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On the Ranks of the Alternating Group A_n

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Abstract Let G be a finite group and X be a conjugacy class of G. The *rank* of X in G, denoted by $rank(G{:}X)$, is defined to be the minimal number of elements of X generating G. In this paper we establish some general results on the ranks of certain conjugacy classes of elements for simple alternating group A_n. We apply these general results together with the structure constants method to determine the ranks of all the non-trivial classes of A_8 and A_9.

Keywords Conjugacy classes · Rank · Generation · Structure constant · Alternating group

Mathematics Subject Classification 20C15 · 20C40 · 20D08

1 Introduction

Generation of finite groups by suitable subsets is of great interest and has many applications to groups and their representations. For example, the computations of the genus of simple groups can be reduced to the generations of the relevant simple

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groups (see Woldar [1] for details). Also Di Martino et al. [2] established a useful connection between generation of groups by conjugates and the existence of elements representable by almost cyclic matrices. Their motivation was to study irreducible projective representations of sporadic simple groups. Recently, more attention was paid to the generation of finite groups by conjugate elements. In his Ph.D. thesis [3], Ward considered the generation of a simple group by conjugate involutions satisfying certain conditions.

We are interested in the generation of finite simple groups by the minimal number of elements from a given conjugacy class of the group. This motivates the following definition.

Definition 1 Let G be a finite simple group and X be a conjugacy class of G. The rank of X in G, denoted by $rank(G{:}X)$, is defined to be the minimal number of elements of X generating G.

One of the applications of ranks of conjugacy classes of a finite group is that they are involved in the computations of the covering number of the finite simple group (see Zisser [4]).

In [5–7], the second author computed the ranks of involuntary classes of the Fischer sporadic simple group Fi_{22}. He found that $rank(Fi_{22}{:}2B) = rank(Fi_{22}{:}2C) = 3$, while $rank(Fi_{22}{:}2A) \in \{5, 6\}$. The work of Hall and Soicher [8] implies that $rank(Fi_{22}{:}2A) = 6$. Then in a considerable number of publications (for example, but not limited to, see [9–14] or [7]) Moori, Ali and Ibrahim explored the ranks for various sporadic simple groups. In this article we prove some general results on the ranks of certain conjugacy classes of elements for the simple alternating group A_n. Then we apply these general results together with the structure constants method to determine the ranks for all non-trivial conjugacy classes of A_8 and A_9. In this paper we follow the notation of [15].

2 Preliminaries

Let G be a finite group and $C_1, C_2, \ldots, C_k$ be $k \geq 3$ (not necessarily distinct) conjugacy classes of G with $g_1, g_2, \ldots, g_k$ being representatives for these classes, respectively.

For a fixed representative $g_k \in C_k$ and for $g_i \in C_i$, $1 \leq i \leq k-1$, denote by $\Delta_G = \Delta_G(C_1, C_2, \ldots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \ldots, g_{k-1})$ such that $g_1 g_2 \ldots g_{k-1} = g_k$. This number is known as *class algebra constant* or *structure constant*. With $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \ldots, \chi_r\}$ being the set of complex irreducible characters of G, the number Δ_G is easily calculated from the character table of G through the formula

$$\Delta_G(C_1, C_2, \ldots, C_k) = \frac{\prod_{i=1}^{k-1} |C_i|}{|G|} \sum_{i=1}^{r} \frac{\chi_i(g_1)\chi_i(g_2)\ldots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{(\chi_i(1_G))^{k-2}}. \quad (1)$$

Also for a fixed $g_k \in C_k$ we denote by $\Delta^*_G(C_1, C_2, \ldots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \ldots, g_{k-1}) \in C_1 \times C_2 \times \cdots \times C_{k-1}$ satisfying

$$g_1 g_2 \ldots g_{k-1} = g_k \quad \text{and} \quad \langle g_1, g_2, \ldots, g_{k-1}\rangle = G. \tag{2}$$

Definition 2 If $\Delta^*_G(C_1, C_2, \ldots, C_k) > 0$, the group G is said to be **$(C_1, C_2, \ldots, C_k)$-generated**.

Furthermore, if $H \le G$ is any subgroup containing a fixed element $g_k \in C_k$, we let $\Sigma_H(C_1, C_2, \ldots, C_k)$ be the total number of distinct $(k-1)$-tuples $(g_1, g_2, \ldots, g_{k-1})$ such that $g_1 g_2 \ldots g_{k-1} = g_k$ and $\langle g_1, g_2, \ldots, g_{k-1}\rangle \le H$. The value of $\Sigma_H(C_1, C_2, \ldots, C_k)$ can be obtained as a sum of the structure constants $\Delta_H(c_1, c_2, \ldots, c_k)$ of H-conjugacy classes $c_1, c_2, \ldots, c_k$ such that $c_i \subseteq H \bigcap C_i$.

Theorem 2.1 *Let G be a finite group and H be a subgroup of G containing a fixed element g such that $\gcd(o(g), [N_G(H){:}H]) = 1$. Then the number $h(g, H)$ of conjugates of H containing g is $\chi_H(g)$, where $\chi_H(g)$ is the permutation character of G with action on the conjugates of H. In particular,*

$$h(g, H) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_{N_G(H)}(x_i)|},$$

where $x_1, x_2, \ldots, x_m$ are representatives of the $N_G(H)$-conjugacy classes fused to the G-class of g.

Proof See, for example, Ganief and Moori [16–18]. □

The above number $h(g, H)$ is useful in giving a lower bound for $\Delta^*_G(C_1, C_2, \ldots, C_k)$, namely $\Delta^*_G(C_1, C_2, \ldots, C_k) \ge \Theta_G(C_1, C_2, \ldots, C_k)$, where

$$\Theta_G(C_1, C_2, \ldots, C_k) = \Delta_G(C_1, C_2, \ldots, C_k) - \sum h(g_k, H)\Sigma_H(C_1, C_2, \ldots, C_k), \tag{3}$$

g_k is a representative of the class C_k, and the sum is taken over all the representatives H of G-conjugacy classes of maximal subgroups containing elements of all the classes $C_1, C_2, \ldots, C_k$.

Since we have all the maximal subgroups of the sporadic simple groups (except for $G = \mathbb{M}$ the Monster group), it is possible to build a small subroutine in GAP [19] or Magma [20] to compute the values of $\Theta_G = \Theta_G(C_1, C_2, \ldots, C_k)$ for any collection of conjugacy classes of a sporadic simple group.

If $\Theta_G > 0$ then certainly G is $(C_1, C_2, \ldots, C_k)$-generated. In the case $C_1 = C_2 = \cdots = C_{k-1} = C$ then G can be generated by $k-1$ elements suitably chosen from C and hence $rank(G{:}C) \le k-1$.

We now quote some results for establishing generation and non-generation of finite simple groups. These results are also important in determining the ranks of the finite simple groups.

Lemma 2.2 (e.g., see Ali and Moori [14] or Conder et al. [21]) *Let G be a finite simple group such that G is* (lX, mY, nZ)*-generated. Then G is* $(\underbrace{lX, lX, \ldots, lX}_{m-times}, (nZ)^m)$*-generated.*

Proof Since G is (lX, mY, nZ)-generated group, it follows that there exist $x \in lX$ and $y \in mY$ such that $xy \in nZ$ and $\langle x, y\rangle = G$. Let $N := \langle x, x^y, x^{y^2}, \ldots, x^{y^{m-1}}\rangle$. Then $N \trianglelefteq G$. Since G is simple group and N is non-trivial subgroup we obtain that $N = G$. Furthermore, we have

$$\begin{aligned} xx^y x^{y^2} x^{y^{m-1}} &= x(yxy^{-1})(y^2xy^{-2})\cdots(y^{m-1}xy^{1-m}) \\ &= (xy)^m \in (nZ)^m. \end{aligned}$$

Since $x^{y^i} \in lX$ for all i, the result follows. □

Corollary 2.3 (e.g., see Ali and Moori [14]) *Let G be a finite simple group such that G is* (lX, mY, nZ)*-generated. Then* $rank(G{:}lX) \leq m$.

Proof Follows immediately by Lemma 2.2. □

Lemma 2.4 (e.g., see Ali and Moori [14]) *Let G be a finite simple* $(2X, mY, nZ)$*-generated group. Then G is* $(mY, mY, (nZ)^2)$*-generated.*

Proof Since G is $(2X, mY, nZ)$-generated group, it is also $(mY, 2X, tK)$-generated group. The result follows immediately by Lemma 2.2. □

Corollary 2.5 *If G is a finite simple* $(2X, mY, nZ)$*-generated group, then rank* $(G{:}mY) = 2$.

Proof By Lemma 2.4 and Corollary 2.3 we have $rank(G{:}mY) \leq 2$. But a non-abelian simple group cannot be generated by one element. Thus, $rank(G{:}mY) = 2$. □

The following two results are in some cases useful in establishing non-generation for finite groups.

Lemma 2.6 (e.g., see Ali and Moori [14] or Conder et al. [21]) *Let G be a finite centerless group. If* $\Delta^*_G(C_1, C_2, \ldots, C_k) < |C_G(g_k)|$, $g_k \in C_k$, *then* $\Delta^*_G(C_1, C_2, \ldots, C_k) = 0$ *and therefore G is not* $(C_1, C_2, \ldots, C_k)$*-generated.*

Proof We prove the contrapositive of the statement, that is if $\Delta^*_G(C_1, C_2, \ldots, C_k) > 0$ then $\Delta^*_G(C_1, C_2, \ldots, C_k) \geq |C_G(g_k)|$, for a fixed $g_k \in C_k$. So let us assume that $\Delta^*_G(C_1, C_2, \ldots, C_k) > 0$. Thus, there exists at least one $(k-1)$-tuple $(g_1, g_2, \ldots, g_{k-1}) \in C_1 \times C_2 \times \cdots \times C_{k-1}$ satisfying Eq. (2). Let $x \in C_G(g_k)$. Then we obtain

$$x(g_1g_2\ldots g_{k-1})x^{-1} = (xg_1x^{-1})(xg_2x^{-1})\cdots(xg_{k-1}x^{-1}) = (xg_kx^{-1}) = g_k.$$

Thus, the $(k-1)$-tuple $(xg_1x^{-1}, xg_2x^{-1}, \ldots, xg_{k-1}x^{-1})$ will generate G. Moreover, if x_1 and x_2 are distinct elements of $C_G(g_k)$, then the $(k-1)$-tuples

$(x_1g_1x_1^{-1}, x_1g_2x_1^{-1}, \ldots, x_1g_{k-1}x_1^{-1})$ and $(x_2g_1x_2^{-1}, x_2g_2x_2^{-1}, \ldots, x_2g_{k-1}x_2^{-1})$ are also distinct since G is centerless. Thus, we have at least $|C_G(g_k)|$ $(k-1)$-tuples $(g_1, g_2, \ldots, g_{k-1})$ generating G. Hence, $\Delta_G^*(C_1, C_2, \ldots, C_k) \geq |C_G(g_k)|$. □

The following result is due to Ree [22].

Theorem 2.7 *Let G be a transitive permutation group generated by permutations $g_1, g_2, \ldots, g_s$ acting on a set of n elements such that $g_1g_2 \ldots g_s = 1_G$. If the generator g_i has exactly c_i cycles for $1 \leq i \leq s$, then $\sum_{i=1}^{s} c_i \leq (s-2)n + 2$.*

Proof See, for example, Ali and Moori [14].

The following result is due to Conder et al. [21] and Scott [23].

Theorem 2.8 (Scott's theorem) *Let $g_1, g_2, \ldots, g_s$ be elements generating a group G with $g_1g_2 \ldots g_s = 1_G$ and $\mathbb{V}$ be an irreducible module for G with* $\dim \mathbb{V} = n \geq 2$. *Let $C_{\mathbb{V}}(g_i)$ denote the fixed point space of $\langle g_i \rangle$ on $\mathbb{V}$, and let d_i be the codimension of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$. Then $\sum_{i=1}^{s} d_i \geq 2n$.*

With χ being the ordinary irreducible character afforded by the irreducible module $\mathbb{V}$ and $\mathbf{1}_{\langle g_i \rangle}$ being the trivial character of the cyclic group $\langle g_i \rangle$, the codimension d_i of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$ can be computed using the following formula [16]:

$$d_i = \dim(\mathbb{V}) - \dim(C_{\mathbb{V}}(g_i)) = \dim(\mathbb{V}) - \langle \chi \downarrow_{\langle g_i \rangle}^{G}, \mathbf{1}_{\langle g_i \rangle} \rangle$$
$$= \chi(1_G) - \frac{1}{|\langle g_i \rangle|} \sum_{j=0}^{o(g_i)-1} \chi(g_i^j). \tag{4}$$

3 Some General Results on Ranks of A_n

In this section we give some general results on the ranks for certain conjugacy classes of elements of the simple group A_n.

The alternating group A_n, $n \geq 5$ has $\lfloor \frac{n}{3} \rfloor$ conjugacy classes of elements of order 3. The cycle structures of these classes are $3^m 1^{n-3m}$, $1 \leq m \leq \lfloor \frac{n}{3} \rfloor$. For $m = 1$, let $3A$ denote the class of elements of A_n of cycle structure (a, b, c). In this section we determine the rank of this class in A_n. We will use $(A_n)_{[k_1, k_2, \ldots, k_r]}$ to denote the subgroup of A_n fixing the points $k_1, k_2, \ldots, k_r$, and if it fixes a single point k_i, we use $(A_n)_{k_i}$.

Lemma 3.1 $rank(A_5{:}3A) = 2$.

Proof We claim that $A_5 = \langle (1,2,3), (1,4,5) \rangle$. We have $(1,2,3)(1,4,5) = (1,4,5,2,3)$, which has order 5. This implies that $15 || \langle (1,2,3), (1,4,5) \rangle|$. By looking at the maximal subgroups of A_5 (see the $\mathbb{ATLAS}$ [24] for example) we can see that there is no maximal subgroup of A_5 with order divisible by 15. It follows that $\langle (1,2,3), (1,4,5) \rangle = A_5$ and hence $rank(A_5{:}3A) = 2$. □

Lemma 3.2 $rank(A_n{:}3A) \neq 2, \ \forall n \geq 6$.

Proof Suppose that $x, y \in 3A$ of $A_n,\ n \geq 6$, and let $x = (a, b, c)$ and $y = (d, e, f)$. If $xy = yx$ then $\langle x, y\rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. If $xy \neq yx$, then x and y are not disjoint cycles and have some common points, i.e., $\{a, b, c\} \bigcap \{d, e, f\} \neq \phi$. Thus, the number of moved points by $\langle x, y\rangle$ is at most 5 and it follows that $\langle x, y\rangle \leq A_5$. Hence, $rank(A_n{:}3A) \neq 2$ for $n \geq 6$. □

Lemma 3.3 $rank(A_6{:}3A) = 3$.

Proof We show that $A_6 = \langle(1, 2, 3), (1, 4, 5), (1, 5, 6)\rangle$. Let $H = \langle(1, 2, 3), (1, 4, 5)\rangle$. Then $H \cong A_5$ and $H = (A_6)_6$, which is a maximal subgroup of A_6. Since $(1, 5, 6) \notin H$, we have $\langle H, (1, 5, 6)\rangle = A_6$. Since $rank(A_6{:}3A) \neq 2$ by Lemma 3.2, it follows that $rank(A_6{:}3A) = 3$. □

Now we state and prove an important theorem on the rank of the class $3A$ of $A_n,\ n \geq 5$.

Theorem 3.4 *For the alternating group $A_n,\ n \geq 5$, we have*

$$rank(A_n{:}3A) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

Proof We use the mathematical induction on n. The result is true for $n = 5$ and $n = 6$ by Lemmas 3.1 and 3.3, respectively. We will show that

$$A_n = \begin{cases} \langle(1, 2, 3), (1, 4, 5), (1, 6, 7), \ldots, (1, n-3, n-2), (1, n-1, n)\rangle & \text{if } n \text{ is odd,} \\ \langle(1, 2, 3), (1, 4, 5), (1, 6, 7), \ldots, (1, n-2, n-1), (1, n-1, n)\rangle & \text{if } n \text{ is even.} \end{cases} \quad (5)$$

Suppose that the result is true for n odd; then, we will show that the result will be true for $n + 1$ and $n + 2$. So assume that Eq. (5) is true for n odd. Note that if $H = \langle(1, 2, 3), (1, 4, 5), (1, 6, 7), \ldots, (1, n-1, n)\rangle$, then Eq. (5) implies that $A_n \cong (A_{n+1})_{n+1} = H$. Since $(1, n, n+1) \in A_{n+1} \backslash A_n$ and H is a maximal subgroup of A_{n+1}, we have

$$K = \langle H, (1, n, n + 1)\rangle = A_{n+1}. \quad (6)$$

Since $n + 1$ is even, we have proven the result for the even case. Now since K is a maximal subgroup of A_{n+2} ($K = (A_{n+2})_{n+2}$), then $\langle K, (1, n + 1, n + 2)\rangle = A_{n+2}$ as $(1, n + 1, n + 2) \in A_{n+2} \backslash A_{n+1}$. Thus,

$$\langle H, (1, n, n + 1), (1, n + 1, n + 2)\rangle = A_{n+2}. \quad (7)$$

We now show that we do not need the element $(1, n, n+1)$ in Eq. (7) to generate A_{n+2}; that is, $(1, n, n+1)$ is redundant. Let $\alpha = (1, n-1, n) \in H$ and $\beta = (1, n+1, n+2)$. Then $\alpha^\beta = (n-1, n, n+1) := \gamma$ and $\gamma^\alpha = (1, n+1, n) = (1, n, n+1)^{-1}$. This shows that $\langle H, (1, n, n+1), (1, n+1, n+2)\rangle$ can actually be reduced to $\langle H, (1, n+1, n+2)\rangle$. That is $\langle H, (1, n + 1, n + 2)\rangle = A_{n+2}$, i.e.,

$$\langle(1, 2, 3), (1, 4, 5), (1, 6, 7), \ldots, (1, n - 1, n), (1, n + 1, n + 2)\rangle = A_{n+2},$$

completing the proof. □

Next we turn to the generation of A_n by the classes of n-cycles for n odd and $(n-1)$-cycles for n even.

Note 1 (i). It is well known that for n odd, the class of the n-cycles of the symmetric group S_n splits into two classes (same size) in A_n. For n even, we have no n-cycles in A_n, but the class of the $(n-1)$-cycles splits into two classes (same size) in A_n. The reason for both cases is obvious since if α is an n-cycle (for n odd) or an $(n-1)$-cycle (for n even), then $C_{S_n}(\alpha) = C_{A_n}(\alpha) = \langle\alpha\rangle$.

(ii). Assume that α is an n-cycle (for n odd) or an $(n-1)$-cycle (for n even) in A_n. If $h \in S_n \backslash A_n$, then α and α^h are not conjugate in A_n. If $\alpha \sim \alpha^h$ in A_n, then we have $g \in A_n$ such that $\alpha = (\alpha^h)^g$ which implies that $hg \in C_{S_n}(\alpha) = C_{A_n}(\alpha)$, a contradiction since hg is an odd permutation and cannot be in A_n.

The following lemma is useful in the determination of the ranks of the classes of n- or $(n-1)$-cycles of A_n.

Lemma 3.5 *Let G be a primitive subgroup of S_n. If G contains a 3-cycle, then $G \geq A_n$.*

Proof See Cameron [25]. □

Now we give an important result showing the ranks for the classes of n-cycles and $(n-1)$-cycles of A_n.

Theorem 3.6 *For $n \geq 5$, we have* $\underbrace{rank(A_n{:}nX)}_{n \text{ is odd}} = 2 = \underbrace{rank(A_n{:}(n-1)X)}_{n \text{ is even}}$, $X \in \{A, B\}$.

Proof We consider the case when n is odd and the even case follows easily. For $n \geq 5$ odd, let $\alpha_1 = (1, 2, 3, \ldots, n)$ and $\alpha_2 = (1, 2, 3, \ldots, n, n-1)$ be representatives of the two classes nA and nB of the n-cycle classes of A_n, respectively. (Note that α_1 and α_2 are conjugate in S_n, but not in A_n as $\alpha_2 = \alpha_1^{(n-1,n)}$, see Note 1 (ii).) Also let $\beta_1 = (1, 4, 5, 6, 7, \ldots, n, 2, 3)$ and $\beta_2 = (1, 4, 5, 6, 7, \ldots, n-2, n, n-1, 2, 3)$. We handle the class nA, and the result for the other class nB follows similarly. Let $H := \langle\alpha_1, \beta_1\rangle$. Clearly, $H \leq S_n$ and in fact, since H contains even permutations only, it follows that $H \leq A_n$. We are aiming to show that the equality holds; that is, $H = A_n$. To establish the converse inequality ($H \geq A_n$) we need to show that H is primitive in S_n and contains a 3-cycle element. Since $\alpha_1,\ \beta_1 \in H$, we have

$$\begin{aligned} H \ni \alpha_1^{-1}\beta_1 &= (1, n, n-1, n-2, \ldots, 4, 3, 2)(1, 4, 5, 6, 7, \ldots, n, 2, 3) \\ &= (1, 3, n),\ \text{a 3-cycle element.} \end{aligned}$$

Now since H contains n-cycle elements (at least α_1 and β_1), it follows by O'Nan–Scott theorem (see, for example, Theorem 2.4 of Wilson [26]) that H cannot be of type (i) or (ii) of maximal subgroups of S_n (as subgroups of these two types cannot have n-cycle elements). Hence, H is a primitive subgroup of S_n, and since H contains a 3-cycle element ($\alpha_1^{-1}\beta_1 = (1, 3, n)$), it follows by Lemma 3.5 that $H \geq A_n$. This together with the information $H \leq A_n$ implies that $H = A_n$.

So far, we proved that the group H, generated by the two n-cycles α_1 and β_1, is just A_n. Next we prove that α_1 and β_1 are conjugate in H. But this follows as $(1, 2, 3) \in 3A \subset A_n = H$, and it is easy to see that $\beta_1 = \alpha_1^{(1,2,3)}$, i.e., $\beta_1 \in nA$. Thus, $rank(A_n{:}nA) = 2$. The result holds similarly for the class nB of A_n, n odd, by letting $H := \langle \alpha_2, \beta_2 \rangle$, where in a similar manner it will be shown that $H = A_n$ and also one can see that $\beta_2 = \alpha_2^{(1,2,3)}$, i.e., $\beta_2 \in nB$. Thus, $rank(A_n{:}nB) = 2$.

The even case follows consequently from the odd case, because if n is even then the treatment of the two classes $(n-1)A$ and $(n-1)B$ of the $(n-1)$-cycles of A_n reduce to the odd case. Hence, the result. □

4 Ranks of the Classes of A_8 and A_9

In this section we apply the general results discussed in previous sections, namely Sects. 2 and 3, to the groups A_8 and A_9. We determine the ranks for all conjugacy classes.

4.1 Ranks of A_8

The group A_8 is a simple group of order $20160 = 2^6 \times 3^2 \times 5 \times 7$. By the $\mathbb{ATLAS}$ the group A_8 has exactly 14 conjugacy classes of its elements and 6 conjugacy classes of its maximal subgroups. Representatives of these classes of maximal subgroups can be taken as follows:

$$\begin{array}{ll} H_1 = A_7 & H_2 = 2^3{:}L_3(2) \\ H_3 = 2^3{:}L_3(2) & H_4 = S_6 \\ H_5 = 2^4{:}(S_3 \times S_3) & H_6 = (A_5 \times 3){:}2. \end{array}$$

We firstly list in Table 1 the values of $h(g, H_i)$ for all the non-identity classes and maximal subgroups H_i, $1 \le i \le 6$, of A_8.

We start our investigation on the ranks of the non-trivial classes of A_8 by looking at the two classes of involutions $2A$ and $2B$. It is well known that two involutions generate a dihedral group. Thus, the lower bound of the rank of an involuntary class in a finite group $G \ne D_{2n}$ (the dihedral group of order $2n$) is 3.

In this subsection we let $G = A_8$.

Lemma 4.1 *$rank(G{:}2Z) \ne 3$, for $Z \in \{A, B\}$.*

Proof We show that the group G is not $(2Z, 2Z, 2Z, nX)$-generated group for $Z \in \{A, B\}$ and for any non-trivial conjugacy class nX of G. We start with the case $Z = A$. The direct computations yield $\Delta_G(2A, 2A, 2A, nX) = 0$ for $nX \in T_1 := \{3A, 5A, 6A, 15A, 15B\}$. Thus, G is not $(2A, 2A, 2A, nX)$-generated group for any class nX in T_1. The group A_8 has a 14-dimensional complex irreducible module $\mathbb{V}$. For any conjugacy class nX, let $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$ denote the codimension of the fixed space (in $\mathbb{V}$) of a representative of nX. Using Eq. (4) together with the power maps associated with the character table of A_8 given in the $\mathbb{ATLAS}$,

Table 1 Values $h(g, H_i)$, $1 \leq i \leq 6$ for non-identity classes and maximal subgroups of A_8

	H_1	H_2	H_3	H_4	H_5	H_6
$2A$	0	7	7	4	11	0
$2B$	4	3	3	8	7	12
$3A$	5	0	0	10	5	11
$3B$	2	3	3	1	2	2
$4A$	0	3	3	0	3	0
$4B$	2	1	1	2	1	2
$5A$	1	0	0	3	0	1
$6A$	1	0	0	2	1	3
$6B$	0	1	1	1	2	0
$7A$	1	1	1	0	0	0
$7B$	1	1	1	0	0	0
$15A$	0	0	0	0	0	1
$15B$	0	0	0	0	0	1

Table 2 $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$, nX is a non-trivial class of G and $\dim(\mathbb{V}) = 14$

nX	$2A$	$2B$	$3A$	$3B$	$4A$	$4B$	$5A$	$6A$	$6B$	$7A$	$7B$	$15A$	$15B$
d_{nX}	4	6	10	8	8	10	12	12	12	0	0	14	14

Table 3 $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$, nX is a non-trivial class of G and $\dim(\mathbb{V}) = 7$

nX	$2A$	$2B$	$3A$	$3B$	$4A$	$4B$	$5A$	$6A$	$6B$	$7A$	$7B$	$15A$	$15B$
d_{nX}	3	2	2	4	6	4	4	4	4	6	6	6	6

we were able to compute all the values of d_{nX} for all non-trivial classes nX of G, and we list these values in Table 2.

Now if A_8 is $(2A, 2A, 2A, nX)$-generated group, then by Scott's theorem (see Theorem 2.8) we must have $d_{2A}+d_{2A}+d_{2A}+d_{nX} \geq 2\times 14$. However, it is clear from Table 2 that $3 \times d_{2A} + d_{nX} < 28$, for each $nX \in T_2 := \{2A, 2B, 3B, 4A, 4B, 6B, 7A, 7B\}$ and therefore A_8 is not $(2A, 2A, 2A, nX)$-generated group, for any $nX \in T_2$. Since G is not $(2A, 2A, 2A, nX)$-generated group, for any $nX \in T_1 \bigcup T_2$, it follows that $rank(G{:}2A) \neq 3$.

We now turn to the other case $Z = B$ to show that G is not $(2B, 2B, 2B, nX)$-generated group, for all non-trivial conjugacy classes nX of G. We use similar arguments to the above case. We know that the group A_8 has a 7-dimensional complex irreducible module $\mathbb{V}$. Let d_{nX} be the codimension of the fixed space in $\mathbb{V}$, $\dim(\mathbb{V}) = 7$ of a representative of nX. Similarly, we list in Table 3 the values of d_{nX} for all non-trivial classes nX of G.

Now if A_8 is $(2B, 2B, 2B, nX)$-generated group, then we must have $d_{2B} + d_{2B} + d_{2B} + d_{nX} \geq 2 \times 7$. However, it is clear from Table 3 that $3 \times d_{2B} + d_{nX} < 14$, for all non-trivial classes nX of A_8 and therefore A_8 is not $(2B, 2B, 2B, nX)$-generated group, for any class nX of A_8. This establishes the non-generation of A_8 by three conjugate involutions from class $2B$. Thus, $rank(G{:}2B) \neq 3$, completing the proof. □

Note 2 Observe that the non-generation of A_8 by three conjugate involutions from class $2A$ can be established without the need of computing the structure constants $\Delta_G(2A, 2A, 2A, nX)$, $nX \in T_1$, as it is clear from Table 2 that $3 \times d_{2A} + d_{nX} < 28$ for all the non-trivial classes nX of A_8.

Lemma 4.2 *The group* A_8 *is* $(2A, 4B, 15A)$- *and* $(2B, 4A, 15B)$-*generated group.*

Proof Let $a_1 := (1, 2)(3, 4)(5, 6)(7, 8) \in 2A$, $a_2 := (5, 6)(7, 8) \in 2B$, $b_1 := (2, 3)(4, 6, 7, 5) \in 4B$ and $b_2 := (1, 2, 6, 8)(3, 5, 4, 7) \in 4A$. Then $\langle a_1, b_1\rangle = \langle a_2, b_2\rangle = A_8$ with $a_1b_1 = (1, 3, 6, 4, 2)(5, 7, 8) \in 15A$ and $a_2b_2 = (1, 2, 6, 4, 7)(3, 5, 8) \in 15B$. Thus, A_8 is $(2A, 4B, 15A)$- and $(2B, 4A, 15B)$-generated group. □

Proposition 4.3 $rank(G{:}2Z) = 4$, *for* $Z \in \{A, B\}$.

Proof Since by Lemma 4.2, A_8 is $(2A, 4B, 15A)$- and $(2B, 4A, 15B)$-generated group, it follows by applications of Lemma 2.2 that A_8 is $(2A, 2A, 2A, 2A, (15A)^4)$- and $(2B, 2B, 2B, 2B, (15B)^4)$-generated group. Thus, $rank(A_8{:}2Z) \leq 4$, for $Z \in \{A, B\}$. Since $rank(A_8{:}2Z) \neq 3$ by Lemma 4.1, it follows that $rank(A_8{:}2Z) = 4$, for $Z \in \{A, B\}$. □

Remark 1 The generation of A_8 by four suitable involutions from class $2A$ or $2B$ can be established using the structure constant method. For example, the direct computations show that $\Delta_G(2Z, 2Z, 2Z, 2Z, 15A) = 3375$ (74250) for $Z = A$ and B, respectively. From Table 1 we see that $h(g, H_i) = 0$ for $g \in 15A$ and $i \in \{1, 2, 3, 4, 5\}$, while $h(g, H_6) = 1$ for $g \in 15A$. The computations show that $\Sigma_{H_6}(2Z, 2Z, 2Z, 2Z, 15A) = 0$ (10125) for $Z = A$ and B, respectively. It follows that

$$\begin{aligned}\Theta_G(2Z, 2Z, 2Z, 2Z, 15A) &= \Delta_G(2Z, 2Z, 2Z, 2Z, 15A)\\ &\quad - \sum_{i=1}^{6} h(g_i, H_i)\Sigma_{H_i}(2Z, 2Z, 2Z, 2Z, 15A)\\ &= 3375 - 0 = 3375\ (74250 - 10125 = 64125)\\ &\quad \text{for } Z = A \text{ and } B\text{ , respectively,}\end{aligned}$$

establishing the generation of G by the tuple $(2Z, 2Z, 2Z, 2Z, 15A)$. Hence, the result.

Proposition 4.4 $rank(G{:}3A) = 4$.

Proof Direct application of Theorem 3.4. □

Table 4 Some information on the classes $nX \in T$

	$\Delta_G(nX, nX, 15A)$	$h(15A, H_6)$	$\Sigma_{H_6}(nX, nX, 15A)$	$\Theta_G(nX, nX, 15A)$
$3B$	35	1	5	30
$4A$	45	1	0	45
$4B$	270	1	45	225
$5A$	45	1	0	45
$6A$	155	1	20	135
$6B$	510	1	0	510
$7A$	405	1	0	405
$7B$	405	1	0	405
$15B$	125	1	5	120

Remark 2 We can prove Proposition 4.4 without using Theorem 3.4. The rank of class $3A$ in G can still be established using the structure constant method together with the results of Sect. 2. The direct computations together with applications of Lemma 2.6 and Theorem 2.8 reveal that G is neither $(3A, 3A, nX)$- nor $(3A, 3A, 3A, nX)$-generated group for any non-trivial class nX of G. Thus, $rank(G{:}3A) \notin \{2, 3\}$. It is easy to show that G is $(3A, 4A, 15A)$-generated group. Now it follows by applications of Lemma 2.2 that G is $(3A, 3A, 3A, 3A, (15A)^4)$-generated group. Thus, $rank(G{:}3A) \leq 4$. Since $rank(G{:}3A) \notin \{2, 3\}$, we deduce that $rank(G{:}3A) = 4$.

Proposition 4.5 *$rank(G{:}7X) = 2$, for $X \in \{A, B\}$.*

Proof Direct application of Theorem 3.6. □

The above result can be established using the structure constant method. In the next proposition, we give the ranks of all the remaining non-trivial classes of A_8 including the classes $7A$ and $7B$.

Proposition 4.6 *Let $T := \{3B, 4A, 4B, 5A, 6A, 6B, 7A, 7B, 15A, 15B\}$. Then $rank(G{:}nX) = 2$ for any $nX \in T$.*

Proof The aim here is to show that G is an $(nX, nX, 15A)$-generated group for any $nX \in T\backslash\{15A\}$. For all the classes $nX \in T\backslash\{15A\}$ we give in Table 4 some information about $\Delta_G(nX, nX, 15A)$, $h(15A, H_6)$, $\Sigma_{H_6}(nX, nX, 15A)$ and $\Theta_G(nX, nX, 15A)$, where by $h(15A, H_6)$ we mean the number of conjugate subgroups of H_6 that contain a fixed element of $15A$.

The last column of Table 4 establishes the generation of G by the triple $(nX, nX, 15A)$ for all $nX \in T\backslash\{15A\}$. Thus, $rank(G{:}nX) = 2$ for all $nX \in T\backslash\{15A\}$.

Now it is possible to show that G is $(15A, 15A, nX)$ for any non-trivial class nX of G. For example, we have $\Delta_G(15A, 15A, 7Y) = 91$, $h(g, H_1) = h(g, H_2) = h(g, H_3) = 1$ and $\Sigma_{H_i}(15A, 15A, 7Y) = 0$ for all $1 \leq i \leq 6$ and $Y \in \{A, B\}$. It follows that $\Theta_G(15A, 15A, 7Y) = 91 - 0 = 91$, showing the generation of G by the triple $(15A, 15A, 7Y)$. Thus, $rank(G{:}15A) = 2$. Hence, the result. □

Table 5 Values $h(g, H_i)$, $1 \le i \le 8$, for non-identity classes and maximal subgroups of A_9

	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
$2A$	5	12	20	0	0	26	20	0
$2B$	1	4	4	8	8	6	16	8
$3A$	6	15	21	0	0	21	10	0
$3B$	0	0	3	3	3	0	10	21
$3C$	3	3	3	6	6	6	1	6
$4A$	3	4	4	0	0	4	2	0
$4B$	1	0	0	0	0	2	0	4
$5A$	4	6	4	0	0	1	0	0
$6A$	2	3	5	0	0	5	2	0
$6B$	1	1	1	2	2	0	1	2
$7A$	2	1	0	1	1	0	0	0
$9A$	0	0	0	3	0	0	1	0
$9B$	0	0	0	0	3	0	1	0
$10A$	0	2	0	0	0	1	0	0
$12A$	0	1	1	0	0	1	2	0
$15A$	1	0	1	0	0	1	0	0
$15B$	1	0	1	0	0	1	0	0

Remark 3 For most of the classes $nX \in T$ of Proposition 4.6, the result can also be proved using Lemma 2.5 together with the facts that the group A_8 is $(2B, 4A, 15B)$-, $(2A, 4B, 15A)$-, $(2A, 5A, 15A)$-, $(2A, 6A, 15B)$-, $(2A, 6B, 15A)$-, $(2A, 7A, 15A)$-, $(2A, 7B, 15B)$-, $(2A, 15A, 15B)$- and $(2A, 15B, 15A)$-generated group.

Now we gather the results on ranks of the non-trivial classes of A_8.

Theorem 4.7 *Let G be the alternating group A_8. Then*

1. $rank(G{:}2A) = rank(G{:}2B) = rank(G{:}3A) = 4$.
2. $rank(G{:}nX) = 2$ *for all* $nX \notin \{1A, 2A, 2B, 3A\}$.

Proof The result follows by Propositions 4.3, 4.4 and 4.6. □

4.2 Ranks of A_9

The group A_9 is a simple group of order $181440 = 2^6 \times 3^4 \times 5 \times 7$. By the $\mathbb{ATLAS}$ the group A_9 has exactly 18 conjugacy classes of its elements and 8 conjugacy classes of its maximal subgroups. Representatives of these classes of maximal subgroups can be taken as follows:

$$\begin{array}{ll} H_1 = A_8 & H_2 = S_7 \\ H_3 = (A_6 \times 3){:}2 & H_4 = L_2(8){:}3 \\ H_5 = L_2(8){:}3 & H_6 = (A_5 \times A_4){:}2 \\ H_7 = 3^3{:}S_4 & H_8 = 3^2{:}2A_4. \end{array}$$

We firstly list in Table 5 the values of $h(g, H_i)$ for all the non-identity classes and maximal subgroups H_i, $1 \le i \le 8$, of A_9.

We start our investigation on the ranks of the non-trivial classes of A_9 by looking at the two classes of involutions $2A$ and $2B$.

From now on let $G = A_9$.

Lemma 4.8 $rank(G{:}2A) \neq 3$.

Proof We show that the group G is not $(2A, 2A, 2A, nX)$-generated group for any non-trivial conjugacy class nX of G. The group A_9 has an 8-dimensional complex irreducible module $\mathbb{V}$. For any conjugacy class nX, let $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$ denote the codimension of the fixed space (in $\mathbb{V}$) of a representative of nX. Using Eq. (4) together with the power maps associated with the character table of A_9 given in the $\mathbb{ATLAS}$, we were able to compute all the values of d_{nX} for all non-trivial classes nX of G, and we list these values in Table 6.

Now if A_9 is $(2A, 2A, 2A, nX)$-generated group, then by Scott's theorem (see Theorem 2.8) we must have $d_{2A} + d_{2A} + d_{2A} + d_{nX} \geq 2 \times 8$. However, it is clear from Table 6 that $3 \times d_{2A} + d_{nX} < 16$, for all the non-trivial classes nX of G. Therefore, A_9 is not $(2A, 2A, 2A, nX)$-generated group for any non-trivial class nX of G and it follows that $rank(G{:}2A) \neq 3$.

Lemma 4.9 *The group* A_9 *is* $(2A, 4B, 9B)$*-generated group.*

Proof Let $a := (6,7)(8,9) \in 2A$ and $b := (1,3,6,9)(2,5,4,7) \in 4B$. Then $\langle a, b\rangle = A_9$ with $ab = (1,3,6,2,5,4,7,9,8) \in 9B$. Thus A_9 is $(2A, 4B, 9B)$-generated group. □

Proposition 4.10 $rank(G{:}2A) = 4$.

Proof Since by Lemma 4.9, A_9 is $(2A, 4B, 9B)$-generated group, it follows by applications of Lemma 2.2 that A_9 is $(2A, 2A, 2A, 2A, (9B)^4)$-generated group. Thus, $rank(A_9{:}2A) \leq 4$. Since $rank(A_9{:}2A) \neq 3$ by Lemma 4.8, it follows that $rank(A_9{:}2A) = 4$. □

Remark 4 The generation of A_9 by four suitable involutions from class $2A$ can be established using the structure constant method. For example, the direct computations show that $\Delta_G(2A, 2A, 2A, 2A, 9A) = 59049$. From Table 5 we see that $h(g, H_i) = 0$ for $g \in 9A$ and $i \in \{1, 2, \ldots, 8\}\backslash\{4, 7\}$, while $h(g, H_4) = 3$ and $h(g, H_7) = 1$ for $g \in 9A$. However, the computations show that $\Sigma_{H_4}(2A, 2A, 2A, 2A, 9A) = \Sigma_{H_7}(2A, 2A, 2A, 2A, 9A) = 0$. It follows that

$$\begin{aligned}\Theta_G(2A, 2A, 2A, 2A, 9A) &= \Delta_G(2A, 2A, 2A, 2A, 9A) \\ &\quad - \sum_{i=1}^{8} h(g_i, H_i)\Sigma_{H_i}(2A, 2A, 2A, 2A, 9A) \\ &= 59049 - 0 = 59049,\end{aligned}$$

establishing the generation of G by the tuple $(2A, 2A, 2A, 2A, 9A)$. Hence, the result.

Lemma 4.11 *The group* A_9 *is* $(2B, 3C, 15A)$*-generated group.*

Table 6 $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$, nX is a non-trivial class of G and $\dim(\mathbb{V}) = 8$

nX	$2A$	$2B$	$3A$	$3B$	$3C$	$4A$	$4B$	$5A$	$6A$	$6B$	$7A$	$9A$	$9B$	$10A$	$12A$	$15A$	$15B$
d_{nX}	2	4	2	6	4	4	6	4	4	6	6	8	8	6	6	6	6

Proof Let $a := (2, 4)(3, 6)(5, 7)(8, 9) \in 2B$ and $b := (1, 8, 2)(3, 5, 7)(4, 6, 9) \in 3C$. Then $\langle a, b\rangle = A_9$ with $ab = (1, 8, 4)(2, 6, 5, 3, 9) \in 15A$. Thus, A_9 is $(2B, 3C, 15A)$-generated group. □

Proposition 4.12 $rank(G{:}2B) = 3$.

Proof Since by Lemma 4.11, A_9 is $(2B, 3C, 15A)$-generated group, and it follows by applications of Lemma 2.2 that A_9 is $(2B, 2B, 2B, (15A)^4)$-generated group. Thus, $rank(A_9{:}2B) \leq 3$. Since two involutions generate dihedral group, it follows that $rank(A_9{:}2B) \neq 2$ and hence we have $rank(A_9{:}2B) = 3$. □

Proposition 4.13 $rank(G{:}3A) = 4$.

Proof Direct application of Theorem 3.4. □

Remark 5 Similarly to Remark 2, generation of $G = A_9$ by four suitable elements from class $3A$ can be established using the structure constant method together with the results of Sect. 2. The direct computations together with applications of Lemma 2.6 and Theorem 2.8 reveal that G is neither $(3A, 3A, nX)$- nor $(3A, 3A, 3A, nX)$-generated group for any non-trivial class nX of G. Thus, $rank(G{:}3A) \notin \{2, 3\}$. Now the direct computations yield $\Delta_G(3A, 3A, 3A, 3A, 9A) = 729$. From Table 5 we see that $h(g, H_i) = 0$ for $g \in 9A$ and $i \in \{1, 2, \ldots, 8\}\backslash\{4, 7\}$, while $h(g, H_4) = 3$ and $h(g, H_7) = 1$ for $g \in 9A$. However, the computations show that $\Sigma_{H_4}(3A, 3A, 3A, 3A, 15A) = \Sigma_{H_7}(3A, 3A, 3A, 3A, 15A) = 0$ and it follows that

$$\begin{aligned}\Theta_G(3A, 3A, 3A, 3A, 9A) &= \Delta_G(3A, 3A, 3A, 3A, 9A) \\ &\quad - \sum_{i=1}^{8} h(g_i, H_i)\Sigma_{H_i}(3A, 3A, 3A, 3A, 9A) \\ &= 729 - 0 = 729,\end{aligned}$$

establishing the generation of A_9 by the tuple $(3A, 3A, 3A, 3A, 9A)$. Thus, $rank(G{:}3A) \leq 4$. Since $rank(G{:}3A) \notin \{2, 3\}$, we deduce that $rank(G{:}3A) = 4$.

Proposition 4.14 $rank(G{:}9X) = 2$, *for* $X \in \{A, B\}$.

Proof Direct application of Theorem 3.6. □

The above result can be established using the structure constant method. In the next proposition, we give the ranks of all the remaining non-trivial classes of A_9 including the classes $9A$ and $9B$.

Proposition 4.15 *Let* $T := \{3B, 3C, 4A, 4B, 5A, 6A, 6B, 7A, 9A, 9B, 10A, 12A, 15A, 15B\}$. *Then* $rank(G{:}nX) = 2$ *for any* $nX \in T$.

Proof The aim here is to show that G is an $(nX, nX, 9A)$-generated group for any $nX \in T\backslash\{9A\}$. For all the classes $nX \in T\backslash\{9A\}$ we give in Table 7 some information about $\Delta_G = \Delta_G(nX, nX, 9A)$, $h(9A, H_4)$, $h(9A, H_7)$, $\Sigma_{H_4} =$

Table 7 Some information on the classes $nX \in T$

	Δ_G	$h(9A, H_4)$	Σ_{H_4}	$h(9A, H_7)$	Σ_{H_7}	$\sum_{i\in\{4,7\}} h(9A, H_i)$ $\Sigma_{H_i}(nX, nX, 9A)$	Θ_G
3*B*	24	3	0	1	15	15	9
3*C*	36	3	9	1	0	27	9
4*A*	144	3	0	1	9	9	135
4*B*	729	3	0	1	0	0	729
5*A*	9	3	0	1	0	0	9
6*A*	324	3	0	1	0	0	324
6*B*	5220	3	144	1	36	468	4752
7*A*	3240	3	0	1	0	0	3240
9*A*	1872	3	0	1	9	9	1863
9*B*	1872	3	0	1	9	9	1863
10*A*	405	3	0	1	0	0	405
12*A*	1116	3	0	1	36	36	1080
15*A*	792	3	0	1	0	0	792
15*B*	792	3	0	1	0	0	792

$\Sigma_{H_4}(nX, nX, 9A)$, $\Sigma_{H_7} = \Sigma_{H_7}(nX, nX, 9A)$ and $\Theta_G = \Theta_G(nX, nX, 9A)$, where by $h(9A, H_4)$, $h(9A, H_7)$ we mean the number of conjugate subgroups of H_4 (resp. H_7) that contain a fixed element of $9A$.

The last column of Table 7 establishes the generation of G by the triple $(nX, nX, 9A)$ for all $nX \in T\backslash\{9A\}$. Thus, $rank(G{:}nX) = 2$ for all $nX \in T\backslash\{9A\}$.

Now it is possible to show that G is $(9A, 9A, nX)$ for any non-trivial class nX of G. For example, we have $\Delta_G(9A, 9A, 15A) = 2280$, $h(g, H_1) = h(g, H_3) = h(g, H_6) = 1$ and $\Sigma_{H_i}(9A, 9A, 15A) = 0$ for all $1 \leq i \leq 8$. It follows that $\Theta_G(9A, 9A, 15A) = 2280 - 0 = 2280$, showing the generation of G by the triple $(9A, 9A, 15A)$. Thus, $rank(G{:}9A) = 2$. Hence, the result. □

Remark 6 For most of the classes $nX \in T$ of Proposition 4.15, the result can also be proved using Corollary 2.5 together with the facts that the group A_9 is $(2B, 3C, 15A)$-, $(2B, 4A, 15A)$-, $(2A, 4B, 9B)$-, $(2B, 5A, 9B)$-, $(2B, 6A, 9A)$-, $(2B, 6B, 15B)$-, $(2B, 7A, 5A)$-, $(2A, 9A, 7A)$-, $(2A, 9B, 7A)$-, $(2B, 10A, 7A)$-, $(2B, 15A, 5A)$- and $(2B, 15B, 5A)$-generated group. We also note that A_9 is not $(2X, 3B, nY)$-generated group for $X \in \{A, B\}$ and for any non-trivial conjugacy class nY of G.

We now gather the results on the ranks of all non-trivial classes of A_9.

Theorem 4.16 *Let G be the alternating group A_9. Then*

1. $rank(G{:}nA) = 4$ *for* $n \in \{2, 3\}$.
2. $rank(G{:}2B) = 3$.
3. $rank(G{:}nX) = 2$ *for all* $nX \notin \{1A, 2A, 2B, 3A\}$.

Proof The result follows by Propositions 4.10, 4.12, 4.13 and 4.15. □

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