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On the Burning Number of Generalized Petersen Graphs

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Abstract The burning number $b(G)$ of a graph G is used for measuring the speed of contagion in a graph. In this paper, we study the burning number of the generalized Petersen graph $P(n, k)$. We show that for any fixed positive integer k, $\lim_{n\to\infty} \frac{b(P(n,k))}{\sqrt{\frac{n}{k}}} = 1$. Furthermore, we give tight bounds for $b(P(n, 1))$ and $b(P(n, 2))$.

Keywords Burning number · Generalized Petersen graphs

Mathematics Subject Classification 05C57 · 05C80

1 Introduction

Graph burning is a discrete-time process that can be used to model the spread of social contagion in social networks. It was introduced by Bonato et al. [2,3,8]. This process is defined on the vertex set of a simple finite graph. Throughout the process, each vertex is either *burned* or *unburned*. Initially, at time step $t = 0$, all vertices are unburned. At the beginning of every time step $t \geq 1$, an unburned vertex is chosen to burn (if

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such a vertex is available). After that, if a vertex is burned in time step $t-1$, then in time step t, each of its unburned neighbours becomes burned. A burned vertex will remain burned throughout the process. The process ends when all vertices are burned, in which case we say the graph is *burned*.

Suppose a graph G is burned in m time steps in a burning process. For $1 \leq i \leq m$, we denote the vertex we choose to burn at the beginning of time step i by x_i. The sequence $(x_1, x_2, \ldots, x_m)$ is called a *burning sequence* for G. Each x_i is called a *burning source* of G. The *burning number* of a graph G, denoted by $b(G)$, is the length of a shortest burning sequence for G. It is straightforward to see that $b(K_n) = 2$. For paths and cycles, Bonato et al. [3] determined their burning numbers exactly.

Theorem 1.1 [3, Theorem 9 and Corollary 10] *Let P_n be a path with n vertices and C_n be a cycle with n vertices. Then,*

$$b(P_n) = \left\lceil n^{1/2} \right\rceil = b(C_n).$$

For general graphs, they showed that the burning number of any graph G can be bounded by its radius r and diameter d, giving $\lceil (d+1)^{1/2} \rceil \leq b(G) \leq r+1$. In the same paper, they also gave an upper bound on the burning number of any connected graph G of order n, showing that $b(n) \leq 2\sqrt{n}-1$. This upper bound was later improved to roughly $\frac{\sqrt{6}}{2}\sqrt{n}$ by Land and Lu [5]. It was conjectured in [3] that $b(G) \leq \lceil \sqrt{n} \rceil$ for any connected graph G of order n. Very recently, Bonato and Lidbetter [4] verified this conjecture for spider graphs, which are trees with exactly one vertex of degree at least 3.

Determining $b(G)$ for general graphs is a non-trivial problem. It is known that computing the burning number of a graph is NP-complete [1]. The burning number of the hypercube Q_n is asymptotically $\frac{n}{2}$ [7], but the exact value of $b(Q_n)$ is still unknown. Several other results on burning number of graphs have also been studied recently. For example, Mitsche, Pralat and Roshanbin investigated the burning number of graph products in [7] and they also focused on the probabilistic aspects of the burning number in [6].

In this paper, we are interested in the burning number of the *generalized Petersen graphs*. Let $n \geq 3$ and k be integers such that $1 \leq k \leq n-1$. The generalized Petersen graph $P(n, k)$ is defined to be the graph on $2n$ vertices with vertex set

$$V(P(n,k)) = \{u_i, v_i \colon i = 0, 1, 2, \ldots, n-1\}$$

and edge set

$$E(P(n,k)) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} \colon i = 0, 1, 2, \ldots, n-1\},$$

where subscripts are taken modulo n. Let $D_1 = \{u_i : i = 0, 1, 2, \ldots, n-1\}$ and $D_2 = \{v_i : i = 0, 1, 2, \ldots, n-1\}$. The subgraph induced by D_1 is called the *outer rim*, while the subgraph induced by D_2 is called the *inner rim*. A *spoke* of $P(n, k)$ is an edge of the form $u_i v_i$ for some $0 \leq i \leq n-1$.

The following are the main results of this paper.

Theorem 1.2 *Let k be a fixed positive integer. Then,*

$$\left\lceil \sqrt{\left\lfloor \frac{n}{k} \right\rfloor} \right\rceil \leq b(P(n,k)) \leq \left\lceil \sqrt{\left\lfloor \frac{n}{k} \right\rfloor} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor + 2.$$

In particular,

$$\lim_{n\to\infty} \frac{b(P(n,k))}{\sqrt{\frac{n}{k}}} = 1.$$

Theorem 1.3 *For $n \geq 3$,*

$$\lceil \sqrt{n} \rceil \leq b(P(n,1)) \leq \lceil \sqrt{n} \rceil + 1.$$

Furthermore, the bounds are tight, and if n is a square, then $b(P(n,1)) = \sqrt{n} + 1$.

Theorem 1.4 *For $n \geq 3$,*

$$\left\lceil \sqrt{\frac{n}{2}} \right\rceil + 1 \leq b(P(n,2)) \leq \left\lceil \sqrt{\frac{n}{2}} \right\rceil + 2.$$

Furthermore, the bounds are tight, and if $\frac{n}{2}$ is a square, then $b(P(n,2)) = \sqrt{\frac{n}{2}} + 2$.

We use standard graph terminology throughout the paper. The *distance* between two vertices u and v in a graph G, denoted by $\text{dist}_G(u,v)$, is the length of a shortest path from u to v in the graph G. By convention, $\text{dist}_G(u,u) = 0$. Furthermore, we shall write $\text{dist}(u,v)$ for $\text{dist}_G(u,v)$ if the graph in question is clear. Given a non-negative integer s, the *s-th closed neighbourhood* of a vertex u, denoted by $N_s^G[u]$, is the set of vertices whose distance from u is at most s, i.e.

$$N_s^G[u] = \{v \in V(G)\colon \text{dist}_G(u,v) \leq s\}.$$

Again, if the graph in question is clear, we shall write $N_s[u]$ for $N_s^G[u]$.

Let $(x_1, x_2, \ldots, x_m)$ be a burning sequence of a graph G. As in [3, Section 2], for each pair i and j, with $1 \leq i < j \leq m$, we have $\text{dist}(x_i, x_j) \geq j - i$ and

$$V(G) = N_{m-1}[x_1] \cup N_{m-2}[x_2] \cup \cdots \cup N_0[x_m]. \tag{1}$$

The plan of the paper is as follows. In Sect. 2, we provide bounds for the burning number of $P(n,k)$ and show that $b(P(n,k))$ is asymptotically $\sqrt{\frac{n}{k}}$. In Sect. 3, we determine the exact values of $b(P(n,k))$ for $1 \leq n \leq 8$. Then, we prove Theorems 1.3 and 1.4 in Sect. 4.

2 General Case

Lemma 2.1 *For $n \geq 3$ and $1 \leq k < n$,*

$$b(P(n,k)) \geq \left\lceil \sqrt{\left\lfloor \frac{n}{k} \right\rfloor} \right\rceil .$$

Proof Let C be a cycle with $\lfloor \frac{n}{k} \rfloor$ vertices, $V(C) = \{0, 1, 2, \ldots, \lfloor \frac{n}{k} \rfloor - 1\}$ and $E(C) = \{i(i+1)\colon 0, 1, \ldots, \lfloor \frac{n}{k} \rfloor - 1\}$, where the integers are taken modulo $\lfloor \frac{n}{k} \rfloor$. Recall that the outer rim and inner rim of $P(n,k)$ are $D_1 = \{u_0, u_1, \ldots, u_{n-1}\}$ and $D_2 = \{v_0, v_1, \ldots, v_{n-1}\}$, respectively.

For each $m \in \{0, 1, 2, \ldots, n-1\}$, let

$$f(m) = \begin{cases} p, & \text{if } \; m = pk + q, \quad 0 \leq p < \lfloor \frac{n}{k} \rfloor, \quad 0 \leq q \leq k-1; \\ \lfloor \frac{n}{k} \rfloor - 1, & \text{if } \; m = \lfloor \frac{n}{k} \rfloor k + q, \quad 0 \leq q < k-1 . \end{cases} \tag{2}$$

Let $\varphi : V(P(n,k)) \to V(C)$ be defined by

$$\varphi(u_i) = f(i) = \varphi(v_i), \quad \forall i \in \{0, 1, 2, \ldots, n-1\}. \tag{3}$$

Clearly, φ is surjective.

Let $(x_1, x_2, \ldots, x_s)$ be a burning sequence of $P(n,k)$. We construct a burning sequence for C using the map φ as follows:

(a) At the beginning of time step 1, burn $y_1 = \varphi(x_1)$;
(b) At the beginning of time step t $(2 \leq t \leq s)$, if $\varphi(x_t)$ is still unburned, then burn $y_t = \varphi(x_t)$; otherwise, burn any unburned vertex $y_t \in V(C)$.

Note that in (b) above, if at the beginning of time step t $(2 \leq t \leq s)$, no unburned vertex can be found, then $(y_1, y_2, \ldots, y_{t-1})$ is a burning sequence of C. So, we may assume that such an unburned vertex can be found at the beginning of every time step. We shall show that $(y_1, y_2, \ldots, y_s)$ is a burning sequence of C. This follows from φ is surjective and the following claim.

Claim If $z \in V(P(n,k))$ is burned at time step t_0, then its image $\varphi(z)$ in C is burned at time step $t_1 \leq t_0$.

Proof If $z = x_1$, then it is burned at time step 1. Its image $\varphi(z) = y_1$ is also burned at time step 1. The claim is true. Assume that the claim is true for a $t_0 < s$.

Suppose z is burned at time step $t_0 + 1$. If z is a burning source, then $z = x_{t_0+1}$. By (b), $\varphi(z)$ is burned at time step $t_0 + 1$ provided that $\varphi(x_{t_0+1})$ is unburned. If $\varphi(x_{t_0+1})$ is burned, then it must be burned at an earlier time step. So, the claim holds.

We may assume that $z \neq x_{t_0+1}$. Note that for any two distinct vertices $w_1, w_2 \in V(P(n,k))$ such that $\varphi(w_1), \varphi(w_2) \in V(C)$ and $|\varphi(w_1) - \varphi(w_2)| \leq 1$ or $|\varphi(w_1) - \varphi(w_2)| = \lfloor \frac{n}{k} \rfloor - 1$, then $\varphi(w_1) = \varphi(w_2)$ or $\varphi(w_1)$ and $\varphi(w_2)$ are adjacent in C. We shall distinguish two cases.

Case 1 Let $z = u_l$. Then, it is adjacent to v_l, u_{l+1} and u_{l-1} where the subscript are taken modulo n. Furthermore, either v_l, u_{l+1} or u_{l-1} is burned at time step t_0. So, by induction, $\varphi(v_l), \varphi(u_{l+1})$ or $\varphi(u_{l-1})$ is burned at time step $t_1 \le t_0$ respectively. By Eqs. (2) and (3), $|\varphi(u_l)-\varphi(v_l)| = 0, |\varphi(u_l)-\varphi(u_{l-1})| \le 1$ and $|\varphi(u_l)-\varphi(u_{l+1})| \le 1$ where $l = 1, 2, \dots, n-2$ and $|\varphi(u_0) - \varphi(u_{n-1})| = \lfloor \frac{n}{k} \rfloor - 1$. This means that $\varphi(z) = \varphi(u_l)$ is burned at time step $t_1 + 1 \le t_0 + 1$.

Case 2 Let $z = v_l$. It is adjacent to u_l, v_{l+k} and u_{l-k} where the subscript are taken modulo n. Either u_l, v_{l-k} or v_{l+k} is burned at time step t_0. Here, we denote $v_{-i} = v_{n-i}$ for a non-negative i. So, by induction, $\varphi(u_l), \varphi(v_{l+k})$ or $\varphi(v_{l-k})$ is burned at time step $t_1 \le t_0$ respectively. By Eqs. (2) and (3), we have $|\varphi(v_l) - \varphi(u_l)| = 0$,

$$|\varphi(v_l) - \varphi(v_{l-k})| = \begin{cases} \lfloor \frac{n}{k} \rfloor - 1, & \text{if } \ l = 0, 1, 2, \dots, k-1; \\ 1, & \text{if } \ l = k, k+1, \dots, \lfloor \frac{n}{k} \rfloor k - 1; \\ 0, & \text{if } \ l = \lfloor \frac{n}{k} \rfloor k, \lfloor \frac{n}{k} \rfloor k + 1, \dots, n-1. \end{cases}$$

and

$$|\varphi(v_l) - \varphi(v_{l+k})| \\ = \begin{cases} 1, & \text{if } \ l = 0, 1, 2, \dots, \left(\lfloor \frac{n}{k} \rfloor - 1\right) k - 1; \\ 0, & \text{if } \ l = \left(\lfloor \frac{n}{k} \rfloor - 1\right) k, \left(\lfloor \frac{n}{k} \rfloor - 1\right) k + 1, \dots, n-1-k; \\ \lfloor \frac{n}{k} \rfloor - 1, & \text{if } \ l = n-k, n-k+1, \dots, n-1. \end{cases}$$

This means that $\varphi(z) = \varphi(v_l)$ is burned at time step $t_1 + 1 \le t_0 + 1$.

This completes the proof of the claim. □

Therefore, given any burning sequence of $P(n, k)$, we can construct a burning sequence for C with shorter or the same length. Hence, $b(P(n, k)) \ge b(C) = \left\lceil \sqrt{\lfloor \frac{n}{k} \rfloor} \right\rceil$, where the last equality follows from Theorem 1.1. □

Lemma 2.2 *For $n \ge 3$ and $1 \le k < n$,*

$$b(P(n,k)) \le \left\lceil \sqrt{\left\lfloor \frac{n}{k} \right\rfloor} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor + 2.$$

Proof Recall that the outer rim and inner rim of $P(n, k)$ are $D_1 = \{u_0, u_1, \dots, u_{n-1}\}$ and $D_2 = \{v_0, v_1, \dots, v_{n-1}\}$, respectively. Let $r = \lfloor \frac{n}{k} \rfloor$. We shall construct a burning sequence for $P(n, k)$ of length at most $\lceil \sqrt{r} \rceil + \lfloor \frac{k}{2} \rfloor + 2$. Note that a subgraph G induced by the vertices $v_0, v_k, v_{2k}, \dots, v_{(r-1)k}$ in $P(n, k)$ is a path or cycle of order r. By Theorem 1.1, $b(G) = \lceil \sqrt{r} \rceil$. So, there is a burning sequence $(x_1, x_2, \dots, x_{\lceil \sqrt{r} \rceil})$ of G. We shall take $x_1, x_2, \dots, x_{\lceil \sqrt{r} \rceil}$ as the first part of our burning sequence for $P(n, k)$.

Note that at time step $\lceil \sqrt{r} \rceil$, all $v_0, v_k, v_{2k}, \dots, v_{(r-1)k}$ are burned. If at time step $\lceil \sqrt{r} \rceil$, u_{rk} is unburned, then we set $x_{\lceil \sqrt{r} \rceil + 1} = u_{rk}$. Otherwise, we set $x_{\lceil \sqrt{r} \rceil + 1}$ to be any unburned vertex. Since u_{ik} is adjacent to v_{ik} for $0 \le i \le (r-1)$, at time step $\lceil \sqrt{r} \rceil + 1$, all $u_0, u_k, u_{2k}, \dots, u_{(r-1)k}, u_{rk}$ are burned. Furthermore, at most $k-1$ vertices are unburned in the path $u_{ik} u_{ik+1} u_{ik+2} \dots u_{(i+1)k}$ in the outer rim (see Fig. 1).

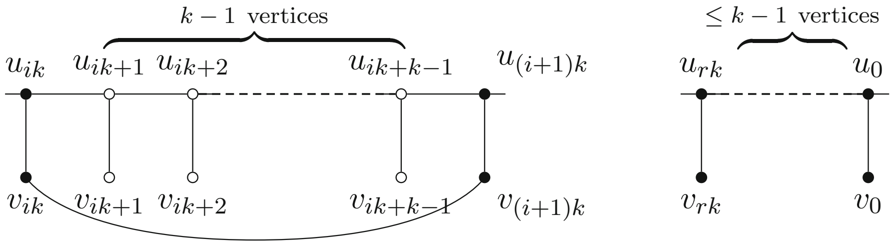

Fig. 1 Filled vertices are burned, whereas empty vertices are unburned

Now, for $j \geq \lceil \sqrt{r} \rceil + 2$, we can choose x_j to be any unburned vertex. Note that at time step $\lceil \sqrt{r} \rceil + 1 + \lfloor \frac{k}{2} \rfloor$, all the vertices in the outer rim are burned. Since u_i and v_i are adjacent, at time step $\lceil \sqrt{r} \rceil + 2 + \lfloor \frac{k}{2} \rfloor$, all vertices in the inner rim are also burned. Hence, the lemma follows. □

Proof of Theorem 1.2 By Lemmas 2.1 and 2.2, we have

$$\left\lceil \sqrt{\left\lfloor \frac{n}{k} \right\rfloor} \right\rceil \leq b(P(n,k)) \leq \left\lceil \sqrt{\left\lfloor \frac{n}{k} \right\rfloor} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor + 2.$$

By noting that $\lim_{n\to\infty} \frac{\left\lceil \sqrt{\lfloor \frac{n}{k} \rfloor} \right\rceil}{\sqrt{\frac{n}{k}}} = 1$ and $\lim_{n\to\infty} \frac{\lfloor \frac{k}{2} \rfloor + 2}{\sqrt{\frac{n}{k}}} = 0$, we conclude

$$\lim_{n\to\infty} \frac{b(P(n,k))}{\sqrt{\frac{n}{k}}} = 1.$$

□

3 Case $1 \leq N \leq 8$

We shall give the exact burning numbers for the case $1 \leq n \leq 8$ in this section. Note that $P(n,k)$ is isomorphic to $P(n, n-k)$. So, we may assume that $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$. Recall that the sth closed neighbourhood of a vertex $x \in V(P(n,k))$ is

$$N_s[x] = \{y \in V(P(n,k)) : \text{dist}(y,x) \leq s\},$$

and the outer rim and inner rim of $P(n,k)$ are $D_1 = \{u_0, u_1, \ldots, u_{n-1}\}$ and $D_2 = \{v_0, v_1, \ldots, v_{n-1}\}$, respectively.

Proposition 3.1 *Let* $3 \leq n \leq 8$ *and* $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$. *Then,*

$$b(P(n,k)) = \begin{cases} 3, & \text{if } \ 3 \leq n \leq 6 \text{ or } n = 7, k \neq 1, \\ 4, & \text{if } \ n = 8 \text{ or } n = 7, k = 1. \end{cases}$$

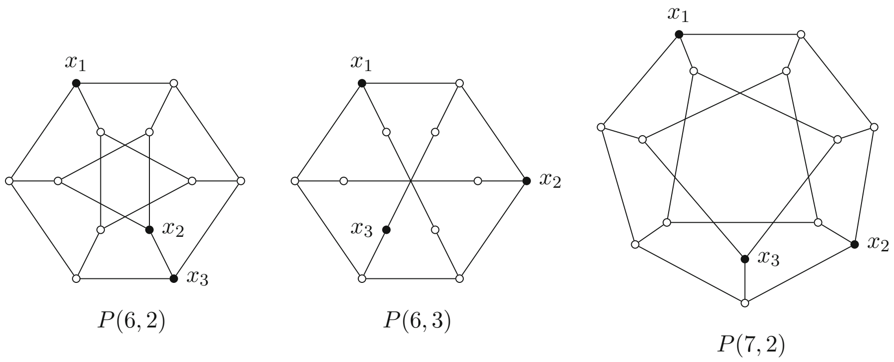

Fig. 2 Burning sequences

Proof Since each vertex $x \in V(P(n,k))$ is of degree 3, $|N_0[x]| = 1$, $|N_1[x]| \leq 4$ and $|N_2[x]| \leq 10$.

Let $3 \leq n \leq 7$. If (x_1, x_2) is a burning sequence of $P(n,k)$, then by Eq. (1),

$$2n \leq |N_1[x_1]| + |N_0[x_2]| \leq 4 + 1 = 5,$$

implying that $n < 3$, which is a contradiction. Hence, $b(P(n,k)) \geq 3$. Similarly, if (x_1, x_2, x_3) is a burning sequence of $P(8,k)$, then

$$16 \leq |N_2[x_1]| + |N_1[x_2]| + |N_0[x_3]| \leq 10 + 4 + 1 = 15,$$

again is a contradiction. Hence, $b(P(8,k)) \geq 4$.

Note that for each $x \in V(P(7,1))$, $|N_2[x]| = 8$. So, if (x_1, x_2, x_3) is a burning sequence of $P(7,1)$, then

$$14 \leq |N_2[x_1]| + |N_1[x_2]| + |N_0[x_3]| \leq 8 + 4 + 1 = 13,$$

which is a contradiction. Hence, $b(P(7,1)) \geq 4$.

Now, the proposition can be verified easily from the burning sequences in the following table (see also Fig. 2).

Burning sequence	Graph
(u_0, v_1, v_2)	$P(3,1)$, $P(4,1)$, $P(4,2)$
(u_0, v_3, u_3)	$P(5,1)$, $P(6,1)$, $P(6,2)$
(u_0, u_2, v_4)	$P(5,2)$, $P(6,3)$
(u_0, u_2, u_4, v_4)	$P(7,1)$
(u_0, u_3, v_4)	$P(7,2)$
(v_0, v_2, u_5)	$P(7,3)$
(u_0, u_2, v_4, u_4)	$P(8,1)$, $P(8,2)$, $P(8,3)$, $P(8,4)$

□

4 Case $1 \leq K \leq 2$

4.1 Proof of Theorem 1.3

Note that for each $x \in V(P(n,1))$, $|N_m[x]| \leq 4m$ for $m \geq 1$ and $|N_0[x]| = 1$. So, if $(x_1, x_2, \ldots, x_l)$ is a burning sequence of $P(n,1)$, then by Eq. (1),

$$2n \leq |N_0[x_l]| + \sum_{i=1}^{l-1} |N_{l-i}[x_i]| \leq 1 + \sum_{i=1}^{l-1} 4(l-i) = 2l^2 - 2l + 1.$$

Since $l \geq 1$, by completing the square, we conclude that $l \geq \frac{2+\sqrt{4-8(1-2n)}}{4} = \frac{1}{2} + \sqrt{n - \frac{1}{4}} > \sqrt{n}$. Hence, $b(P(n,1)) \geq \lceil \sqrt{n} \rceil$, and if n is a square, then $b(P(n,1)) \geq \lceil \sqrt{n} \rceil + 1$.

The subgraph C induced by the vertices in the outer rim $D_1 = \{u_0, u_1, \ldots, u_{n-1}\}$ is a cycle of length n. By Theorem 1.1, $b(C) = \lceil \sqrt{n} \rceil$. So, C has a burning sequence $(y_1, y_2, \ldots, y_{\lceil \sqrt{n} \rceil})$. We shall take $y_1, y_2, \ldots, y_{\lceil \sqrt{n} \rceil}$ as the first part of our burning sequence for $P(n,1)$. Note that at time step $\lceil \sqrt{n} \rceil$, all the vertices in the outer rim are burned. Choose any unburned vertex z in the inner rim. Let $y_{\lceil \sqrt{n} \rceil + 1} = z$. Since $u_i v_i$ are adjacent for $1 \leq i \leq n-1$, at time step $\lceil \sqrt{n} \rceil + 1$ all vertices in the inner rim are also burned. Hence, $b(P(n,1)) \leq \lceil \sqrt{n} \rceil + 1$, and if n is a square, then $b(P(n,1)) = \sqrt{n} + 1$.

Finally, by Proposition 3.1, $b(P(5,1)) = 3 = \left\lceil \sqrt{5} \right\rceil$. So the bounds are tight. This completes the proof of Theorem 1.3. □

4.2 Proof of Theorem 1.4

We shall first define an isomorphic graph of $P(n,2)$, say $H(n)$. Let

$$W_1 = \left\{ s_i, s_i', t_i, t_i' : i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \right\};$$
$$W_2 = \left\{ t_i t_{i+1}, t_i' t_{i+1}', s_i s_{i+1}', s_j t_j, s_j' t_j', s_j s_j' : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor \right\}.$$

If n is even, then let

$$V(H(n)) = W_1;$$
$$E(H(n)) = W_2 \cup \left\{ t_1 t_{\frac{n}{2}}, t_1' t_{\frac{n}{2}}', s_{\frac{n}{2}} s_1' \right\}. \tag{4}$$

If n is odd, then let

$$V(H(n)) = W_1 \cup \{s_0, t_0\};$$
$$E(H(n)) = W_2 \cup \left\{ s_0 s_{\frac{n-1}{2}}, s_0 s_1', s_0 t_0, t_0 t_1, t_0 t_{\frac{n-1}{2}}', t_{\frac{n-1}{2}} t_1' \right\}. \tag{5}$$

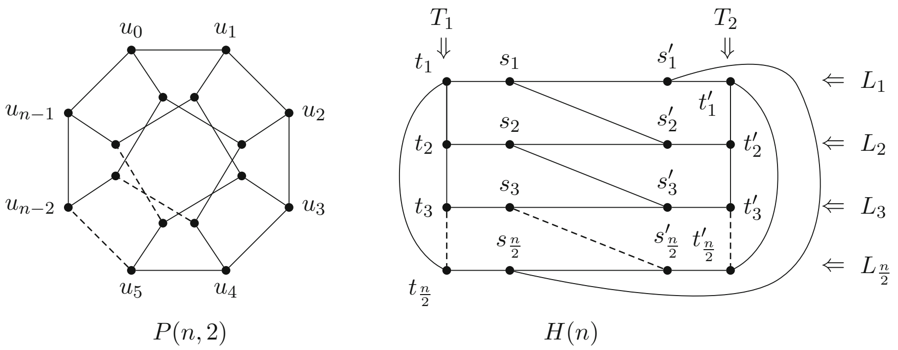

Fig. 3 $H(n)$ is isomorphic to $P(n, 2)$ where n is even

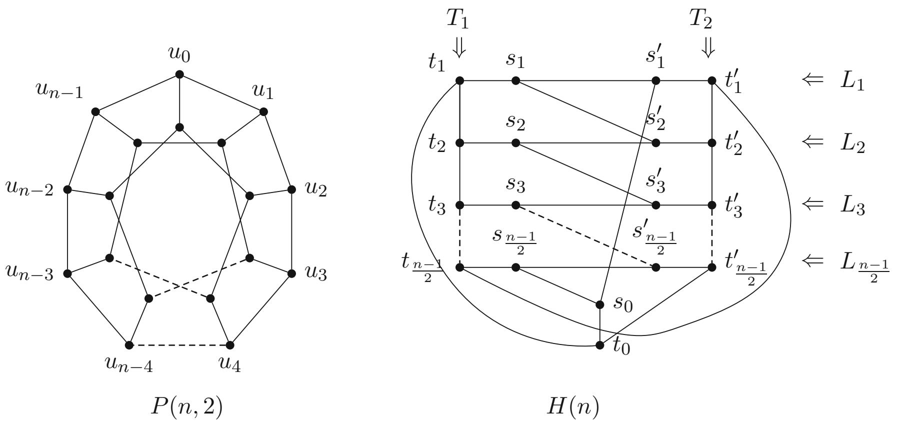

Fig. 4 $H(n)$ is isomorphic to $P(n, 2)$ where n is odd

We now show that $P(n, 2)$ is isomorphic to $H(n)$ (see Figs. 3 and 4). Define $\phi : V(P(n, 2)) \to V(H(n))$ as follows: Let $\phi(u_i) = s'_{\frac{i}{2}+1}$ if i is even and $i \neq n-1$; $\phi(u_i) = s_{\frac{i-1}{2}+1}$ if i is odd; $\phi(u_{n-1}) = s_0$ if $n-1$ is even. Let $\phi(v_i) = t'_{\frac{i}{2}+1}$ if i is even and $i \neq n-1$; $\phi(v_i) = t_{\frac{i-1}{2}+1}$ if i is odd; $\phi(v_{n-1}) = t_0$ if $n-1$ is even. Note that the subgraph induced by all the vertices s_i, s'_i in $H(n)$ is isomorphic to the outer rim in $P(n, 2)$, and the subgraph induced by all the vertices t_i, t'_i in $H(n)$ is isomorphic to the inner rim in $P(n, 2)$. Furthermore, $s_i t_i, s'_i t'_i$ are the spokes in $P(n, 2)$. So $P(n, 2)$ is isomorphic to $H(n)$.

Let $T_1 = \{t_i : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$, $T_2 = \{t'_i : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$, and level $L_i = \{s_i, s'_i, t_i, t'_i\}$ for $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$.

Lemma 4.1 *For $n \geq 3$,*

$$b(P(n, 2)) \geq \left\lceil \sqrt{\frac{n}{2}} \right\rceil + 1.$$

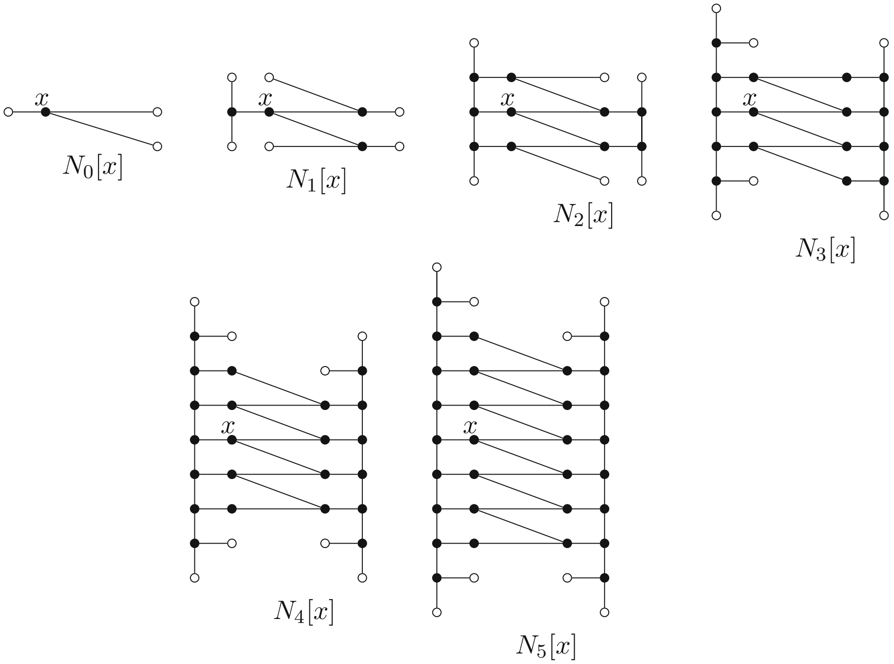

Fig. 5 Spreading of fire from $x \notin T_1 \cup T_2$. Filled vertices are burned, whereas empty vertices are unburned

Furthermore, if $\frac{n}{2}$ *is a square, then* $b(P(n,2)) \geq \sqrt{\frac{n}{2}} + 2$.

Proof Note that if $x \notin T_1 \cup T_2$, then $|N_0[x]| = 1$, $|N_1[x]| \leq 4$, $|N_2[x]| \leq 10$, $|N_3[x]| \leq 16$, $|N_4[x]| \leq 22$, $|N_5[x]| \leq 30$ and $|N_r[x]| \leq 30 + 8(r-5)$ for $r \geq 6$ (see Fig. 5). After 5 steps, a maximum of 8 vertices are newly burned in each following step.

If $x \in T_1 \cup T_2$, then $|N_0[x]| = 1$, $|N_1[x]| \leq 4$, $|N_2[x]| \leq 10$, $|N_3[x]| \leq 18$ and $|N_r[x]| \leq 18 + 8(r-3)$ for $r \geq 4$ (see Fig. 6). After 4 steps, a maximum of 8 vertices are newly burned in each following step.

In either case, we have $|N_0[x]| = 1$, $|N_1[x]| \leq 4$, $|N_2[x]| \leq 10$, $|N_3[x]| \leq 18$ and $|N_r[x]| \leq 18 + 8(r-3) = 8r - 6$ for $r \geq 4$.

By Proposition 3.1, $b(P(n,2)) = 3 = \left\lceil \sqrt{\frac{n}{2}} \right\rceil + 1$ for $3 \leq n \leq 7$ and

$$b(P(8,2)) = 4 = \sqrt{\frac{n}{2}} + 2.$$

Hence, the lemma holds for $3 \leq n \leq 8$. So, we may assume $n \geq 9$. Suppose $9 \leq n \leq 16$, then $\left\lceil \sqrt{\frac{n}{2}} \right\rceil \leq 3$. If $P(n,2)$ has a burning sequence of length 3, say (x_1, x_2, x_3), then by Eq. (1), $18 \leq 2n \leq \sum_{i=1}^{3} |N_{3-i}[x_i]| \leq 1 + 4 + 10 = 15$, a contradiction. Suppose $17 \leq n \leq 32$, then $\left\lceil \sqrt{\frac{n}{2}} \right\rceil \leq 4$. If $P(n,2)$ has a burning sequence of length

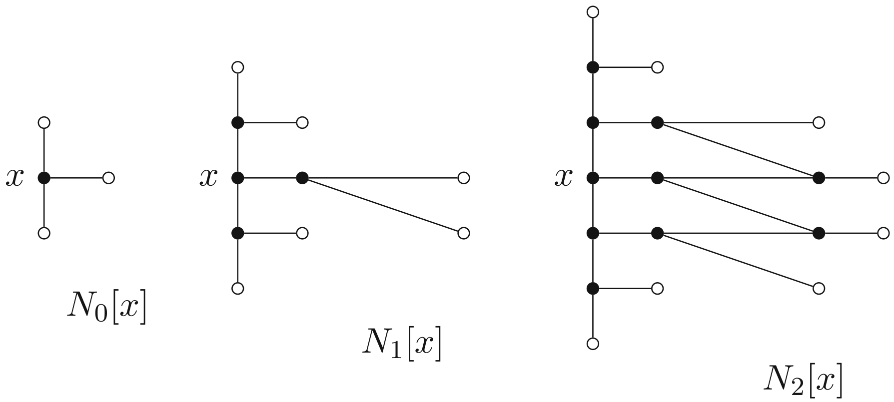

Fig. 6 Spreading of fire from $x \in T_1 \cup T_2$. Filled vertices are burned, whereas empty vertices are unburned

4, say (x_1, x_2, x_3, x_4), then $34 \leq 2n \leq \sum_{i=1}^{4} |N_{4-i}[x_i]| \leq 1 + 4 + 10 + 18 = 33$, a contradiction. So, $b(P(n, 2)) \geq \left\lceil \sqrt{\frac{n}{2}} \right\rceil + 1$ for $3 \leq n \leq 32$.

Note that for $9 \leq n \leq 32$, $\frac{n}{2}$ is a square if and only if $n = 18, 32$. When $n = 18$, $\sqrt{\frac{n}{2}} + 2 = 5$. If $P(18, 2)$ has a burning sequence of length 4, then $\sum_{i=1}^{4} |N_{4-i}[x_i]| \leq 33$, but $|V(P(18, 2))| = 36$. When $n = 32$, $\sqrt{\frac{n}{2}} + 2 = 6$. If $P(32, 2)$ has a burning sequence of length 5, then $\sum_{i=1}^{5} |N_{5-i}[x_i]| \leq 1 + 4 + 10 + 18 + 26 = 59$, but $|V(P(32, 2))| = 64$. Thus, if $\frac{n}{2}$ is a square and $9 \leq n \leq 32$, then $b(P(n, 2)) \geq \sqrt{\frac{n}{2}} + 2$.

Suppose $n \geq 33$. If $P(n, 2)$ has a burning sequence of length l, say $(x_1, x_2, \ldots, x_l)$, then by Eq. (1),

$$\begin{aligned} 2n &\leq \sum_{i=1}^{l} |N_{l-i}[x_i]| \leq |N_0[x_l]| + |N_1[x_{l-1}]| + |N_2[x_{l-2}]| + \sum_{i=1}^{l-3} |N_{l-i}[x_i]| \\ &\leq 1 + 4 + 10 + \sum_{r=3}^{l-1} (8r - 6) \\ &= 4l^2 - 10l + 9. \end{aligned}$$

Since $l \geq 1$, by completing the square, we conclude that

$$l \geq \frac{10 + \sqrt{100 - 16(9 - 2n)}}{8} = \frac{5}{4} + \sqrt{\frac{n}{2} - \frac{11}{16}} > \sqrt{\frac{n}{2}} + 1.$$

Hence, $b(P(n, 2)) \geq \left\lceil \sqrt{\frac{n}{2}} \right\rceil + 1$, and if $\frac{n}{2}$ is a square, then $b(P(n, 2)) \geq \sqrt{\frac{n}{2}} + 2$. This completes the proof of the lemma. □

Lemma 4.2 *For $n \geq 3$,*

$$b(P(n,2)) \leq \left\lceil \sqrt{\frac{n}{2}} \right\rceil + 2.$$

Proof Let $l = \left\lceil \sqrt{\frac{n}{2}} \right\rceil$. It is sufficient to show that there is a burning sequence $(x_1, x_2, \ldots, x_l, x_{l+1}, x_{l+2})$ in $H(n)$.

Note that for $2 \leq j \leq l$, the term $(2j-1)l-(j-1)^2$ is increasing. Let m_0 be the largest positive integer such that $(2m_0-1)l-(m_0-1)^2 \leq \lfloor \frac{n}{2} \rfloor$. Since

$$(2l-1)l-(l-1)^2 = l^2+l-1 \geq \left(\sqrt{\frac{n}{2}}\right)^2 + \left(\sqrt{\frac{n}{2}}-1\right) > \frac{n}{2},$$

we must have $m_0 \leq l-1$.

Now, we construct the first part of a burning sequence for $H(n)$, say $x_1, x_2, \ldots, x_l$, as follows:

(a) Let $x_1 = t_l$;
(b) For each $2 \leq j \leq m_0$, set $x_j = t_{(2j-1)l-(j-1)^2}$ if j is odd, or $x_j = t'_{(2j-1)l-(j-1)^2}$ if j is even;
(c) For $j \geq m_0+1$:
  (i) Suppose $m_0 \leq l-2$. If $x_{m_0} = t_{(2m_0-1)l-(m_0-1)^2}$, then set $x_{m_0+1} = t'_{\lfloor \frac{n}{2} \rfloor}$, whereas if $x_{m_0} = t'_{(2m_0-1)l-(m_0-1)^2}$, then set $x_{m_0+1} = t_{\lfloor \frac{n}{2} \rfloor}$. For $m_0+2 \leq w \leq l$, choose x_w to be any unburned vertex (if possible).
  (ii) Suppose $m_0 = l-1$. If $x_{l-1} = t_{(2l-3)l-(l-2)^2}$, then set $x_l = t'_{\lfloor \frac{n}{2} \rfloor}$, whereas if $x_{l-1} = t'_{(2l-3)l-(l-2)^2}$, then set $x_l = t_{\lfloor \frac{n}{2} \rfloor}$.

In Fig. 7, the filled vertices are $N_{l-i}[x_i]$ and the shaded vertices are $N_{l+2-i}[x_i] \setminus N_{l-i}[x_i]$. In particular, $L_4 \cup L_5 \cup \cdots \cup L_l \subseteq N_{l-1}[x_1]$. So $(L_1 \cup L_2 \cup \cdots \cup L_l) \setminus \{t'_1\} \subseteq N_{l+1}[x_1]$ (see Fig. 7a).

Suppose $2 \leq j \leq m_0$. Note that x_j is contained in level $L_{(2j-1)l-(j-1)^2}$ and x_{j-1} is contained in level $L_{(2j-3)l-(j-2)^2}$. There are exactly $2l-2j+4 = ((2j-1)l-(j-1)^2)-((2j-3)l-(j-2)^2)+1$ levels between $L_{(2j-1)l-(j-1)^2}$ and $L_{(2j-3)l-(j-2)^2}$ (inclusive). All these levels are contained in $N_{l-j+3}[x_{j-1}] \cup N_{l-j+2}[x_j]$ (see Fig. 7b).

Suppose $m_0 \leq l-2$. By the choice of m_0, $(2m_0+1)l-m_0^2 > \lfloor \frac{n}{2} \rfloor$. So, the number of levels between $L_{\lfloor \frac{n}{2} \rfloor}$ and $L_{(2m_0-1)l-(m_0-1)^2}$ (inclusive) is at most

$$\begin{aligned}\left\lfloor \frac{n}{2} \right\rfloor - ((2m_0-1)l-(m_0-1)^2)+1 &< (2m_0+1)l-m_0^2 \\ &\quad - ((2m_0-1)l-(m_0-1)^2)+1 \\ &= 2l-2m_0+2.\end{aligned}$$

All these levels are contained in $N_{l-m_0+2}[x_{m_0}] \cup N_{l-m_0+1}[x_{m_0+1}]$ (see Fig. 7b).

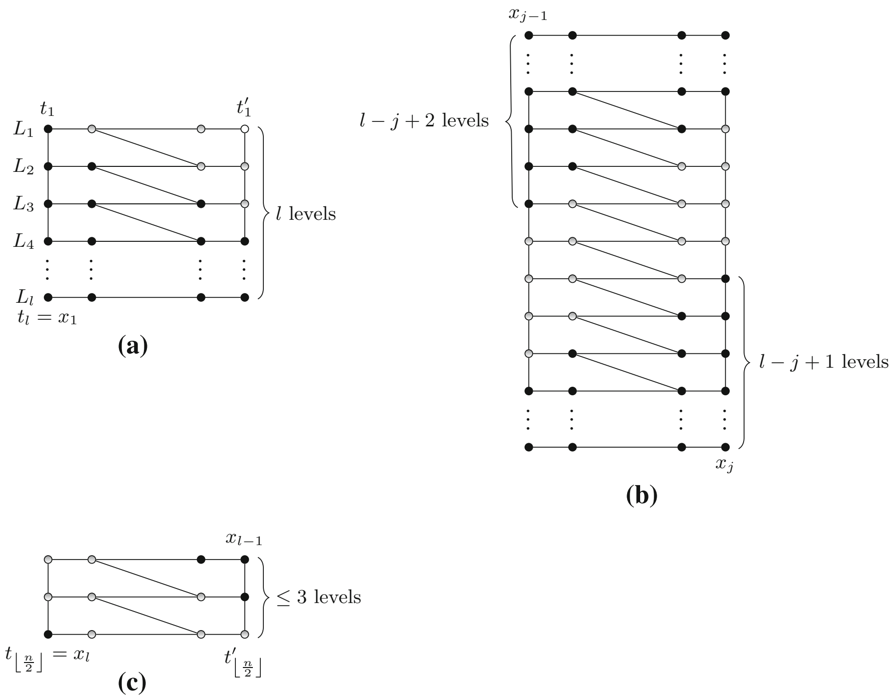

Fig. 7 Construction

Suppose $m_0 = l - 1$. Then, x_{l-1} is in level $L_{(2l-3)l-(l-2)^2}$ and x_l is in level $L_{\lfloor \frac{n}{2} \rfloor}$. Note that

$$(2l-3)l - (l-2)^2 + 2 = l^2 + l - 2 > \frac{n}{2} - 1 \geq \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

Hence, we have

$$(2l-3)l - (l-2)^2 + 2 \geq \left\lfloor \frac{n}{2} \right\rfloor.$$

Therefore,

$$L_{(2l-3)l-(l-2)^2} \cup L_{(2l-3)l-(l-2)^2+1} \cup \cdots \cup L_{\lfloor \frac{n}{2} \rfloor} \subseteq N_3[x_{l-1}] \cup N_2[x_l],$$

(see Fig. 7c).

If we set $x_{l+1} = t_1'$ and x_{l+2} to be any unburned vertex at time step $l+1$ (if possible), then $(x_1, x_2, \ldots, x_l, x_{l+1}, x_{l+2})$ is a burning sequence of $H(n)$ when n is even. If n is odd, it is also a burning sequence by noticing that $\{s_0, t_0\} \in N_{l+1}[x_1]$ (see Figs. 4 and 7a). This completes the proof of the lemma. □

The first part of Theorem 1.4 follows from Lemmas 4.1 and 4.2. Furthermore, if $\frac{n}{2}$ is a square, then $b(P(n, 2)) = \sqrt{\frac{n}{2}} + 2$. Finally, by Proposition 3.1, $b(P(3, 2)) = 3 = \left\lceil \sqrt{\frac{3}{2}} \right\rceil + 1$. So the bounds are tight. This completes the proof of Theorem 1.4. □

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