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Acyclic Edge Coloring of 4-Regular Graphs (II)

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Abstract A proper edge coloring is called acyclic if no bichromatic cycles are produced. It was conjectured that every simple graph G with maximum degree Δ is acyclically edge-$(\Delta + 2)$-colorable. In this paper, combining some known results, we confirm the conjecture for graphs with $\Delta = 4$.

Keywords Acyclic edge coloring · 4-Regular graph · Maximum degree

AMS Subject Classification 05C15

1 Introduction

Only simple graphs are considered in this paper. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *proper edge-k-coloring* is a mapping $c : E(G) \rightarrow \{1, 2, \ldots, k\}$ such that any two adjacent edges receive different colors. The graph G is *edge-k-colorable* if it has an edge-k-coloring. The *chromatic index* $\chi'(G)$ of G is the smallest

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integer k such that G is edge-k-colorable. A proper edge-k-coloring of G is called *acyclic* if there are no bichromatic cycles in G, that is, the union of any two color classes induces a subgraph of G that is a forest. The *acyclic chromatic index* of G, denoted $a'(G)$, is the smallest integer k such that G is acyclically edge-k-colorable.

Let $\Delta = \Delta(G)$ denote the maximum degree of a graph G. By Vizing's theorem [17], $\Delta \leq \chi'(G) \leq \Delta + 1$. So it holds automatically that $a'(G) \geq \chi'(G) \geq \Delta$. Fiamčik [9] and later Alon et al. [2] put forward the following conjecture:

Conjecture 1 *For any graph G, $a'(G) \leq \Delta + 2$.*

Using probabilistic method, Alon et al. [1] proved that $a'(G) \leq 64\Delta$ for any graph G. This upper bound was gradually improved to that $a'(G) \leq 16\Delta$ in [12], that $a'(G) \leq \lceil 9.62(\Delta - 1)\rceil$ in [13], that $a'(G) \leq 4\Delta$ in [8], and that $a'(G) \leq \lceil 3.74(\Delta - 1)\rceil + 1$ in [10]. For the class of subcubic graphs, Conjecture 1 was affirmed to be true, see [3,4]. Other results about this topic can be seen in [6,7,11,14,16].

In 2009, Basavaraju and Chandran [5] showed that if G is a graph with $\Delta = 4$ and $|E(G)| \leq 2|V(G)| - 1$, then $a'(G) \leq 6$. Namely, every non-regular graph of $\Delta = 4$ satisfies Conjecture 1. More recently, Shu et al. [15] extended this result by showing that every 4-regular graph G without 3-cycles is acyclically edge-6-colorable. In this paper, we solve the case of 4-regular graphs having at least one 3-cycle. Hence, combining the previously known results, Conjecture 1 is confirmed for all graphs with $\Delta = 4$.

2 Main Results

Assume that c is a partial acyclic edge-k-coloring of a graph G using the color set $C = \{1, 2, \ldots, k\}$. For a vertex $v \in V(G)$, we use $C(v)$ to denote the set of colors assigned to edges incident to v under c. If the edges of a cycle $ux \cdots vu$ are alternately colored with colors i and j, then we call such cycle an *$(i, j)_{(u,v)}$-cycle*. If the edges of a path $ux \cdots v$ are alternately colored with colors i and j, then we call such path an *$(i, j)_{(u,v)}$-path*. For simplicity, we use $\{e_1, e_2, \ldots, e_m\} \to a$ to express that all edges $e_1, e_2, \ldots, e_m$ are colored or recolored with same color a. In particular, when $m = 1$, we write simply $e_1 \to a$. Moreover, we use $(e_1, e_2, \ldots, e_m)_c = (a_1, a_2, \ldots, a_m)$ to denote that $c(e_i) = a_i$ for $i = 1, 2, \ldots, m$. Let $(e_1, e_2, \ldots, e_n) \to (b_1, b_2, \ldots, b_n)$ denote that e_i is colored or recolored with color b_i for $i = 1, 2, \ldots, n$. Note that b_i and b_j may be same for some $i \neq j$.

For a graph G, let $X = \{v_1, v_2, \ldots, v_j\} \subset V(G)$ and $S = \{e_1, e_2, \ldots, e_k\}$ be an edge subset. We use $(G - X) \cup S$ or $G - \{v_1, v_2, \ldots, v_j\} + \{e_1, e_2, \ldots, e_k\}$ to denote the graph obtained by deleting from G the vertices in X together with all the edges incident with some vertex in X and adding the edges in S together with all the new vertices incident with some edge in S. We write $G - v_1 + \{e_1, e_2, \ldots, e_k\}$ if $j = 1$ or $G - \{v_1, v_2, \ldots, v_j\} + e_1$ if $k = 1$.

Several lemmas below will be frequently used in the proof of the main result.

Lemma 1 ([15]) *Suppose that a graph G has an edge-6-coloring c. Let $P = uv_1v_2 \cdots v_kv_{k+1}$ be a maximal $(a, b)_{(u,v_{k+1})}$-path in G with $c(uv_1) = a$ and $b \notin C(u)$. If $w \notin V(P)$, then there is no $(a, b)_{(u,w)}$-path in G under c.*

Lemma 2 ([3,4]) *If* G *is a graph with* $\Delta \leq 3$*, then* $a'(G) \leq 5$*, and* $a'(G) = 5$ *if and only if* $G \in \{K_4, K_{3,3}\}$.

Lemma 3 ([5]) *If* G *is a graph with* $\Delta = 4$ *that is not 4-regular, then* $a'(G) \leq 6$.

Lemma 4 ([15]) *If* G *is a 4-regular graph without 3-cycles, then* $a'(G) \leq 6$.

Theorem 1 *If* G *is a 4-regular graph, then* $a'(G) \leq 6$.

Proof The proof is proceeded by induction on the number $\sigma(G) = |V(G)| + |E(G)|$. If $\sigma(G) = 15$, that is, $|V(G)| = 5$, then G is the complete graph K_5, and it is easy to show that $a'(G) \leq 6$. Let G be a 4-regular graph with $\sigma(G) \geq 16$, so $|V(G)| \geq 6$. Obviously, we may assume that G is 2-connected by Lemma 3. If G contains no 3-cycles, then $a'(G) \leq 6$ by Lemma 4. So assume that G contains at least one 3-cycle. For any graph H with $\Delta(H) \leq 4$ and $\sigma(H) < \sigma(G)$, by the induction hypothesis or Lemmas 3 and 4, H admits an acyclic edge-6-coloring c using the color set $C = \{1, 2, \ldots, 6\}$. Before constructing an acyclic edge-6-coloring of G, we first prove the following lemma. □

Lemma 5 *Let* $\emptyset \neq X \subset V(G)$ *and put* $S = [X, \overline{X}]$ *where* $\overline{X} = V(G) \setminus X$*. If* $S = \{x_1y_1, x_2y_2, x_3y_3, x_4y_4\}$ *with* $x_i \in X$*,* $y_i \in \overline{X}$*,* $i = 1, 2, 3, 4$*, where* x_1, x_2, x_3, x_4 *are pairwise distinct, but some of* y_i*'s may be identical. Then* $(G - X) \cup S$ *has an acyclic edge-6-coloring* c *using the color set* $C = \{1, 2, \ldots, 6\}$ *such that* $c(x_iy_i) = i$ *for each* $i \in \{1, 2, 3, 4\}$.

Proof Note that for any graph H with $\Delta(H) \leq 4$ and $\sigma(H) < \sigma(G)$, H has an acyclic edge-6-coloring c using $C = \{1, 2, \ldots, 6\}$ by the foregoing discussion. By symmetry, we have to consider the following cases. Let $u \notin V(G)$ be a new vertex.

- If y_1, y_2, y_3, y_4 are identical to a vertex, say v, then we define $H = G - X + uv$ and can assume $c(uv) = 1$ with $2, 3, 4 \notin C(v)$.
- If y_1, y_2, y_3 are identical to a vertex v and $y_4 \neq v$, then we define $H = G - X + \{uv, uy_4\}$ and assume that $(uv, uy_4)_c = (1, 4)$ with $2, 3 \notin C(v)$.
- If y_1, y_2 are identical to a vertex v_1, and y_3, y_4 are identical to a vertex v_4, let $H = G - X + \{uv_1, uv_4\}$ and we can assume that $(uv_1, uv_4)_c = (1, 4)$ with $2 \in C\backslash(C(v_1) \cup \{1, 4\})$, $3 \in C\backslash(C(v_4) \cup \{1, 4, 2\})$.
- If y_1, y_2 are identical to a vertex v, and $y_3 \neq y_4 \neq v$, let $H = G - X + \{uv, uy_3, uy_4\}$ and assume that $(uv, uy_3, uy_4)_c = (1, 3, 4)$ with $2 \in C\backslash(C(v) \cup \{3, 4\})$.
- If y_1, y_2, y_3, y_4 are all distinct, let $H = G - X + \{uy_1, uy_2, uy_3, uy_4\}$ and assume that $(uy_1, uy_2, uy_3, uy_4)_c = (1, 2, 3, 4)$.

Now we only need to let $x_iy_i \to i$ for $i = 1, 2, 3, 4$ for all cases above to complete the proof. □

Let $v \in V(G)$ be a vertex adjacent to v_0, v_1, v_2, v_3. By Lemma 4, we may assume that v lies in a 3-cycle. To obtain an acyclic edge-6-coloring of G, the proof is divided into the following five cases by symmetry.

Case 1 $v_0v_1, v_1v_2, v_2v_3, v_3v_0 \in E(G)$.

For $i \in \{0, 1, 2, 3\}$, let v_i' be the neighbor of v_i different from v, v_{i-1}, v_{i+1}, where all indices are taken modulo 4. Since $|V(G)| \geq 6$, $G \neq K_5$. We only need to consider the following subcases by symmetry:

Case 1.1 $v_1v_3 \in E(G)$ and $v_2v_0 \notin E(G)$.

Take $X = \{v, v_1, v_2, v_3\}$ and $S = \{v_0v_1, v_2v_2', v_0v_3, vv_0\}$, and let $H = (G-X)\cup S$. By Lemma 5, H has an acyclic edge-6-coloring c with $(v_0v_1, v_2v_2', v_0v_3, vv_0)_c = (1, 2, 3, 4)$. To extend c to the whole graph G, we let $(v_1v_3, vv_2, v_1v_2, vv_3, vv_1, v_2v_3) \rightarrow (2, 3, 4, 5, 6, 6)$.

Case 1.2 $v_1v_3, v_2v_0 \notin E(G)$.

Take $X = \{v, v_0, v_1, v_2, v_3\}$ and $S = \{v_1v_1', v_2v_2', v_3v_3', v_0v_0'\}$, and let $H = (G - X)\cup S$. By Lemma 5, H has an acyclic edge-6-coloring c with $(v_1v_1', v_2v_2', v_3v_3', v_0v_0')_c = (1, 2, 3, 4)$. To extend c to the whole graph G, we let $(v_0v_3, vv_3, v_1v_2, vv_1, vv_2, v_0v_1, vv_0, v_2v_3) \rightarrow (1, 2, 3, 4, 5, 5, 6, 6)$.

Case 2 $v_0v_1, v_1v_2, v_2v_3 \in E(G)$ and $v_0v_3 \notin E(G)$.

If $v_1v_3, v_0v_2 \in E(G)$, then since $v_0v_1, v_0v_2, v_2v_3, v_1v_3 \in E(G)$, v_2 lies in four 3-cycles and the proof can be reduced to Case 1. Thus, without loss of generality, assume that $v_0v_2 \notin E(G)$. Let v_2' be the neighbor of v_2 other than v, v_1, v_3.

Case 2.1 $v_1v_3 \in E(G)$.

Let v_3' be the neighbor of v_3 other than v, v_1, v_2. Take $X = \{v, v_1, v_2, v_3\}$ and $S = \{v_0v_1, v_2v_2', v_3v_3', vv_0\}$, and let $H = (G - X) \cup S$. By Lemma 5, H has an acyclic edge-6-coloring c with $(v_0v_1, v_2v_2', v_3v_3', vv_0)_c = (1, 2, 3, 4)$. It suffices to define $(v_2v_3, vv_3, v_1v_2, v_1v_3, vv_2, vv_1) \rightarrow (1, 2, 3, 4, 5, 6)$.

Case 2.2 $v_1v_3 \notin E(G)$

Let v_5 be the neighbor of v_1 other than v, v_0, v_2. There are two possibilities as follows.

Case 2.2.1 $v_2' = v_5$.

By Case 1, we may assume that $v_0v_5, v_3v_5 \notin E(G)$. Let $H = G - \{v, v_1, v_2\} + \{v_0v_5, v_3v_5\}$ and assume that $(v_0v_5, v_3v_5)_c = (1, 2)$. First, let $\{v_1v_5, vv_0\} \rightarrow 1$ and $\{v_2v_5, vv_3\} \rightarrow 2$. Next, if $C(v_0) = \{3, 4\}$ and $C(v_3) = \{5, 6\}$, then let $(v_1v_2, v_2v_3, vv_1, v_0v_1, vv_2) \rightarrow (3, 4, 5, 6, 6)$. Otherwise, w.l.o.g., assume that $3 \notin C(v_3) \cup C(v_0)$ and $4 \notin C(v_0)$. Let $(v_0v_1, v_2v_3, v_1v_2, vv_1, vv_2) \rightarrow (3, 3, 4, 5, 6)$. Obviously, there is neither a $(1, 5)_{(v,v_1)}$-cycle nor a $(2, 6)_{(v,v_2)}$-cycle in G.

Case 2.2.2 $v_2' \neq v_5$.

Let $H = G - \{v, v_1, v_2\} + \{uv_5, uv_2', uv_3, uv_0, v_0v_3\}$ and assume that $(uv_0, uv_3, v_0v_3)_c = (1, 2, 3)$, where u is a new vertex. First, let $v_2v_3 \rightarrow 2$ and $(v_1v_5, v_2v_2') \rightarrow (c(uv_5), c(uv_2'))$.

- Assume that $(uv_5, uv_2')_c = (3, 4)$.

Let $(v_0v_1, vv_1, vv_0, v_1v_2) \to (1, 2, 3, 6)$. If $\{5, 6\}\backslash C(v_3) \neq \emptyset$, say $5 \notin C(v_3)$, let $(vv_2, vv_3) \to (1, 5)$; otherwise, $C(v_3) = \{5, 6\}$, we let $(vv_2, vv_3) \to (5, 1)$.

- Assume that $(uv_5, uv_2')_c = (4, 5)$.

If $4 \notin C(v_0)$, then let $(vv_0, v_1v_2, v_0v_1, vv_3, vv_2, vv_1) \to (1, 1, 3, 3, 4, 6)$. Otherwise, $4 \in C(v_0)$. Then first let $(v_0v_1, vv_1, vv_0) \to (1, 2, 3)$. Next, if $4 \notin C(v_3)$, then let $(vv_3, vv_2, v_1v_2) \to (4, 1, 6)$. Or else, $4 \in C(v_3)$, and similarly assume $5 \in C(v_3) \cap C(v_0)$. It is enough to let $(vv_3, v_1v_2, vv_2) \to (1, 3, 6)$.

Case 3 $v_1v_2, v_2v_3 \in E(G)$ and $v_1v_0, v_3v_0 \notin E(G)$.

If $v_1v_3, v_0v_2 \in E(G)$, then the proof is reduced to Case 2. Thus, assume that $v_1v_3 \notin E(G)$, or $v_0v_2 \notin E(G)$.

Case 3.1 $v_1v_3 \in E(G)$ and $v_0v_2 \notin E(G)$.

Let v_5, v_6, v_7 be the forth neighbor of v_1, v_2, v_3 other than v, v_0, v_1, v_2, v_3, respectively. Take $X = \{v, v_1, v_2, v_3\}$ and $S = \{v_1v_5, v_2v_6, v_3v_7, vv_0\}$, and let $H = (G - X) \cup S$. By Lemma 5, H has an acyclic edge-6-coloring c with $(v_1v_5, v_2v_6, v_3v_7, vv_0)_c = (1, 2, 3, 4)$. To extend c to G, we let $(v_2v_3, vv_3, vv_1, vv_2, v_1v_3, v_1v_2) \to (1, 2, 3, 5, 5, 6)$.

Case 3.2 $v_1v_3 \notin E(G)$ and $v_0v_2 \in E(G)$.

Let $H = G - \{v, v_2\} + \{v_0v_1, v_0v_3, v_1v_3\}$ and assume that $(v_0v_1, v_0v_3, v_1v_3)_c = (1, 2, 3)$ with $4 \in C\backslash(C(v_1) \cup \{1, 2, 3\})$. First, let $(vv_0, vv_3, v_1v_2, vv_1) \to (1, 3, 3, 4)$. If $\{5, 6\}\backslash C(v_3) \neq \emptyset$, say $5 \notin C(v_3)$, let $(v_0v_2, v_2v_3, vv_2) \to (2, 5, 6)$; otherwise, $C(v_3) = \{5, 6\}$, we let $v_2v_3 \to 1$. Then if $\{5, 6\}\backslash C(v_0) \neq \emptyset$, say $5 \notin C(v_0)$, let $(v_0v_2, vv_2) \to (2, 5)$; or else, $C(v_0) = \{5, 6\}$, let $(vv_2, v_0v_2) \to (2, 4)$.

Case 3.3 $v_1v_3, v_0v_2 \notin E(G)$.

Let $v_5 \notin \{v, v_1, v_3, v_0\}$ be the forth neighbor of v_2, and $v_6, v_7 \notin \{v, v_2, v_0, v_3\}$ be the other two neighbors of v_1. Note that v_5, v_6, v_7 are pairwise distinct by Case 2. Let $H = G - \{v, v_2\} + \{v_0v_1, v_1v_5\}$ and suppose that $(v_1v_5, v_0v_1, v_1v_6, v_1v_7)_c = (1, 2, 3, 4)$. Let $(v_2v_5, vv_0) \to (1, 2)$.

Case 3.3.1 $C(v_3) \cap \{1, 2\} \neq \emptyset$, say $1 \in C(v_3)$.

- Assume that $2 \notin C(v_3)$.

Since $\{3, 4\}\backslash C(v_3) \neq \emptyset$ and $\{5, 6\}\backslash C(v_3) \neq \emptyset$, we may assume that $4, 6 \notin C(v_3)$. If $6 \notin C(v_5)$, then let $(v_1v_2, vv_3, vv_2, vv_1, v_2v_3) \to (2, 4, 5, 6, 6)$. If $4 \notin C(v_5)$, then let $(vv_1, v_1v_2, v_2v_3, vv_2, vv_3) \to (1, 2, 4, 5, 6)$. Otherwise, $4, 6 \in C(v_5)$ and $\{3, 5\}\backslash C(v_3) \subseteq C(v_5)$. It follows that $2 \notin C(v_5)$, and hence, let $(vv_2, v_2v_5, vv_3, v_1v_2, vv_1, v_2v_3) \to (1, 2, 4, 5, 6, 6)$.

- Assume that $C(v_3) = \{1, 2\}$.

Assume that $6 \notin C(v_5)$. Let $(v_1v_2, vv_2, v_2v_3) \to (2, 4, 6)$. If $5 \notin C(v_0)$, then let $(vv_1, vv_3) \to (1, 5)$. If $3 \notin C(v_0)$, let $(vv_1, vv_3) \to (5, 3)$. Otherwise, we may assume that $C(v_0) = \{2, 3, 4, 5\}$, and hence, let $(vv_1, vv_3, vv_0) \to (1, 5, 6)$.

Assume that $\{5,6\} \subseteq C(v_5)$, and $\{5,6\} \subseteq C(v_0)$ similarly. If $3, 4 \notin C(v_0) \cup C(v_5)$, then let $(v_1v_2, v_2v_5, vv_3, vv_1, v_2v_3, vv_2) \to (1,3,4,5,5,6)$. Otherwise, we suppose that $C(v_0) = \{2,4,5,6\}$. Let $(vv_1, vv_2, vv_0, vv_3, v_1v_2) \to (1,2,3,5,6)$ and color v_2v_3 with a color in $\{3,4\} \backslash C(v_5)$.

Case 3.3.2 $1, 2 \notin C(v_3)$.

If $\{5,6\} \backslash C(v_3) \neq \emptyset$, say $6 \notin C(v_3)$, then let $(vv_3, v_1v_2, vv_1, v_2v_3) \to (1,2,6,6)$, and color vv_2 with a color in $\{3,4,5\} \backslash C(v_3)$. Otherwise, $C(v_3) = \{5,6\}$, and it suffices to let $(vv_1, v_2v_3, vv_3, vv_2, v_1v_2) \to (1,2,3,4,5)$.

Case 4 $v_0v_3, v_1v_2 \in E(G)$ and $v_0v_1, v_2v_3, v_1v_3, v_0v_2 \notin E(G)$.

Let $V_i = \{v_{i1}, v_{i2}\}$ be the set of other neighbors of v_i for $i \in \{0,1,2,3\}$. By Cases 1–3, we assume that $V_1 \cap V_2 = \emptyset$ and $V_0 \cap V_3 = \emptyset$. Let $H = G - \{v, v_0, v_1, v_2, v_3\} + \{uv_{11}, uv_{12}, uv_{21}, uv_{22}, wv_{31}, wv_{32}, wv_{01}, wv_{02}\}$, where u and w are new vertices added. Assume that $(uv_{11}, uv_{12}, uv_{21}, uv_{22})_c = (1,2,3,4)$, $\{c(wv_{01}), c(wv_{02}), c(wv_{31}), c(wv_{32})\} = \{a, b, c, d\}$. Let $(v_1v_{11}, v_1v_{12}, v_2v_{21}, v_2v_{22}, v_0v_{01}, v_0v_{02}, v_3v_{31}, v_3v_{32}) \to (1,2,3,4,a,b,c,d)$. Since $|\{1,2,3,4\} \cap \{a,b,c,d\}| \geq 2$, we may assume that $1 \in \{a, b\}$. By symmetry, let us handle the following subcases.

Case 4.1 $\{a,b\} = \{1,2\}$ and $\{c,d\} \in \{\{3,4\}, \{3,5\}, \{5,6\}\}$.

If $\{c,d\} \in \{\{3,5\}, \{5,6\}\}$, then let $(vv_2, vv_1, vv_3, vv_0, v_1v_2) \to (1,3,4,5,6)$ and color v_0v_3 with a color in $\{3,6\} \backslash \{c,d\}$. Otherwise, $\{c,d\} = \{3,4\}$. If $1 \notin C(v_{21})$, let $(vv_2, vv_0, v_1v_2, vv_3, vv_1, v_0v_3) \to (1,3,5,5,6,6)$. Or else, $1 \in C(v_{21})$, and furthermore assume that $1, 2 \in C(v_{21}) \cap C(v_{22})$. Hence, $\{5,6\} \backslash C(v_{21}) \neq \emptyset$, say $5 \notin C(v_{21})$. Let $(vv_0, vv_1, vv_2, v_0v_3, v_1v_2, vv_3) \to (3,4,5,5,6,6)$. If $6 \notin C(v_{22})$, we are done. Or else, $C(v_{22}) = \{4,1,2,6\}$, we recolor $\{vv_1, v_2v_{22}\}$ with 5, and vv_2 with 4.

Case 4.2 $\{a,b\} = \{1,3\}$ and $\{c,d\} \in \{\{2,4\}, \{2,5\}, \{4,5\}, \{5,6\}\}$.

Note that G contains no $(2,3)_{(v_0,v_3)}$-path. We first let $(vv_2, vv_0, vv_3, v_1v_2, vv_1) \to (1,2,3,5,6)$, then let $v_0v_3 \to 5$ if $\{c,d\} = \{2,4\}$; $v_0v_3 \to 6$ if $\{c,d\} = \{4,5\}$; and $v_0v_3 \to 4$ if $\{c,d\} = \{2,5\}$ or $\{5,6\}$.

Case 4.3 $\{a,b\} = \{1,5\}$ and $\{c,d\} \in \{\{2,6\}, \{3,6\}\}$.

In this case, it suffices to let $(vv_2, vv_0, v_0v_3, v_1v_2, vv_3, vv_1) \to (1,2,4,5,5,6)$.

Case 5 $v_1v_2 \in E(G)$ and $v_2v_3, v_0v_3, v_0v_1, v_1v_3, v_0v_2 \notin E(G)$.

Let V_1, V_2 be defined similarly as in Case 4. By Cases 1–4, $V_1 \cap V_2 = \emptyset$. Let $H = G - \{v, v_1, v_2\} + \{uv_{11}, uv_{12}, uv_{21}, uv_{22}, v_0v_3\}$ and assume $(uv_{11}, uv_{12}, uv_{21}, uv_{22})_c = (1,2,3,4)$, where u is a new vertex. Let $(v_1v_{11}, v_1v_{12}, v_2v_{21}, v_2v_{22}) \to (1,2,3,4)$. Without loss of generality, we assume that $c(v_0v_3) \in \{1,5\}$.

Case 5.1 $c(v_0v_3) = 1$.

- Assume that $2 \notin C(v_3)$.

If G contains neither a $(2, 3)_{(v_1,v_3)}$-path nor a $(1, 3)_{(v_1,v_0)}$-path, let $(vv_0, vv_3, vv_1, v_1v_2, vv_2) \rightarrow (1, 2, 3, 5, 6)$. Otherwise, G contains a $(2, 3)_{(v_1,v_3)}$-path or a $(1, 3)_{(v_1,v_0)}$-path. If $3 \in C(v_3)\backslash C(v_0)$, then let $(vv_2, vv_3, vv_0) \rightarrow (1, 2, 3)$ and $vv_1 \rightarrow a \in \{4, 5, 6\}\backslash C(v_3)$, $v_1v_2 \rightarrow b \in \{5, 6\}\backslash\{a\}$. If $3 \in C(v_0)\backslash C(v_3)$, then let $(vv_0, vv_2, vv_3) \rightarrow (1, 2, 3)$ and $vv_1 \rightarrow c \in \{4, 5, 6\}\backslash C(v_0)$, $v_1v_2 \rightarrow d \in \{5, 6\}\backslash\{c\}$. Otherwise, $3 \in C(v_3) \cap C(v_0)$ and $4 \in C(v_3) \cap C(v_0)$ similarly. Hence, $\{5, 6\}\backslash C(v_3) \neq \emptyset$, say $5 \notin C(v_3)$. Let $(vv_0, vv_2, v_1v_2, vv_3, vv_1) \rightarrow (1, 2, 5, 5, 6)$. If there is no $(1, 6)_{(v_1,v_0)}$-path in G, we are done. Otherwise, G contains a $(1, 6)_{(v_1,v_0)}$-path, which cannot pass through v_3 and $C(v_0) = \{3, 4, 6\}$. It suffices to let $(vv_3, vv_0) \rightarrow (1, 5)$.

- Assume that $2 \in C(v_3) \cap C(v_0)$ and $3 \notin C(v_3)$.

If $5 \notin C(v_0)$, let $(vv_3, vv_2, vv_1, vv_0, v_1v_2) \rightarrow (1, 2, 3, 5, 6)$. Otherwise, $5 \in C(v_0)$ and $6 \in C(v_0)$. Let $(vv_0, vv_3, vv_1, vv_2, v_1v_2) \rightarrow (1, 3, 4, 5, 6)$. If there is no $(3, 5)_{(v_2,v_3)}$-path in G, we are done. Otherwise, it suffices to let $(vv_0, vv_3) \rightarrow (3, 1)$.

- Assume that $C(v_3) = C(v_0) = \{2, 3, 4\}$.

Let $(vv_3, vv_2, v_1v_2, vv_0, vv_1) \rightarrow (1, 2, 5, 5, 6)$. If G contains no $(2, 5)_{(v_1,v_0)}$-path, we are done. Otherwise, it suffices to let $(vv_0, vv_3) \rightarrow (1, 5)$.

Case 5.2 $c(v_0v_3) = 5$.

Note that $\{1, 2, 3, 4\}\backslash C(v_3) \neq \emptyset$, say $1 \notin C(v_3)$, by symmetry. If G contains no $(1, i)_{(v_1,v_3)}$-path for some $i \in \{3, 4\}$, let $(vv_3, vv_2, vv_1, vv_0, v_1v_2) \rightarrow (1, 2, i, 5, 6)$. Otherwise, for any $i \in \{3, 4\}$, G contains a $(1, i)_{(v_1,v_3)}$-path, implying that $3, 4 \in C(v_{11}) \cap C(v_3)$ and G contains no $(1, i)_{(v_2,v_3)}$-path. If $1 \notin C(v_0)$, let $(vv_0, vv_2, vv_1, vv_3, v_1v_2) \rightarrow (1, 2, 3, 5, 6)$. Otherwise, $1 \in C(v_0)$. If G contains neither a $(2, 5)_{(v_1,v_0)}$-path nor a $(1, 6)_{(v_1,v_3)}$-path, let $(vv_3, vv_2, vv_0, v_1v_2, vv_1) \rightarrow (1, 2, 5, 5, 6)$.

Assume that G contains a $(1, 6)_{(v_1,v_3)}$-path. Thus, $6 \in C(v_{11}) \cap C(v_3)$, and $C(v_{11}) = \{1, 3, 4, 6\}$ and $C(v_3) = \{3, 4, 6\}$. Let $(vv_3, v_1v_2, vv_2, v_1v_{11}, vv_0) \rightarrow (1, 1, 2, 5, 5)$, and color vv_1 with a color in $\{3, 4, 6\}\backslash C(v_0)$.

Assume that G contains a $(2, 5)_{(v_1,v_0)}$-path. Then $2 \in C(v_0)$, and $\{3, 4\}\backslash C(v_0) \neq \emptyset$, say $3 \notin C(v_0)$. If $5 \notin C(v_{11})$, then let $(vv_3, vv_2, vv_1, vv_0, v_1v_{11}, v_1v_2) \rightarrow (1, 2, 3, 5, 5, 6)$. Otherwise, $C(v_{11}) = \{1, 3, 4, 5\}$, it suffices to let $(v_1v_2, vv_3, vv_0, vv_1, vv_2, v_1v_{11}) \rightarrow (1, 1, 3, 4, 5, 6)$. □

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