \int_{\mathbb{R}} \widehat{g}(z_1)\widehat{\Phi}_1(z_1, z) * \widehat{\mathrm{e}^{-2\pi i z_1 u\cdot t}}\mathrm{d}\omega \\ &= \int_{\mathbb{R}} \widehat{g}(z_1)\widehat{\Phi}_1(z_1, z) * \delta(z_1 u + z)\mathrm{d}\omega \\ &= \int_{\mathbb{R}} \widehat{\Phi}_1(z_1, z_1 u + z)\widehat{g}(z_1)\mathrm{d}\omega, \end{aligned} \tag{12}$$

$z = \xi + i\eta \in \mathbb{C}^n$, $z_1 = \omega + i\mu \in \mathbb{C}$.

We now prove the continuity of the bilinear synthesis mapping. Since the Fourier transform $g \mapsto \widehat{g}$ is a topological isomorphism from $\mathscr{K}_1(\mathbb{R}^n)$ onto $\mathscr{U}(\mathbb{C}^n)$, the family of seminorms

$$\sigma_k(g) = \theta_k(\widehat{g}), \quad g \in \mathscr{K}_1(\mathbb{R}^n), \quad k \in \mathbb{N}_0,$$

is a bases of seminorms for the topology of $\mathscr{K}_1(\mathbb{R}^n)$.

We also know that the Fourier transform with respect to the first variable, $\Phi(x, z) \to \hat{\Phi}_1(z_1, z)$, is a topological isomorphism from $\mathscr{K}_1(\mathbb{R})\hat{\otimes}\mathscr{U}(\mathbb{C}^n)$ onto $\mathscr{U}(\mathbb{C})\hat{\otimes}\mathscr{U}(\mathbb{C}^n)$. Therefore, the family of seminorms

$$\theta_{l,k}(\Phi) = \sup_{(z_1,z)\in\Pi_l^1\times\Pi_k} (1+|z_1|^2)^{l/2}(1+|z|^2)^{k/2}\left|\hat{\Phi}_1(z_1, z)\right|, \quad l, k \in \mathbb{N}_0,$$

$\Pi_l^1 = \mathbb{R} + i[-l, l]$, is a basis of seminorms for the topology of $\mathscr{K}_1(\mathbb{R})\hat{\otimes}\mathscr{U}(\mathbb{C}^n)$ .

We show that for a given $N \in \mathbb{N}_0$ there is $C > 0$ such that

$$\sigma_N\left(\mathrm{DS}^*_{g,u}\Phi\right) \le C\sigma_{N+2}(g)\theta_{0,N}(\Phi).$$

Now, setting again $\varphi(x) := \mathrm{DS}^*_{g,u}\Phi(x)$ and using expression (12), we get

$$\begin{aligned}(1+|z|)^N\left|\hat{\varphi}(z)\right| &= (1+|z|)^N\left|\int_{\mathbb{R}}\hat{g}(z_1)\hat{\Phi}_1(z_1, z_1u+z)\mathrm{d}\omega\right| \\ &\le \int_{\mathbb{R}}|\hat{g}(z_1)||\hat{\Phi}_1(z_1, z_1u+z)|(1+|z+z_1u|)^N(1+|z_1|)^N\mathrm{d}\omega \\ &\le \sigma_{N+2}(g)\theta_{0,N}(\Phi)\int_{\mathbb{R}}\frac{1}{(1+|z_1|)^2}\mathrm{d}\omega,\end{aligned}$$

where $z_1 = \omega + i\mu \in \mathbb{C}$. □

### 3.2 The STFT in the Direction of *u* on $\mathscr{K}_1'(\mathbb{R}^n)$

Let $u \in \mathbb{S}^{n-1}$. The continuity results allow us to define the STFT in the direction of $u$ of $f \in \mathscr{K}_1'(\mathbb{R}^n)$ with respect to $g \in \mathscr{K}_1(\mathbb{R})$ as the element $\mathrm{DS}_{g,u}f \in \mathscr{K}_1'(\mathbb{R})\hat{\otimes}\mathscr{U}'(\mathbb{C}^n)$ whose action on test functions is given by

$$\langle \mathrm{DS}_{g,u}f, \Phi\rangle := \langle f, \mathrm{DS}^*_{\overline{g},u}\Phi\rangle, \quad \Phi \in \mathscr{K}_1(\mathbb{R})\hat{\otimes}\mathscr{U}(\mathbb{C}^n). \tag{13}$$

Then, the synthesis operator $\mathrm{DS}^*_{g,u} : \mathscr{K}_1'(\mathbb{R})\hat{\otimes}\mathscr{U}'(\mathbb{C}^n) \to \mathscr{K}_1'(\mathbb{R}^n)$ can be defined as

$$\langle \mathrm{DS}^*_{g,u}F, \varphi\rangle := \langle F, \mathrm{DS}_{\overline{g},u}\varphi\rangle, \quad F \in \mathscr{K}_1'(\mathbb{R})\hat{\otimes}\mathscr{U}'(\mathbb{C}^n),\ \varphi \in \mathscr{K}_1(\mathbb{R}^n). \tag{14}$$

We immediately obtain:

**Proposition 1** *Let $g \in \mathscr{K}_1(\mathbb{R})$. The short-time Fourier transform in the direction of $u$, $DS_{g,u} : \mathscr{K}_1'(\mathbb{R}^n) \to \mathscr{K}_1'(\mathbb{R})\hat{\otimes}\mathscr{U}'(\mathbb{C}^n)$ and the synthesis operator $DS^*_{g,u} : \mathscr{K}_1'(\mathbb{R})\hat{\otimes}\mathscr{U}'(\mathbb{C}^n) \to \mathscr{K}_1'(\mathbb{R}^n)$ are continuous linear maps.*

### 3.3 Direct Definition of STFT in the Direction of $u$ on $\mathscr{S}'(\mathbb{R}^n)$

In [10], DSTFT on $\mathscr{S}'(\mathbb{R}^n)$ is defined as transposed mapping. The same definition holds for $\mathrm{DS}_{g,u} f$. In this subsection we will consider a direct definition of $\mathrm{DS}_{g,u} f$ on $\mathscr{S}'(\mathbb{R}^n)$ as follows. Let $g \in \mathscr{S}(\mathbb{R})$, $u \in \mathbb{S}^{n-1}$, and $x \in \mathbb{R}$. Then

$$\mathscr{S}'(\mathbb{R}^n) \ni f(t) \mapsto f(t)\overline{g(t \cdot u - x)} \in \mathscr{S}'(\mathbb{R}^n), \tag{15}$$

and

$$\mathscr{S}'(\mathbb{R}^n) \ni f(t)\overline{g(t \cdot u - x)} \mapsto \mathscr{F}(f(t)\overline{g(t \cdot u - x)})(\xi) \in \mathscr{S}'(\mathbb{R}^n)$$

defines $\mathrm{DS}_{g,u} f(x, \xi)$.

**Proposition 2** *The direct definition of the STFT in the direction of u and the one given via the transposed mapping coincide.*

*Proof* Let $f \in \mathscr{S}'(\mathbb{R}^n)$ and $(f_k)$ be a sequence from $\mathscr{S}'(\mathbb{R}^n)$ which converges to $f$ in $\mathscr{S}'(\mathbb{R}^n)$. Since both definitions agree on $f_k$, for every $k$, the assertion follows by the continuity. □

## 4 Multidimensional STFT in the Direction of $u^k$

We will extend our transform by introducing the STFT in the direction of $u^k$, $1 \le k \le n$. The case $k = 1$ is explained in the previous part of the paper.

Note that the $k$-th tensor product completed in $\pi$- or $\varepsilon$-topology $\mathscr{K}_1(\mathbb{R})\hat{\otimes} \cdots \hat{\otimes} \mathscr{K}_1(\mathbb{R})$ equals to $\mathscr{K}_1(\mathbb{R}^k)$. The same holds for $\mathscr{S}(\mathbb{R}^k)$. Below we will use notations $(\mathscr{K}_1(\mathbb{R}))^k = \mathscr{K}_1(\mathbb{R}) \times \cdots \times \mathscr{K}_1(\mathbb{R})$ and $(\mathscr{S}(\mathbb{R}))^k = \mathscr{S}(\mathbb{R}) \times \cdots \times \mathscr{S}(\mathbb{R})$.

Let $1 \le k \le n$. Let $u^k = (u_1, \ldots, u_k)$ where $u_i, i = 1, \ldots, k$ are independent vectors of $\mathbb{S}^{n-1}$, and $x^k = (x_1, \ldots, x_k) \in \mathbb{R}^k$. Let the nontrivial functions $g_1, \ldots, g_k$ belong to $\mathscr{K}_1(\mathbb{R})$ (resp., $\mathscr{S}(\mathbb{R})$), $g^k = g_1 \ldots g_k \in (\mathscr{K}_1(\mathbb{R}))^k$ (resp., $(\mathscr{S}(\mathbb{R}))^k$) and $\xi \in \mathbb{R}^n$.

Let $f \in \mathscr{K}_1(\mathbb{R}^n)$ (resp., $\mathscr{S}(\mathbb{R}^n)$). Then, we define the STFT in the direction of $u^k$ by

$$\mathrm{DS}_{g^k,u^k} f(x^k, \xi) := \int_{\mathbb{R}^n} f(t)\overline{g_1(u_1 \cdot t - x_1)} \cdots \overline{g_k(u_k \cdot t - x_k)} \mathrm{e}^{-2\pi i t \cdot \xi} \mathrm{d}t. \tag{16}$$

**Proposition 3** *By* (16) *is defined a continuous linear mapping of* $\mathscr{K}_1(\mathbb{R}^n)$ *(resp.,* $\mathscr{S}(\mathbb{R}^n)$*) into* $(\mathscr{K}_1(\mathbb{R}))^k \hat{\otimes} \mathscr{U}(\mathbb{C}^n)$ *(resp.,* $(\mathscr{S}(\mathbb{R}))^k \hat{\otimes} \mathscr{U}(\mathbb{C}^n)$*).*

*In particular, when* $k = n$, (16) *is the short-time Fourier transform.*

*Proof* Let $A = [u_{i,j}]$ be a $k \times n$ matrix with rows $u_i$, $i = 1, \ldots, k$ and $I_{n-k,n-k}$ be the identity matrix. Let $B$ be an $n \times n$ matrix determined by $A$ and $I_{n-k,n-k}$ so that $Bt = s$, where

$$s_1 = u_{1,1}t_1 + \cdots + u_{1,n}t_n, \ldots, s_k = u_{k,1}t_1 + \cdots + u_{k,n}t_n,$$

$s_{k+1} = t_{k+1}, \ldots, s_n = t_n$. Clearly, it is regular. Put $C = B^{-1}$ and $e^k = (e_1, \ldots, e_k)$ where $e_1 = (1, 0, \ldots, 0), \ldots, e_k = (0, \ldots, 1)$ are unit vectors of the coordinate system of $\mathbb{R}^k$. Then, with the change of variables $t = Cs$, and $\eta = C^t \xi$ ($C^t$ is the transposed matrix for $C$), one obtains, for $f \in \mathscr{K}_1(\mathbb{R}^n)$,

$$\begin{aligned} \mathrm{DS}_{g^k,u^k} f(x^k, \xi) &= (\mathrm{DS}_{g^k,e^k} |C| f(C\cdot))(x^k, \eta) \\ &= \int_{\mathbb{R}^n} f(s) \overline{g_1(s_1 - x_1)} \cdots \overline{g_k(s_k - x_k)} \mathrm{e}^{-2\pi i s \cdot \eta} \mathrm{d}s, \end{aligned} \tag{17}$$

where $|C|$ is the determinant of $C$.

Now, we immediately see the proof of the proposition since $\tilde{f}(s) = |C| f(Cs)$, $s \in \mathbb{R}^n$ is an element of $\mathscr{K}_1(\mathbb{R}^n)$. □

Put

$$g^k_{u^k, x^k, \xi}(t) = g_1(u_1 \cdot t - x_1) \cdots g_k(u_k \cdot t - x_k) \mathrm{e}^{2\pi i t \cdot \xi}, \, t \in \mathbb{R}^n.$$

Let $\psi_i \in \mathscr{K}_1(\mathbb{R})$ (resp., $\psi_i \in \mathscr{S}(\mathbb{R})$) be the synthesis window for $g_i \in \mathscr{K}_1(\mathbb{R})$ (resp., $g_i \in \mathscr{S}(\mathbb{R})$), $i = 1, \ldots, k$ and let

$$(g^k, \psi^k) = (g_1, \psi_1) \cdots (g_k, \psi_k) \neq 0.$$

We will prove the inversion formula.

**Proposition 4** *Let $f \in \mathscr{K}_1(\mathbb{R}^n)$ (resp., $\mathscr{S}(\mathbb{R}^n)$), $g^k, \psi^k \in (\mathscr{K}_1(\mathbb{R}))^k$ (resp., $(\mathscr{S}(\mathbb{R}))^k$). Then*

$$f(t) = \frac{1}{(g^k, \psi^k)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} DS_{g^k,u^k} f(x^k, \xi) \psi^k_{u^k,x^k,\xi}(t) \mathrm{d}x^k \mathrm{d}\xi \tag{18}$$

*pointwisely.*

*Proof* The proof is the same as for the short-time Fourier transform (see [3], Theorem 3.2.1 and Corollary 3.2.3). We will use, after the change of variables representation (17). Let $\tilde{f}_i(\cdot) = |C| f_i(C\cdot)$, $i = 1, 2$. Actually, by the Parseval identity we have that for given $f_1, f_2 \in L^2(\mathbb{R}^n)$ and $g^k, \psi^k \in (\mathscr{K}_1(\mathbb{R}))^k$,

$$\begin{aligned} &\left(\mathrm{DS}_{g^k,u^k} f_1(x^k, \xi), \mathrm{DS}_{\psi^k,u^k} f_2(x^k, \xi)\right)_{L^2(\mathbb{R}^k \times \mathbb{R}^n)} \\ &= \left(\mathrm{DS}_{g^k,e^k} \tilde{f}_1(x^k, \xi), \mathrm{DS}_{\psi^k,e^k} \tilde{f}_2(x^k, \xi)\right)_{L^2(\mathbb{R}^k \times \mathbb{R}^n)} \\ &= (\tilde{f}_1, \tilde{f}_2)_{L^2(\mathbb{R}^n)} (\overline{g^k}, \overline{\psi^k})_{L^2(\mathbb{R}^k)}. \end{aligned} \tag{19}$$

We obtain reconstruction formula (18) as a consequence of (19), as in the quoted corollary of [3]. □

Let $f \in \mathscr{K}_1(\mathbb{R}^n)$. We have that (16) extends to a holomorphic function, i.e., $\mathrm{DS}_{g^k,u^k} f(x^k, z)$ is entire in $z \in \mathbb{C}^n$. As in the case $k = 1$, if $\Phi \in (\mathscr{K}_1(\mathbb{R}))^k \widehat{\otimes} \mathscr{U}(\mathbb{C}^n)$ and $g^k \in (\mathscr{K}_1(\mathbb{R}))^k$, for arbitrary $\eta \in \mathbb{R}^n$ and by the Cauchy theorem, we can write

$$\mathrm{DS}^*_{g^k,u^k} \Phi(t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} \Phi(x^k, \xi + i\eta)$$

$$g_1(u_1 \cdot t - x_1) \cdots g_k(u_k \cdot t - x_k) \mathrm{e}^{2\pi i (\xi + i\eta)\cdot t} \mathrm{d}x^k \mathrm{d}\xi, \quad t \in \mathbb{R}^n. \tag{20}$$

The STFT in the direction of $u^k$ on dual spaces $\mathscr{K}_1'(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$ can be defined as in the case $k = 1$ (cf. Sects. 3.2 and 3.3).

The next theorem connects STFTs in the direction of $u^k$ with respect to different windows. It is crucial for the main theorem of Sect. 5.

**Theorem 3** *Let $u_1, \ldots, u_k \in \mathbb{S}^{n-1}$ be independent. Let $h_1, \ldots, h_k, g_1, \ldots, g_k, \gamma_1, \ldots, \gamma_k$ belong to $\mathscr{S}(\mathbb{R})$ where $\gamma_i$ is synthesis window for $g_i$, $i = 1, \ldots, k$. Let $f \in \mathscr{S}'(\mathbb{R}^n)$. Then*

$$DS_{h^k,u^k} f(y^k, \eta) = \left(DS_{g^k,u^k} f(s^k, \zeta)\right) * \left(DS_{h^k,u^k} \gamma^k(s^k, \zeta)\right)(y^k, \eta).$$

*Proof* By the use of the change of variables given in the proof of Proposition 3, it follows that it is enough to prove the assertion for $u^k = e^k$. Let $F \in \mathscr{S}'(\mathbb{R}^k)\hat{\otimes}\mathscr{U}'(\mathbb{C}^n)$. Then

$$\begin{aligned}
&\mathrm{DS}_{h^k,e^k}(\mathrm{DS}^*_{\gamma^k,e^k} F)(y^k, \eta) \\
&= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} F(x^k, \xi) \gamma_1(t_1 - x_1) \cdots \gamma_k(t_k - x_k) \mathrm{e}^{2\pi i \xi \cdot t} \mathrm{d}x^k \mathrm{d}\xi \right) \\
&\quad \overline{h_1(t_1 - y_1)} \cdots \overline{h_k(t_k - y_k)} \mathrm{e}^{-2\pi i t \cdot \eta} \mathrm{d}t \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} \left( \int_{\mathbb{R}^n} \overline{h_1(t_1 - (y_1 - x_1))} \cdots \overline{h_k(t_k - (y_k - x_k))} \right. \\
&\quad \left. \gamma_1(t_1) \cdots \gamma_k(t_k) \mathrm{e}^{-2\pi i t \cdot (\eta - \xi)} \mathrm{d}t \right) F(x^k, \xi) \mathrm{d}x^k \mathrm{d}\xi \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} \left( \int_{\mathbb{R}^n} \gamma^k(t^k) \overline{h^k(t^k - (y^k - x^k))} \mathrm{e}^{-2\pi i t \cdot (\eta - \xi)} \mathrm{d}t \right) F(x^k, \xi) \mathrm{d}x^k \mathrm{d}\xi \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} F(x^k, \xi) \mathrm{DS}_{h^k,e^k} \gamma^k(y^k - x^k, \eta - \xi) \mathrm{d}x^k \mathrm{d}\xi.
\end{aligned}$$

Now, we put $F = \mathrm{DS}_{g^k,e^k} f$ and obtain

$$\mathrm{DS}_{h^k,e^k} f(y^k, \eta) = (\mathrm{DS}_{g^k,e^k} f(s^k, \zeta)) * (\mathrm{DS}_{h^k,e^k} \gamma^k(s^k, \zeta))(y^k, \eta). \tag{21}$$

This completes the proof of the theorem. □

## 5 Directional Wave Fronts

The STFT in the direction of $u^k$ can be used in the detection of singularities determined by the hyperplanes orthogonal to vectors $u_1, \dots, u_k$. For this purpose, we introduce (multi)directional regular sets and wave front sets for tempered distributions using the STFT in the direction of $u^k$.

The proofs of Proposition 3 and Theorem 3 show that we can simplify our exposition by the use of the linear transformation $C$ of Proposition 3 and transfer the STFT in $u^k$ to STFT in $e^k$. Thus, in order to simplify our exposition of this section, we will consider regularity properties in the framework of the direction $u^k = e^k$.

If $k = 1$, we consider direction $e^1 = e_1$, while for $1 < k \leq n$, we consider direction $e^k = (e_1, \dots, e_k)$. Let $k = 1$ and $x_0 = x_{0,1} \in \mathbb{R}$. Put $\Pi_{e^1,x_0,\varepsilon} = \Pi_{x_0,\varepsilon} := \{t \in \mathbb{R}^n; |t_1 - x_0| < \varepsilon\}$. It is a part of $\mathbb{R}^n$ between two hyperplanes orthogonal to $e_1$, that is,

$$\Pi_{x_0,\varepsilon} = \bigcup_{x \in (x_0-\varepsilon, x_0+\varepsilon)} P_x, \quad (x_0 = (x_0, 0, \dots, 0)),$$

and $P_x$ denotes the hyperplane orthogonal to $e_1$ passing through $x$.

We keep the notation of Sect. 4. Put

$$\Pi_{e^k,x^k,\varepsilon} = \Pi_{e_1,x_1,\varepsilon} \cap \ldots \cap \Pi_{e_k,x_k,\varepsilon}, \ \Pi_{e^k,x^k} = \Pi_{e_1,x_1} \cap \ldots \cap \Pi_{e_k,x_k}.$$

The first set is a parallelepiped determined by $2k$ finite edges, while the other edges are infinite. The set $\Pi_{e^k,x^k}$ equals $\mathbb{R}^{n-k}$ translated by vectors $\mathbf{x_1}, \dots, \mathbf{x_k}$. We will call it $n-k$-dimensional element of $\mathbb{R}^n$ and denote it as $P_{e^k,x^k} \in \mathbb{R}^k$. If $k = n$, then this is just the point $x^n = (x_1, \dots, x_n)$.

**Definition 1** Let $f \in \mathscr{S}'(\mathbb{R}^n)$. It is said that $f$ is $k$-directionally regular at $(P_{e^k,x_0^k}, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ if there exists $g^k \in (\mathscr{D}(\mathbb{R}))^k$, $g^k(0) \neq 0$, a product of open balls $L_r(x_0^k) = L_r(x_{0,1}) \times \cdots \times L_r(x_{0,k}) \in \mathbb{R}^k$ and a cone $\Gamma_{\xi_0}$ such that for every $N \in \mathbb{N}$ there exists $C_N > 0$ such that

$$\sup_{x^k \in L_r(x_0^k),\, \xi \in \Gamma_{\xi_0}} |\mathrm{DS}_{g^k,e^k} f(x^k, \xi)| \\ = \sup_{x^k \in L_r(x_0^k),\, \xi \in \Gamma_{\xi_0}} |\mathscr{F}(f(t)\overline{g^k(t^k - x^k)})(\xi)| \leq C_N(1 + |\xi|^2)^{-N/2}. \quad (22)$$

Note that for $k = n$ our definition implies classical Hörmander's regularity.

*Remark 1* (a) If $f$ is $k$-directionally regular at $(P_{e^k,x_0^k}, \xi_0)$, then there exist an open ball (with radius $r$ and center $x_0^k$) $L_r(x_0^k)$ and an open cone $\Gamma \subset \Gamma_{\xi_0}$ so that $f$ is $k$-directionally regular at $(P_{e^k,z_0^k}, \theta_0)$ for any $z_0^k \in L_r(x_0^k)$ and $\theta_0 \in \Gamma$. This implies that the union of all $k$-directional regular points $(P_{e^k,z_0^k}, \theta_0)$, $(z_0^k, \theta_0) \in L_r(x_0^k) \times \Gamma$ is an open set of $\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$.

(b) Denote by $Pr_k$ the projection of $\mathbb{R}^n$ onto $\mathbb{R}^k$. Then, the $k$-directional regular point $(P_{e^k,x_0^k},\xi_0)$, considered in $\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}$ with respect to the first $k$ variables, equals $(Pr_k^{-1}\times I_\xi)(P_{e^k,x_0^k},\xi_0)$ ($I_\xi$ is the identity matrix on $\mathbb{R}^n$).

We define the $k$-directional wave front as the complement in $\mathbb{R}^k\times\mathbb{R}^n\setminus\{0\}$ of all $k$-directional regular points $(P_{e^k,x_0^k},\xi_0)$. This set is denoted as $WF_{e^k}f$. In $\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}$, this is $(Pr_k^{-1}\times I_\xi)(WF_{e^k}f)$.

**Proposition 5** *The set $WF_{e^k}(f)$ is closed in $\mathbb{R}^k\times\mathbb{R}^n\setminus\{0\}$ (and $\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}$).*

We will use notation $B_r(0^k)$ to denote a closed ball in $\mathbb{R}^k$ with center at zero $0^k$ and radius $r>0$. Our main theorem relates directional regular sets for two STFTs in the direction of $u^k$.

**Theorem 4** *If (22) holds for some $g^k\in(\mathscr{D}(\mathbb{R}))^k$, then it holds for every $h^k\in(\mathscr{D}(\mathbb{R}))^k$, ($h^k(0)\neq 0$) supported by a ball $B_\rho(0^k)$, where $\rho\le\rho_0$ and $\rho_0$ depends on $r$ in (22).*

*Proof* Since $g^k$ and $h^k$ are compactly supported, the integration with respect to $x^k$, which will be performed below, is finite. Moreover, we can assume that $f$ is a continuous polynomially bounded function. If not, let $f=P(D)F$, where $F$ is polynomially bounded and continuous, while $P(D)$ is a differential operator with constant coefficients. In this case we can perform partial integration and transfer the differentiation from $f$ on other factors of the integrand which do not affect the proof. So the analysis can be continued with $f$ continuous and polynomially bounded.

We use Proposition 3, that is, form (21). Assume that (22) holds and that $\gamma^k$ is chosen so that $\operatorname{supp}\gamma^k\subset B_{\rho_1}(0^k)$ and $\rho_1<r-r_0$. Let $h^k\in(\mathscr{D}(\mathbb{R}))^k$ and $\operatorname{supp}h^k\subset B_\rho(0^k)$. We will find $\rho_0$ such that (22) holds for $\mathrm{DS}_{h^k,e^k}f(y^k,\eta)$, with $y^k\in B_{r_0}(x_0^k)$, $\eta\in\Gamma_1\subset\subset\Gamma_{\xi_0}$, for $\rho\le\rho_0$ ($\Gamma_1\subset\subset\Gamma_{\xi_0}$ means that $\Gamma_1\cap\mathbb{S}^{n-1}$ is a compact subset of $\Gamma_{\xi_0}\cap\mathbb{S}^{n-1}$).

We need the next simple observations:

$$\begin{aligned}&|p^k|\le\rho_1,\quad |y^k-x_0^k|\le r_0 \text{ and } \ |p^k-\Big((y^k-x_0^k)-(x^k-x_0^k)\Big)|\le\rho\\ &\quad\Rightarrow |x^k-x_0^k|\le\rho+\rho_1+r_0.\end{aligned}\tag{23}$$

So, we choose $\rho_0$ such that $\rho_0+\rho_1<r-r_0$. Then

$$\rho+\rho_1+r_0<r \ \text{ holds for } \ \rho\le\rho_0.\tag{24}$$

Let $\Gamma_1\subset\subset\Gamma_{\xi_0}$. Then, with a suitable $c\in(0,1)$,

$$\eta\in\Gamma_1,\,|\eta|>1 \text{ and } |\eta-\xi|\le c|\eta|\Rightarrow\xi\in\Gamma_{\xi_0};\quad |\eta-\xi|\le c|\eta|\Rightarrow|\eta|\le(1-c)^{-1}|\xi|.\tag{25}$$

Let $y^k\in B_{r_0}(x_0^k)$, $\eta\in\Gamma_1$. Integrals which will appear below are considered as oscillatory integrals. We have

$$|\mathrm{DS}_{h^k,e^k}f(y^k,\eta)|=\left|\int_{\mathbb{R}^k}\int_{\mathbb{R}^n}\mathrm{DS}_{g^k,e^k}f(x^k,\xi)\mathrm{DS}_{h^k,e^k}\gamma^k(y^k-x^k,\eta-\xi)\mathrm{d}\xi\mathrm{d}x^k\right|.$$

Consider

$$J_1=\int_{\mathbb{R}^n}\mathrm{DS}_{g^k,e^k}f(x^k,\eta-\xi)\mathrm{d}\xi \text{ and } J_2=\int_{\mathbb{R}^n}\mathrm{DS}_{h^k,e^k}\gamma^k(y^k-x^k,\xi)\mathrm{d}\xi.$$

Then, by the use of partial integration, we have, in the oscillatory sense, (with the assumption that $n$ is odd),

$$J_1=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(t)}{(1+|2\pi t|^2)^{(n+1)/2}}\overline{g^k(t^k-x^k)}(1-\Delta_\xi)^{(n+1)/2}\mathrm{e}^{-2\pi it\cdot(\eta-\xi)}\mathrm{d}t\mathrm{d}\xi$$

(for $n$ even, we take $n+2$ instead of $n+1$). This integral still diverges with respect to $\xi$, while $J_2$ converges, since

$$\begin{aligned}J_2&=\int_{\mathbb{R}^n}\int_{B_{\rho_1}(0^k)}\frac{\gamma^k(p^k)\overline{h^k(p^k-(y^k-x^k))}}{(1+|2\pi\xi|^2)^{s/2}}(1-\Delta_p)^{s/2}\mathrm{e}^{-2\pi ip\cdot\xi}\mathrm{d}p\mathrm{d}\xi\\&=\int_{\mathbb{R}^n}\int_{B_{\rho_1}(0^k)}(1-\Delta_p)^{s/2}\frac{\gamma^k(p^k)\overline{h^k(p^k-(y^k-x^k))}}{(1+|2\pi\xi|^2)^{s/2}}\mathrm{e}^{-2\pi ip\cdot\xi}\mathrm{d}p\mathrm{d}\xi.\end{aligned}$$

Rewrite

$$|\mathrm{DS}_{h^k,e^k}f(y^k,\eta)|=\int_{\mathbb{R}^k}\left|\left(\int_{|\eta-\xi|\le c|\eta|}+\int_{|\eta-\xi|\ge c|\eta|}\right)(\ldots)\mathrm{d}\xi\right|\mathrm{d}x^k=I_1+I_2.$$

Then,

$$I_1\le\int_{\mathbb{R}^k}\left(\sup_{|\eta-\xi|\le c|\eta|}|\mathrm{DS}_{g^k,e^k}f(x^k,\eta-\xi)\int_{|\eta-\xi|\le c|\eta|}|\mathrm{DS}_{h^k,e^k}\gamma(y^k-x^k,\xi)|\mathrm{d}\xi\right)\mathrm{d}x^k.$$

Now, we use (23), (24) and $(1+|\eta|^2)^{N/2}\le C(1+|\xi|^2)^{N/2}$, for $|\xi|\ge(1-c)|\eta|$. This implies

$$\begin{aligned}\sup_{y^k\in B_{r_0}(x_0^k),\ \eta\in\Gamma_1}(1+|\eta|^2)^{N/2}I_1\le\int_{B_r(x_0^k)}&\left(\sup_{\xi\in\Gamma_{\xi_0}}|\mathrm{DS}_{g^k,e^k}f(x^k,\xi)|(1+|\xi|^2)^{N/2}\right.\\&\left.\times\int_{|\xi|\ge(1-c)|\eta|}|\mathrm{DS}_{h^k,e^k}\gamma^k(y^k-x^k,\xi)|\mathrm{d}\xi\right)\mathrm{d}x^k.\end{aligned}$$

Now by the finiteness of $J_2$, we obtain that $I_1$ satisfies the necessary estimate of (22). Let us consider $I_2$.

$$I_2\le\int_{\mathbb{R}^k}\left|\int_{|\xi|\ge c|\eta|}\mathrm{DS}_{g^k,e^k}f(x^k,\eta-\xi)\mathrm{DS}_{h^k,e^k}\gamma^k(y^k-x^k,\xi)\mathrm{d}\xi\right|\mathrm{d}x^k.$$

Let $K = \{\xi : |\xi| \geq c|\eta|\}$. Denote by $\widetilde{\kappa}_d$, $0 < d < 1$, the characteristic function of $K_d = \bigcup_{\xi \in K} L_d(\xi)$, that is, $K_d$ is open $d$-neighborhood of $K$. Then, put

$$\kappa_\eta = \widetilde{\kappa}_d * \varphi_d,$$

where $\varphi_d = \frac{1}{d^n}\varphi(\cdot/d)$, $\varphi \in \mathscr{D}(\mathbb{R}^n)$ is nonnegative, supported by the ball $B_1(0)$ and equals $1/2$ on $B_{1/2}(0)$. This construction implies that $\kappa_\eta$ equals one on $K$, is supported by $K_{2d}$. Moreover, all the derivatives of $\kappa_\eta$ are bounded. Assume that $n$ is odd and that $s$ is even and sufficiently large. Then,

$$\sup_{y^k \in B_{r/2}(x_0^k),\, \eta \in \Gamma_1} I_2 \leq C \int_{\mathbb{R}^k} \left| \int_{\mathbb{R}^n} \kappa_\eta(\xi) \mathrm{DS}_{g^k, e^k} f(x^k, \eta - \xi) \mathrm{DS}_{h^k, e^k} \gamma^k(y^k - x^k, \xi) \mathrm{d}\xi \right| \mathrm{d}x^k$$
$$\leq C \int_{\mathbb{R}^k} \left| \int_{\mathbb{R}^n_\xi} \left( \int_{\mathbb{R}^n_t} \frac{f(t)}{(1 + |2\pi t|^2)^{(n+1)/2}} \overline{g^k(t^k - x^k)} e^{-2\pi i t \cdot (\eta - \xi)} \mathrm{d}t \right) (1 - \Delta_\xi)^{(n+1)/2} \right.$$
$$\left. \left( \frac{\kappa_\eta(\xi)}{(1 + |2\pi\xi|^2)^{s/2}} \int_{\mathbb{R}^n_p} \gamma^k(p^k) \overline{h^k(p^k - (y^k - x^k))} (1 - \Delta_p)^{s/2} \mathrm{e}^{-2\pi i p \cdot \xi} \mathrm{d}p \right) \mathrm{d}\xi \right| \mathrm{d}x^k.$$

Then, for every $y^k \in B_{r_0}(x_0^k)$ and $\eta \in \Gamma_1$, choosing $s > N + n$ (and being even) as well as using the Petree inequality, we see that all the integrals on the right-hand side of

$$(1 + |\eta|^2)^{N/2} I_2 \leq C \int_{\mathbb{R}^k} \int_{\mathbb{R}^n_\xi} \left( \int_{\mathbb{R}^n_t} \frac{|f(t)|}{(1 + |2\pi t|^2)^{(n+1)/2}} |\overline{g^k(t^k - x^k)}| \mathrm{d}t \right.$$
$$\left( \frac{(1 + |\xi|^2)^{N/2}}{(1 + |\eta - \xi|^2)^{N/2}} (1 - \Delta_\xi)^{(n+1)/2} \left( \frac{\kappa_\eta(\xi)}{(1 + |2\pi\xi|^2)^{s/2}} \right) \right.$$
$$\left. \int_{\mathbb{R}^n_p} |(1 - \Delta_p)^{s/2} \left( \gamma^k(p^k) \overline{h^k(p^k - (y^k - x^k))} \right) | dp \right) \mathrm{d}\xi \mathrm{d}x^k$$

are finite. This completes the proof of the theorem. □

*Remark 2* If supp $g^k \subset B_a(0^k)$, then we see that (22) shows the behavior of $f(t)$, $t \in Pr_k^{-1}(B_{a+r}(x_0^k))$ in the direction of $\xi_0$.

**Corollary 1** *Let $g^k \in (\mathscr{S}(\mathbb{R}))^k$, supported by a ball $B_a(0^k)$, have synthesis window $\gamma^k$ supported by $B_{\rho_1}(0^k)$, $\rho_1 \leq a$. Assume that in (22) we have $2r$ instead of $r$, that is,*

$$\sup_{x^k \in L_{2r}(x_0^k),\, \xi \in \Gamma_{\xi_0}} |DS_{g^k, e^k} f(x^k, \xi)| \leq C_N (1 + |\xi|^2)^{-N/2}. \tag{26}$$

*Moreover, assume that $a < r$. Then, for any $h^k \in D(\mathbb{R}))^k$ with support $B_\rho(0^k)$, $\rho < a$, there exists $r_0$ and $\Gamma_1 \subset\subset \Gamma_{\xi_0}$ such that (26) holds for $DS_{h^k, e^k} f(y^k, \eta)$ with the supremum over $y^k \in B_{r_0}(x_0^k)$ and $\eta \in \Gamma_1$.*

*Proof* With the notation of Theorem 4, we have, similarly as in (23),

$$|x^k - x_0^k| \leq \rho + \rho_1 + r_0 < \rho + a - r_0 + r_0 = a + \rho < 2r.$$

This implies $|x^k - x_0^k| < 2r$, so that the supremum in the estimate of $I_1$ holds. In the same way as in Theorem 4, we perform the proof. □

**Theorem 5** *If* $(P_{e^k, x^k}, \xi)$ *is a* $k$*-directional regular point of* $f \in \mathscr{S}'(\mathbb{R}^n)$ *for every* $\xi \in \mathbb{R}^n \setminus \{0\}$*, then* $f \in \mathscr{E}(\mathbb{R}^n)$.

*Proof* Assume that supp $g^k \subset B_\rho(0^k)$. Let $x_0^k \in \mathbb{R}^k$. For every $\xi \in \mathbb{S}^{n-1}$ there exist a ball $L_r^\xi(x_0^k)$ and a cone $\Gamma_\xi$ such that (22) holds. As in the classical theory, the compactness of $\mathbb{S}^{n-1}$ implies that there exists $r > 0$ such that for every $N > 0$ there exists $C_N > 0$ such that

$$\sup_{x^k \in L_r(x_0^k)} |\mathscr{F}(f(t)\overline{g_1(t_1 - x_1)} \cdots \overline{g_k(t_k - x_k)})(\xi)| \leq C_N (1 + |\xi|^2)^{-N/2}, \quad \xi \in \mathbb{R}^n.$$

Thus, $f(t)\overline{g^k(t^k - x^k)} \in \mathscr{E}(\mathbb{R}^n)$ for every $x^k \in L_r(x_0^k)$. Since $|t^k - x^k| < \rho$, we see that $t$ must lie in some $Pr_k^{-1}(L_{r+\rho}(x_0^k))$. Thus, for every point of $\mathbb{R}^n$ there exists an open set around it where $f$ is smooth. This completes the proof of the theorem. □

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