Bull. Malays. Math. Sci. Soc. (2019) 42:2091–2104
https://doi.org/10.1007/s40840-017-0593-6

BULLETIN of the MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Characterizations of Finite σ-Nilpotent and σ-Quasinilpotent Groups

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Received: 12 March 2017 / Revised: 7 December 2017 / Published online: 18 December 2017

Abstract Let $\sigma = \{\sigma_i | i \in I\}$ be a partition of the set $\mathbb{P}$ of all primes and G a finite group. A chief factor H/K of G is said to be σ-*central* (in G) if the semidirect product $(H/K) \rtimes (G/C_G(H/K))$ is a σ_i-group for some $i = i(H/K)$; otherwise, it is called σ-*eccentric*. We say that G is: σ-*nilpotent* if every chief factor of G is σ-central; σ-*quasinilpotent* if for every σ-eccentric chief factor H/K of G, every automorphism of H/K induced by an element of G is inner. In this paper, we study properties of σ-nilpotent and σ-quasinilpotent subgroups of finite groups.

Keywords Finite group · σ-Central chief factor · σ-Nilpotent group · σ-Quasinilpotent group · σ-Hypercenter

Mathematics Subject Classification 20D10 · 20D15 · 20D30

Communicated by Kar Ping Shum.

Research is supported by an NNSF Grant of China (Grant No. 11401264) and a TAPP of Jiangsu Higher Education Institutions (PPZY 2015A013).

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1 Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

In what follows, $\sigma = \{\sigma_i | i \in I\}$ is some partition of $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. We say that G is *σ-primary* [1] provided it is a σ_i-group for some i. A chief factor H/K of G is said to be *σ-central* (in G) if the semidirect product $(H/K) \rtimes (G/C_G(H/K))$ is σ-primary; otherwise, it is called *σ-eccentric*. A normal subgroup E of G is said to be *σ-hypercentral* (in G) if either $E = 1$ or every chief factor of G below E is σ-central in G.

The group G is called: *σ-soluble* [1] if every chief factor of G is σ-primary; *σ-decomposable* (Shemetkov [2]) or *σ-nilpotent* (Guo and Skiba [3]) if $G = G_1 \times \cdots \times G_n$ for some σ-primary groups $G_1, \ldots, G_n$.

In fact, σ-nilpotent groups are exactly the groups whose chief factors are σ-central [1], and such groups have proved to be very useful in the formation theory (see [4,5] and the books [6, Ch. 6], [2, Ch. IV]). In the recent years, the σ-nilpotent groups have found new and to some extent unexpected applications in the theories of permutable and generalized subnormal subgroups (see, for example, the recent papers [1,3,7–10] and the survey [11]).

In this paper, we consider the following generalization of σ-nilpotency.

Definition 1.1 We say that G is *σ-quasinilpotent* if for every σ-eccentric chief factor H/K of G, every automorphism of H/K induced by an element of G is inner (cf. [12, Ch.X, Definition 13.2]).

We say that G is *σ-semisimple* if either $G = 1$ or $G = A_1 \times \cdots \times A_t$ is the direct product of simple non-σ-primary groups $A_1, \ldots, A_t$.

Example 1.2 Let $G = (A_5 \wr A_5) \times (A_7 \times A_{11})$ and $\sigma = \{\{2, 3, 5\}, \{2, 3, 5\}'\}$. Then G is σ-quasinilpotent but G is not σ-nilpotent. The group $A_7 \times A_{11}$ is σ-semisimple.

Let $Z_\sigma(G)$ be the product of all normal σ-hypercentral subgroups of G. It is not difficult to show (see Proposition 2.5(i) below) that $Z_\sigma(G)$ is also σ-hypercentral in G. We call the subgroup $Z_\sigma(G)$ the *σ-hypercenter* of G.

The product of all normal σ-nilpotent (respectively σ-quasinilpotent) subgroups of G is said to be the *σ-Fitting subgroup* [1] (respectively the *generalized σ-Fitting subgroup*) of G and denoted by $F_\sigma(G)$ (respectively by $F^*_\sigma(G)$).

Note that the classical case, when $\sigma = \{\{2\}, \{3\}, \ldots\}$, a chief factor H/K of G is central in G (that is, $C_G(H/K) = G$) if and only if it is σ-hypercentral in G. Thus in this case the subgroups $Z_\sigma(G)$, $F_\sigma(G)$ and $F^*_\sigma(G)$ coincide with $Z_\infty(G)$, $F(G)$ and $F^*(G)$, respectively.

In this paper, we study the influence of the subgroups $Z_\sigma(G)$, $F_\sigma(G)$ and $F^*_\sigma(G)$ the structure of G. In particular, using such subgroups, we give some characterizations of σ-nilpotent and σ-quasinilpotent groups.

A set $\mathcal{H}$ of subgroups of G is said to be a *complete Hall σ-set* of G [11] if every member $\neq 1$ of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some $\sigma_i \in \sigma$ and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every i such that $\sigma_i \cap \pi(G) \neq \emptyset$.

A subgroup H of G is said to be a *maximal σ-nilpotent subgroup* of G if H is σ-nilpotent but every subgroup E of G such that $H < E$ is not σ-nilpotent.

In Sect. 2, we study some properties of the subgroup $Z_\sigma(G)$. In particular, we prove in the section the following

Theorem A (i) *The subgroup $Z_\sigma(G)$ coincides with the intersection of all maximal σ-nilpotent subgroups of G.*
(ii) *If G possesses a complete Hall σ-set $\mathcal{H} = \{H_1, \ldots, H_t\}$, then*

$$Z_\sigma(G) = \bigcap_{x \in G} (N_G(H_1^x) \cap \cdots \cap N_G(H_t^x)).$$

G is said to be *π-decomposable* if $G = O_\pi(G) \times O_{\pi'}(G)$. In the case when $\sigma = \{\pi, \pi'\}$, we get from Theorem A the following result.

Corollary 1.3 *Suppose that G possesses a Hall π-subgroup and a Hall π'-subgroup. Then the intersection of all maximal π-decomposable subgroups coincides with the intersection of the normalizers of all Hall π-subgroups and all Hall π'-subgroups of G.*

In the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$ we get from Theorem A the following well-know results.

Corollary 1.4 (Baer) *The hypercenter $Z_\infty(G)$ of G coincides with the intersection of all maximal nilpotent subgroups of G.*

Corollary 1.5 (Baer) *The hypercenter $Z_\infty(G)$ of G coincides with the intersection of the normalizers of all Sylow subgroups of G.*

In Sect. 3, we obtain the following characterization of σ-quasinilpotent groups.

Theorem B *The following are equivalent:*

(i) *G is σ-quasinilpotent.*
(ii) *$G/Z_\sigma(G)$ is σ-semisimple.*
(iii) *$G/F_\sigma(G)$ is σ-semisimple and $G = F_\sigma(G)C_G(F_\sigma(G))$.*

Corollary 1.6 *G is quasinilpotent if and only if $G/F(G)$ is semisimple and $G = F(G)C_G(F(G))$.*

Corollary 1.7 (See Theorem 13.6 in [12, Ch.X]) *G is quasinilpotent if and only if $G/Z_\infty(G)$ is semisimple.*

A *formation* is a class $\mathfrak{F}$ of groups which is closed under taking subdirect products and homomorphic images. The formation $\mathfrak{F}$ is said to be: *hereditary* if $H \in \mathfrak{F}$ whenever $H \leq G \in \mathfrak{F}$, *(solubly) saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(N) \in \mathfrak{F}$ for some (soluble) normal subgroup N of G; a *Fitting formation* if $\mathfrak{F}$ is closed under taking normal subgroups and products of normal subgroups.

As another application of Theorem B, we prove the following result.

Theorem C *The class $\mathfrak{N}_\sigma^*$, of all σ-quasinilpotent groups, is a solubly saturated Fitting formation.*

Corollary 1.8 (Shemetkov [13]) *The class $\mathfrak{N}^*$, of all quasinilpotent groups, is a solubly saturated formation.*

Corollary 1.9 (See Lemma 13.4 and Corollary 13.11 in [12, Ch.X]) *The class $\mathfrak{N}^*$ is a Fitting formation.*

Remark 1.10 Let $\sigma = \{\sigma_1, \sigma_2, \ldots\}$ be any partition of $\mathbb{P}$ with $|\sigma| > 1$. We show that the formation $\mathfrak{N}_\sigma^*$ is not saturated. We can assume without loss of generality that $2 \in \sigma_1$. Let q be the largest prime in σ_2, and let p be a prime such that $p = q$ if $q > 3$ and p is any odd prime in σ_1 in the case when $q = 3$. Finally, let A_p be the alternating group of degree p. Then A_p is a simple non-σ-primary group.

Let $G = V \rtimes A_p$, where V is a projective envelope of a trivial $\mathbb{F}_p A_p$-module. Let C be the intersection of the centralizers in A_p of all chief factors of G below V. Then $\Phi(G) \cap V = \text{Rad}(V)$ by Lemma B.3.14 in [14], and $C = O_{p',p}(A_p) = 1$ by Theorem VII.14.6 in [15]. Hence, since $V/\text{Rad}(V) = V/\Phi(G) \cap V$ is a central chief factor of G (that is, $C_G(V/\text{Rad}(V)) = G$) by [14, B, 4.8], G has a Frattini chief factor K/L (that is, $K/L \leq \Phi(G/L)$) such that $C_G(K/L) = V$ and for every chief factor M/N of G between K and V we have $C_G(M/N) = G$. Then G/K is σ-quasinilpotent by Theorem B. On the other hand, Theorem B implies that G/L is not σ-quasinilpotent. Thus, the formation $\mathfrak{N}_\sigma^*$ is not saturated.

Finally, being based on Theorems B and C, we prove also the following

Theorem D *If G/E is σ-nilpotent and every cyclic subgroup of $F_\sigma^*(E)$ of prime order or order 4 is contained in $Z_\sigma(G)$, then G is σ-nilpotent.*

Corollary 1.11 (Derr et al. [16]) *If G/E is nilpotent and every cyclic subgroup of E of prime order or order 4 is contained in the hypercenter $Z_\infty(G)$ of G, then G is nilpotent.*

Corollary 1.12 (N. Ito) *If every cyclic subgroup of G of prime order or order 4 is contained in the center $Z(G)$ of G, then G is nilpotent.*

2 Proof of Theorem A

The following lemma is evident.

Lemma 2.1 *If H/K and T/L are G-isomorphic chief factors of G, then*

$$(H/K) \rtimes (G/C_G(H/K)) \simeq (T/L) \rtimes (G/C_G(T/L)).$$

Lemma 2.2 (see Proposition 2.3 in [1]) *The following are equivalent:*

(i) *G is σ-nilpotent.*
(ii) *G has a complete Hall σ-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ such that $G = H_1 \times \cdots \times H_t$.*
(iii) *Every chief factor of G is σ-central in G.*

Lemma 2.3 (See Corollary 2.4 and Lemma 2.5 in [1]) *The class $\mathfrak{N}_\sigma$, of all σ-nilpotent groups, is a hereditary saturated Fitting formation.*

Lemma 2.4 *Let N be a normal σ_i-subgroup of G. Then $N \leq Z_\sigma(G)$ if and only if $O^{\sigma_i}(G) \leq C_G(N)$.*

Proof If $O^{\sigma_i}(G) \leq C_G(N)$, then for every chief factor H/K of G below N both groups H/K and $G/C_G(H/K)$ are σ_i-groups since $G/O^{\sigma_i}(G)$ is a σ_i-group. Hence, $(H/K) \rtimes (G/C_G(H/K))$ is σ-primary. Thus $N \leq Z_\sigma(G)$.

Now assume that $N \leq Z_\sigma(G)$. Let $1 = Z_0 < Z_1 < \ldots < Z_t = N$ be a chief series of G below N and $C_i = C_G(Z_i/Z_{i-1})$. Let $C = C_1 \cap \cdots \cap C_t$. Then G/C is a σ_i-group. On the other hand, $C/C_G(N) \simeq A \leq \mathrm{Aut}(N)$ stabilizes the series $1 = Z_0 < Z_1 < \ldots < Z_t = N$, so $C/C_G(N)$ is a $\pi(N)$-group by Theorem 0.1 in [17]. Hence, $G/C_G(N)$ is a σ_i-group and so $O^{\sigma_i}(G) \leq C_G(N)$.

The lemma is proved.

We write $\sigma(G) = \{\sigma_i | \sigma_i \cap \pi(G) \neq \emptyset\}$, and we say that G is a Π-*group* provided $\sigma(G) \subseteq \Pi \subseteq \sigma$.

Proposition 2.5 *Let $Z = Z_\sigma(G)$. Let A, B and N be subgroups of G, where N is normal in G.*

(i) *Z is σ-hypercentral in G.*
(ii) *$Z_\sigma(A)N/N \leq Z_\sigma(AN/N)$.*
(iii) *$Z_\sigma(B) \cap A \leq Z_\sigma(B \cap A)$.*
(iv) *If $N \leq Z$ and N is a Π-group, then N is σ-nilpotent and $G/C_G(N)$ is a σ-nilpotent Π-group.*
(v) *If G/Z is σ-nilpotent, then G is also σ-nilpotent.*
(vi) *If $N \leq Z$, then $Z/N = Z_\sigma(G/N)$.*
(vii) *If A is σ-nilpotent, then ZA is also σ-nilpotent. Hence, Z is contained in each maximal σ-nilpotent subgroup of G. Moreover, if A is a Hall σ_i-subgroup of G, for some $i \in I$, then $Z \leq N_G(A)$.*
(viii) *If $G = A \times B$, then $Z = Z_\sigma(A) \times Z_\sigma(B)$.*

Proof (i) It is enough to consider the case when $Z = A_1A_2$, where A_1 and A_2 are normal σ-hypercentral subgroups of G. Moreover, in view of the Jordan–Hölder theorem, it is enough to show that if $A_1 \leq K < H \leq A_1A_2$, then H/K is σ-central. But in this case we have $H = A_1(H \cap A_2)$, where evidently $H \cap A_2 \nleq K$, so we have the G-isomorphism $(H \cap A_2)/(K \cap A_2) \simeq (H \cap A_2)K/K = H/K$, and hence H/K is σ-central in G by Lemma 2.1.

(ii) First assume that $A = G$, and let H/K be a chief factor of G such that $N \leq K < H \leq NZ$. Then H/K is G-isomorphic to the chief factor $(H \cap Z)/(K \cap Z)$ of G below Z. Therefore, H/K is σ-central in G by Assertion (i) and Lemma 2.1. Consequently, $ZN/N \leq Z_\sigma(G/N)$.

Now let A be any subgroup of G, and let $f : A/A\cap N \to AN/N$ be the canonical isomorphism from $A/A\cap N$ onto AN/N. Then $f(Z_\sigma(A/A\cap N)) = Z_\sigma(AN/N)$ and

$$f(Z_\sigma(A)(A \cap N)/(A \cap N)) = Z_\sigma(A)N/N.$$

Hence, in view of the preceding paragraph, we have

$$Z_\sigma(A)(A \cap N)/(A \cap N) \le Z_\sigma(A/A \cap N).$$

Hence, $Z_\sigma(A)N/N \le Z_\sigma(AN/N)$.

(iii) First assume that $B = G$, and let $1 = Z_0 < Z_1 < \ldots < Z_t = Z$ be a chief series of G below Z and $C_i = C_G(Z_i/Z_{i-1})$. Now consider the series

$$1 = Z_0 \cap A \le Z_1 \cap A \le \ldots \le Z_t \cap A = Z \cap A.$$

We can assume without loss of generality that this series is a chief series of A below $Z \cap A$.

Let $i \in \{1, \ldots, t\}$. Then, by Assertion (i), Z_i/Z_{i-1} is σ-central in G, $(Z_i/Z_{i-1}) \rtimes (G/C_i)$ is a σ_k-group say. Hence, $(Z_i \cap A)/(Z_{i-1} \cap A)$ is a σ_k-group. On the other hand, $A/A \cap C_i \simeq C_i A/C_i$ is a σ_k-group and

$$A \cap C_i \le C_A((Z_i \cap A)/(Z_{i-1} \cap A)).$$

Thus $(Z_i \cap A)/(Z_{i-1} \cap A)$ is σ-central in A. Therefore, in view of the Jordan–Hölder theorem for the chief series, we have $Z \cap A \le Z_\sigma(A)$.

Now assume that B is any subgroup of G. Then, in view of the preceding paragraph, we have

$$Z_\sigma(B) \cap A = Z_\sigma(B) \cap (B \cap A) \le Z_\sigma(B \cap A).$$

(iv) By Assertion (iii) and Lemma 2.2, N is σ-nilpotent, and it has a complete Hall σ-set $\{H_1, \ldots, H_t\}$ such that $N = H_1 \times \cdots \times H_t$. Then

$$C_G(N) = C_G(H_1) \cap \cdots \cap C_G(H_t).$$

It is clear that $H_1, \ldots, H_t$ are normal in G. We can assume without loss of generality that H_i is a σ_i-group. Then , by Assertion (i) and Lemma 2.4, $G/C_G(H_i)$ is a σ_i-group. Hence,

$$G/C_G(N) = G/(C_G(H_1) \cap \cdots \cap C_G(H_t))$$

is a σ-nilpotent Π-group.

(v), (vi) These assertions are corollaries of Assertion (i) and the Jordan–Hölder theorem.

(vii) Since A is σ-nilpotent, $ZA/Z \simeq A/A \cap Z$ is σ-nilpotent by Lemma 2.3. On the other hand, $Z \le Z_\sigma(ZA)$ by Assertion (iii). Hence, ZA is σ-nilpotent by Assertion (v).

Finally, let A be a Hall σ_i-subgroup of G. Then A is σ-nilpotent and so ZA is also σ-nilpotent. Therefore, $Z \le N_G(A)$ by Lemma 2.2.

(viii) Let $Z_1 = Z_\sigma(A)$ and $Z_2 = Z_\sigma(B)$. Since Z_1 is characteristic in A, it is normal in G.
First assume that $Z_1 \neq 1$ and let R be a minimal normal subgroup of G contained in Z_1. Then R is σ-primary, R is a σ_i-group say, by Assertion (iv). Hence, $A/C_A(R)$ is a σ_i-group by Lemma 2.4. But $C_G(R) = B(C_G(R) \cap A) = BC_A(R)$, so

$$G/C_G(R) = AB/C_A(R)B \simeq A/(A \cap C_A(R)B) = A/C_A(R)(A \cap B) = A/C_A(R)$$

is a σ_i-group, and hence R is σ-central in G. Then $R \leq Z_\sigma(G)$, so $Z_\sigma(G)/R = Z_\sigma(G/R)$ by Assertion (vi). On the other hand, we have $Z_1/R = Z_\sigma(A/R)$ and $Z_2R/R = Z_\sigma(BR/R)$, so by induction we have

$$\begin{aligned} Z_\sigma(G/R) &= Z_\sigma((A/R) \times (BR/R)) = Z_\sigma(A/R) \times Z_\sigma(BR/R) \\ &= (Z_1/R) \times (Z_2R/R) = Z_1Z_2/R = Z/R. \end{aligned}$$

Hence $Z = Z_1 \times Z_2$.

Finally, suppose that $Z_1 = 1 = Z_2$. Assume that $Z_\sigma(G) \neq 1$ and let R be a minimal normal subgroup of G contained in $Z_\sigma(G)$. Then, in view of Assertions (i) and (iii), $R \cap A = 1 = R \cap B$ and hence $G = A \times B \leq C_G(R)$. Thus $R \leq Z(G) = Z(A) \times Z(B) = 1$, a contradiction. Hence we have (viii).

The proposition is proved.

Temporarily, we write $I_\sigma(A)$ to denote the intersection of all maximal σ-nilpotent subgroups of a group A; if A possesses a complete Hall σ-set $\mathcal{H} = \{H_1, \dots, H_t\}$, then we use $I_\mathcal{H}(A)$ to denote the intersection

$$\bigcap_{x \in A} (N_A(H_1^x) \cap \dots \cap N_A(H_t^x)).$$

Proof of Theorem A Let $Z = Z_\sigma(G)$. (i) Suppose that this is false and let G be a counterexample of minimal order. Let $I = I_\sigma(G)$. Then $Z < I$ by Proposition 2.5(vii) and the choice of G, so $I \neq 1$ and G is not σ-nilpotent. Let N be a minimal normal subgroup of G and let L be a minimal normal subgroup of G contained in I. Then L is a σ_i-group for some $i \in I$.

(1) $IN/N \leq I_\sigma(G/N)$.

Let U/N be a maximal σ-nilpotent subgroup of G/N, and let V be a minimal supplement to N in U. Then $V \cap N \leq \Phi(V)$ and, by Lemma 2.3, $V/V \cap N \simeq VN/N = U/N$ is σ-nilpotent. Hence V is σ-nilpotent by Lemma 2.3. Let U_0 be a maximal σ-nilpotent subgroup of G such that $V \leq U_0$. Then $U_0N/N \simeq U_0/U_0 \cap N$ is σ-nilpotent and $U/N \leq U_0N/N$. But then $U/N = U_0N/N$, so $IN/N \leq I_\sigma(G/N)$.

(2) G/I *is not* $\sigma - nilpotent$.

Indeed, suppose that G/I is σ-nilpotent, and let V be a minimal supplement to I in G. Then V is σ-nilpotent, so for a maximal σ-nilpotent subgroup U of G such that $V \leq U$ we have $G = IU = U$, a contradiction.

(3) $IN/N \leq Z_\sigma(G/N) = \mathrm{I}_\sigma(G/N)$.

Indeed, $IN/N \leq I_\sigma(G/N)$ by Claim (1). On the other hand, by the choice of G, $Z_\sigma(G/N) = I_\sigma(G/N)$.

(4) $L \nleq Z$.

Suppose that $L \leq Z$. Then $Z/L = Z_\sigma(G/L)$ by Proposition 2.5(vi), so $I/L \leq I_\sigma(G/L) = Z_\sigma(G/L) = Z/L$ by Claim (3). Hence $I \leq Z$ and so $I = Z$, a contradiction.

(5) *If* $L \leq M < G, then L \leq Z_\sigma(M)$.

Let V be any maximal σ-nilpotent subgroup of M, and let H be a maximal σ-nilpotent subgroup of G such that $V \leq H$. Then $V \leq H \cap M$, where $H \cap M$ is σ-nilpotent by Lemma 2.3, which implies $L \leq V = H \cap M$. Hence $L \leq I_\sigma(M)$. But $|M| < |G|$, so $I_\sigma(M) = Z_\sigma(M)$ by the choice of G. Hence $L \leq Z_\sigma(M)$.

(6) $L = N$ is a unique minimal normal subgroup of G.

Suppose that $L \neq N$. From Claim (3) we deduce that $NL/N \leq Z_\sigma(G/N)$, so from the G-isomorphism $NL/N \simeq L$ and Lemma 2.1 we obtain $L \leq Z$, which contradicts to Claim (4).

(7) $L \nleq \Phi(G)$.

Suppose that $L \leq \Phi(G)$. Let $C = C_G(L)$ and M be any maximal subgroup of G. Then $L \leq M$, so $L \leq Z_\sigma(M)$ by Claim (5). Hence $M/M \cap C$ is a σ_i-group by Lemma 2.4. If $C \nleq M$, then $G/C = CM/C \simeq M/M \cap C$ is a σ_i-group, so L is σ-central in G and hence $L \leq Z$, contrary to Claim (4). Hence $C \leq M$ for all maximal subgroups M of G, so C is nilpotent. Therefore, in view of Claim (6), C is a p-group for some $p \in \sigma_i$ since C is normal in G. Thus M is a σ_i-group. But then G is either a σ_i-group or a group of prime order q for some $q \in \sigma_i'$, so G is σ-nilpotent. This contradiction shows that we have (7).

(8) L is not abelian.

Suppose that L is abelian. Then from Claims (6) and (7) we deduce that $G = L \rtimes M$ for some maximal subgroup M of G and, by [14, Ch.A, 15.6], $C = C_G(L) = L$. Let E be a maximal subgroup of M, $V = LE$. Then, by Claim (5), $L \leq Z_\sigma(V)$, so $E \simeq V/L = V/C_V(L)$ is a σ_i-group by Lemma 2.4. Hence M is either a σ_i-group or a group of prime order, contrary to Claim (2). Hence we have (8).

(9) *If* $L \leq M < G$, *then* M *is a* $\sigma_i - group$.

By Claim (5), $L \leq Z_\sigma(M)$. On the other hand, by Claims (6) and (8), $C_G(L) = 1$. Hence $M \simeq M/1 = M/C_M(L)$ is a σ_i-group by Lemma 2.4.

Final contradiction for (i). Let U be a minimal supplement to L in G. Let V be any maximal subgroup of U. Then $LV \neq G$, so LV is a σ_i-group by Claim (9). Hence V is a σ_i-group. Therefore, every maximal subgroup of U is a σ_i-group. Then U is either a σ_i-group or a group of prime order. Hence U is σ-nilpotent and so $G/L = UL/L \simeq U/U \cap L$ is σ-nilpotent. But then $G/I \simeq (G/L)/(I/L)$ is σ-nilpotent by Lemma 2.3, contrary to Claim (2). Thus Assertion (i) is proved.

(ii) Assume that this is false and let G be a counterexample with minimal order. Let $I = I_{\mathcal{H}}(G)$.

First note that if A is a Hall σ_i-subgroup of G, then $Z \leq N_G(A)$ by Proposition 2.5(vii) and so $Z \leq I$. Thus the choice of G implies that $I \neq 1$. Moreover, since $(N_G(A))^x = N_G(A^x)$, I is a normal σ-nilpotent subgroup of G. Let R be a minimal normal subgroup of G contained in I. Then R is a σ_i-group for some $i \in I$. Therefore, for each $j \neq i$ we have $R \leq C_G(H_j)$ since $R \leq N_G(H_j)$, so $G/C_G(R)$ is a σ_i-group. Hence R is σ-central in G.

It is clear that $\mathcal{H}_0 = \{H_1R/R, \ldots, H_tR/R\}$ is a complete Hall σ-set of G/R. Since

$$N_G(H_i^x)R/R \leq N_{G/R}(H_i^xR/R) = N_{G/R}((H_iR/R)^{xR}),$$

$RI/R \leq I_{\mathcal{H}_0}(G/R)$. The choice of G implies that $RI/R \leq I_{\mathcal{H}_0}(G/R) = Z_\sigma(G/R)$. But since R is σ-central in G, $R \leq Z$ and so $Z/R = Z_\sigma(G/R)$ by Proposition 2.5(vi). Thus $RI/R \leq Z/R$, so $I \leq Z$. Thus $I = Z$, as required.

The theorem is proved.

3 Proof of Theorem B

Lemma 3.1 (i) *If G is a σ-quasinilpotent group and N is a normal subgroup of G, then N and G/N are σ-quasinilpotent.*
(ii) *If G/N and G/L are σ-quasinilpotent, then $G/(N \cap L)$ is also σ-quasinilpotent.*

Proof See the proof of Lemma 13.3 in [12, Ch.X].

Lemma 3.2 *Let N be a minimal normal subgroup of the group G. Then every automorphism of N induced by an element of G is inner if and only if $G = NC_G(N)$.*

Proof See the proof of Lemma 13.4 in [12, Ch.X].

Proof of Theorem B Let $Z = Z_\sigma(G)$.

(i) $\Rightarrow$ (ii) Assume that this is false and let G be a counterexample of minimal order. Then the hypothesis holds for G/Z by Lemma 3.1(i). On the other hand, $Z_\sigma(G/Z) = 1$ by Proposition 2.5(vi). Hence in the case when $Z \neq 1$, $G/Z_\sigma(G)$ is σ-semisimple by the choice of G.
Now assume that $Z = 1$ and let R be any minimal normal subgroup of G. Then $R/1$ is a σ-eccentric chief factor of G, so $G = RC_G(R)$ by Lemma 3.2. Therefore, since $Z(G) \leq Z = 1$, $C_G(R) \neq G$ and hence R is σ-semisimple. Thus $G = R \times C_G(R)$. Therefore, $Z_\sigma(R) \times Z_\sigma(C_G(R)) = Z_\sigma(G) = 1$ by Proposition 2.5(viii). Moreover, the choice of G implies that $C_G(R)$ is σ-semisimple, so $G \simeq G/Z = G/1$ is σ-semisimple and hence Assertion (ii) is true, a contradiction.

(ii) $\Rightarrow$ (iii) First note that $Z \leq F_\sigma(G)$ by Proposition 2.5(iv), so $Z = F_\sigma(G)$ since G/Z is σ-semisimple by hypothesis. But then $G/C_G(F_\sigma(G))$ is σ-nilpotent by Proposition 2.5(iv). Hence $G = F_\sigma(G)C_G(F_\sigma(G))$ since $G/F_\sigma(G) = G/Z$ is σ-semisimple.

(iii) $\Rightarrow$ (i) Let H/K be a chief factor of G. If $F_\sigma(G) \leq K$, then every automorphism of H/K induced by an element of G is inner by Lemma 3.2 since $G/F_\sigma(G)$ is σ-semisimple by hypothesis. Now suppose that $H \leq F_\sigma(G)$. Then

$$C_G(H/K) = C_G(H/K) \cap F_\sigma(G)C_G(F_\sigma(G)) = C_G(F_\sigma(G))C_{F_\sigma(G)}(H/K),$$

so

$$\begin{aligned} G/C_G(H/K) &= F_\sigma(G)C_G(F_\sigma(G))/C_G(F_\sigma(G))C_{F_\sigma(G)}(H/K) \\ &\simeq F_\sigma(G)/F_\sigma(G) \cap C_G(F_\sigma(G))C_{F_\sigma(G)}(H/K) \\ &= F_\sigma(G)/C_{F_\sigma(G)}(H/K)Z(F_\sigma(G)) \\ &\simeq (F_\sigma(G)/C_{F_\sigma(G)}(H/K))/(C_{F_\sigma(G)}(H/K)Z \\ &\quad (F_\sigma(G))/C_{F_\sigma(G)}(H/K)) \end{aligned}$$

is σ-primary by Lemma 2.4. Therefore, H/K is σ-central in G. Now applying the Jordan–Hölder theorem, we get that for every σ-eccentric chief factor H/K of G, every automorphism of H/K induced by an element of G is inner. Hence G is σ-quasinilpotent.

The theorem is proved.

Corollary 3.3 *If a σ-quasinilpotent group $G \neq 1$ is σ-soluble, then $G = O_{\sigma_1}(G) \times \cdots \times O_{\sigma_t}(G)$, where $\{\sigma_1, \ldots, \sigma_t\} = \sigma(G)$.*

Proof This directly follows from Theorem B and Lemma 2.2.

We say that G is σ-*perfect* if $O^{\sigma_i}(G) = G$ for all i.

Corollary 3.4 *Let G be σ-quasinilpotent.*

(i) *If G is σ-perfect, then $Z_\sigma(G) = Z(G)$.*
(ii) *If H is a normal σ-soluble subgroup of G, then $H \leq Z_\sigma(G)$.*

Proof (i) This assertion follows from Theorem B and Proposition 2.5(iv).
(ii) This directly follows from Theorem B.

4 Proof of Theorem C

For any function f of the form

$$f : \mathbb{P} \cup \{0\} \to \{\text{group formations}\}, \tag{*}$$

we put, following [18],

$$\begin{aligned} CF(f) = \{&G \text{ is a group} \mid G/C_G(H/K) \in f(0) \\ &\text{for each non-abelian chief factor } H/K \text{ of } G \\ &\text{and } G/C_G(H/K) \in f(p) \text{ for any abelian } p\text{-chief factor } H/K \text{ of } G\}. \end{aligned}$$

In the paper [18], the following useful fact is proved.

Lemma 4.1 *For any function f of the form (∗), the class $CF(f)$ is a solubly saturated formation.*

Proof of Theorem C Let $\mathfrak{M} = CF(f)$, where $f(p) = \mathfrak{G}_{\sigma_i}$ is the class of all σ_i-groups for all $p \in \sigma_i$, and $f(0) = \mathfrak{N}_\sigma^*$. We show that $\mathfrak{M} = \mathfrak{N}_\sigma^*$. First assume that $\mathfrak{M} \nsubseteq \mathfrak{N}_\sigma^*$ and G be a group of minimal order in $\mathfrak{M} \setminus \mathfrak{N}_\sigma^*$ with a minimal normal subgroup R. Then $G/R \in \mathfrak{M}$ by Lemma 4.1, so G/R is σ-quasinilpotent. Hence R is a unique minimal normal subgroup of G by Lemma 3.1(ii). Therefore, in view of Theorem B, R is not σ-central in G. Hence R is non-abelian. But then $C_G(R) = 1$ and so $G \simeq G/C_G(R) \in f(0) = \mathfrak{N}_\sigma^*$, a contradiction. Thus $\mathfrak{M} \subseteq \mathfrak{N}_\sigma^*$.

Now, assume that $\mathfrak{N}_\sigma^* \nsubseteq \mathfrak{M}$ and G be a group of minimal order in $\mathfrak{N}_\sigma^* \setminus \mathfrak{M}$ with a minimal normal subgroup R. Then $G/R \in \mathfrak{N}_\sigma^*$ by Lemma 3.1(i), so $G/R \in \mathfrak{M}$. Hence R is a unique minimal normal subgroup of G by Lemma 4.1. If R is non-abelian, then $G \simeq G/1 = G/C_G(R) \in f(0) = \mathfrak{N}_\sigma^*$. Hence $G \in \mathfrak{M}$ since $G/R \in \mathfrak{M}$, a contradiction. Hence R is a p-group for some $p \in \sigma_i$, so $R \rtimes (G/C_G(R))$ is a σ_i-group by Theorem B. Therefore, $G/C_G(R) \in f(p)$ and so $G \in \mathfrak{M}$. Thus $\mathfrak{M} = \mathfrak{N}_\sigma^*$. Therefore, $\mathfrak{N}_\sigma^*$ is a solubly saturated formation by Lemma 4.1. Lemma 3.1(i) implies that this formation is normally hereditary.

Therefore, in order to complete the proof of the theorem it is enough to show that if $G = AB$, where A and B are normal σ-quasinilpotent subgroups of G, then G is σ-quasinilpotent. Suppose that this is false and let G be a counterexample of minimal order. Let R be a minimal normal subgroup of G and $C = C_G(R)$. By Lemma 3.1(i), the hypothesis holds for G/R and so the choice of G implies that G/R is σ-quasinilpotent. Therefore, in view of Lemma 3.1(ii), R is a unique minimal normal subgroup of G.

Let $Z_1 = Z_\sigma(A)$ and $Z_2 = Z_\sigma(B)$. If $A \cap B = 1$, then $Z_\sigma(G) = Z_1 \times Z_2$ by Proposition 2.5(viii). On the other hand, A/Z_1 and B/Z_2 are σ-semisimple by Theorem B, so

$$G/Z = (A \times B)/(Z_1 \times Z_2) \simeq (A/Z_1) \times (B/Z_2)$$

is σ-semisimple. Hence G is σ-quasinilpotent by Theorem B. Therefore, $A \cap B \neq 1$, so $R \leq A \cap B$. First assume that R is σ-primary, R is a σ_i-group say. Then by Theorem B, $R \leq Z_1 \cap Z_2$ and so $AC/C \simeq A/A \cap C$ and $BC/C \simeq B/B \cap C$ are σ_i-groups. Hence $G/C = (AC/C)(BC/C)$ is a σ_i-group. Thus R is σ-central in G. Therefore, $R \leq Z_\sigma(G)$ and so $Z_\sigma(G/R) = Z_\sigma(G)/R$ by Proposition 2.5(vi). Thus G is σ-quasinilpotent by Theorem B.

Therefore, R is not σ-primary. Hence R is non-abelian, so $C = 1$. Then $R = R_1 \times \cdots \times R_t$, where $R_1, \ldots, R_t$ are minimal normal subgroups of A. Let $C_i = C_A(R_i)$ $(i = 1, \ldots, t)$. Then $C = 1 = C_1 \cap \cdots \cap C_t$. Since A is σ-quasinilpotent by hypothesis, $A = R_i C_i$ for all $i = 1, \ldots, t$ by Lemma 3.2. Hence

$$\begin{aligned} R = RC &= R_1 \ldots R_t(C_t \cap \cdots \cap C_1) = R_1 \ldots R_{t-1}(R_t C_t \cap C_{t-1} \cap \cdots \cap C_1) \\ &= R_1 \ldots R_{t-1}(A \cap C_{t-1} \cap \cdots \cap C_1) = R_1 \ldots R_{t-1}(C_{t-1} \cap \cdots \cap C_1) \\ &= \cdots = R_1 C_1 = A. \end{aligned}$$

Similarly one can get that $B = R$, so $G = R$ is σ-semisimple. Hence G is $\sigma-$ quasinilpotent.

The theorem is proved. □

5 Proof of Theorem D

Recall that G is said to be a *Schmidt group* if G is not nilpotent but every proper subgroup of G is nilpotent.

Lemma 5.1 (See Proposition 1.6 in [8]) *Let G be σ-soluble. If G is not σ-nilpotent but all proper subgroups of G are σ-nilpotent, then G is a Schmidt group.*

Lemma 5.2 (See [19, Ch.III, 5.2] and [19, Ch.IV, 5.4]) *If G is not p-nilpotent but every proper subgroup of G is p-nilpotent, then G is a p-closed Schmidt group and so $G = P \rtimes Q$, where P is a Sylow p-subgroup of G and Q is a Sylow q-subgroup of G for some primes $p \neq q$. Moreover, P is of exponent p or exponent 4 (if P is a non-abelian 2-group).*

Lemma 5.3 (see [19, Ch.IV, 5.12]) *Let P be a p-group, a a p'-automorphism of P.*

(1) *If $[a, \Omega_2(P)] = 1$, then $a = 1$.*
(2) *If $[a, \Omega_1(P)] = 1$ and either p is odd or P is abelian, then $a = 1$.*

Proof of Theorem D Let $Z = Z_\sigma(G)$ and $F^* = F^*_\sigma(E)$. Let $F = F_\sigma(E)$ and $C = C_G(F)$. It is enough to show that if every cyclic subgroup of F^* of prime order or order 4 is contained in Z, then G is σ-nilpotent. Assume that this is false and let G be a counterexample with $|G| + |E|$ minimal.

(1) $F \neq E. Hence E = G.$

Assume that $F = E$. Then G is σ-soluble since G/E is σ-nilpotent by hypothesis. Let M be any maximal subgroup of G. Then $M/M \cap E \simeq EM/E$ is σ-nilpotent and $M \cap E$ is a normal σ-nilpotent subgroup of M by Lemma 2.3. Hence $F^*_\sigma(M \cap E) = F_\sigma(M \cap E)$ by Corollary 3.3. If A is a cyclic subgroup of $M \cap E$ of prime order or order 4, then $A \leq Z \cap M \leq Z_\sigma(M)$ by Proposition 2.5(iii). Therefore, the hypothesis holds for $(M, M \cap E)$, so the choice of G implies that M is σ-nilpotent. Hence G is a Schmidt group by Lemma 5.1, and so by Lemma 5.2, $G = P \rtimes Q$, where P is a Sylow p-subgroup of G and Q is a Sylow q-subgroup of G for some primes $p \neq q$ dividing $|G|$. Moreover, P is of exponent p or exponent 4 (if P is a non-abelian 2-group). Thus $P \leq Z$, which implies that G is σ-nilpotent by Proposition 2.5(v). This contradiction shows that $F \neq E$. Therefore, since the hypothesis holds for (E, E) by Proposition 2.5(iii), the choice of G and Theorem B imply that $E = G$.

(2) If N is a normal subgroup of G, then the hypothesis holds for (N, N).

Indeed, $F^*_\sigma(N) \leq F^*_\sigma(E)$ by Theorem C since $F^*_\sigma(N)$ is characteristic in N. Hence the hypothesis holds for (N, N) by Proposition 2.5(iii).

(3) $F = F^*$. Hence G/F is not σ-nilpotent.

Assume that $F < F^*$. Then F^* is not σ-soluble by Theorem B. Moreover, the hypothesis holds for (F^*, F^*) by Claim (2), so the choice of G implies that $F^* = G$. Then, by Theorem B, $G = FC$. Hence the choice of G and Lemma 2.3 imply that C is not σ-nilpotent. But the hypothesis holds for (C, C) by Claim (2). Hence $C = G$, so $Z = F = Z(G)$. Since $G = F^*$ is not σ-soluble by Theorem B, G is not p-nilpotent for some odd prime p and so G has a p-closed Schmidt subgroup $H = P \rtimes Q$, where P is of exponent p by Lemma 5.2. Hence $P \leq Z(G) \cap H$ and so H is nilpotent.

This contradiction shows that $F = F^*$. Finally, note that if G/F is a σ-nilpotent, the hypothesis holds for (G, F), so the choice of G and Claim (1) imply that G is σ-nilpotent. This contradiction shows that we have (3).

(4) G/F is a simple non-σ-primary group.

Let $F \leq M < G$, where M is a normal subgroup of G with simple quotient G/M. Then the hypothesis holds for (M, M) by Claim (2). Therefore, $M = F$ by the choice of G.

(5) $Z = Z(G)$. Hence F is nilpotent.

The hypothesis holds for $(C_G(Z), C_G(Z))$ by Claim (2). Assume that $C_G(Z) < G$. Then the choice of G implies that $C_G(Z)$ is σ-nilpotent. But $G/C(Z)$ is σ-nilpotent by Proposition 2.5(iv). Hence G is σ-soluble, contrary to Claim (4). Therefore, $C_G(Z) = G$. Arguing now as in the proof of Claim (1), one can show that F is nilpotent.

(6) $F = Z(G)$.

In view of Claim (5), it is enough to show that for any prime p dividing $|F|$ we have $P \leq Z(G)$, where P is the Sylow p-subgroup of F. Let $\Omega = \Omega_1(P)$ if $p > 2$ and $\Omega = \Omega_2(P)$ if $p = 2$. Then $\Omega(P) \leq Z(G)$ by Claims (1) and (5), so $P \leq Z(G)$ by Lemma 5.3.

Final contradiction From Claims (4) and (6), it follows that $G = F^*$ is σ-quasinilpotent by Theorem B, contrary to Claim (3).

The theorem is proved.

Acknowledgements The authors are deeply grateful to the helpful suggestions of the referees.

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