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Double Roman Domination in Digraphs

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Abstract Let D be a finite and simple digraph with vertex set $V(D)$. A double Roman dominating function (DRDF) on a digraph D is a function $f : V(D) \rightarrow \{0, 1, 2, 3\}$ satisfying the condition that if $f(v) = 0$, then the vertex v must have at least two in-neighbors assigned 2 under f or one in-neighbor assigned 3, while if $f(v) = 1$, then the vertex v must have at least one in-neighbor assigned 2 or 3. The weight of a DRDF f is the sum $\sum_{v \in V(D)} f(v)$. The double Roman domination number of a digraph D is the minimum weight of a DRDF on D. In this paper, we initiate the study of the double Roman domination of digraphs, and we give several relations between the double Roman domination number of a digraph and other domination parameters such as Roman domination number, k-domination number and signed domination number. Moreover, various bounds on the double Roman domination number of a digraph are presented, and a Nordhaus–Gaddum type inequality for the parameter is also given.

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1 Introduction

Due to the diversity of its applications to both theoretical and practical problems, domination and its variants have become one of the important research topics in graph theory (see, for example, [2,5,6,11,13]). Our aim in this paper is to initiate the study of the double Roman domination in digraphs.

Throughout this paper, $D = (V, A)$ is a finite digraph with neither loops nor multiple arcs (but pairs of opposite arcs are allowed). For two vertices $u, v \in V(D)$, we use (u, v) to denote the arc with direction from u to v, and we also call v an *out-neighbor* of u and u an *in-neighbor* of v. For $v \in V(D)$, the *out-neighborhood* and *in-neighborhood* of v, denoted by $N_D^+(v) = N^+(v)$ and $N_D^-(v) = N^-(v)$, are the sets of out-neighbors and in-neighbors of v, respectively. The *closed out-neighborhood* and *closed in-neighborhood* of a vertex $v \in V(D)$ are the sets $N_D^+[v] = N^+[v] = N^+(v) \cup \{v\}$ and $N_D^-[v] = N^-[v] = N^-(v) \cup \{v\}$, respectively. In general, for a set $X \subseteq V(D)$, we denote $N_D^+(X) = \bigcup_{v\in X} N_D^+(v)$ and $N_D^+[X] = \bigcup_{v\in X} N_D^+[v]$. The *out-degree* and *in-degree* of a vertex $v \in V(D)$ are defined by $d_D^+(v) = d^+(v) = |N_D^+(v)|$ and $d_D^-(v) = d^-(v) = |N_D^-(v)|$, respectively. The *maximum out-degree*, *maximum in-degree*, *minimum out-degree* and *minimum in-degree* among the vertices of D are denoted by $\Delta^+(D) = \Delta^+$, $\Delta^-(D) = \Delta^-$, $\delta^+(D) = \delta^+$ and $\delta^-(D) = \delta^-$, respectively. For two vertices u and v of D, the *distance* $d(u, v)$ *from* u *to* v is the length of a shortest directed u-v path in D. If D contains no directed u-v path, then $d(u, v) = \infty$. For a subdigraph H of D and $v \in V(D)$, the *distance from* H *to* v in D is $d(H, v) = \min\{d(u, v) : u \in V(H)\}$. Let $\overrightarrow{P}_n$ and $\overrightarrow{C}_n$ denote a directed path and a directed cycle of order n, respectively.

A *rooted tree* is a connected digraph with a vertex of in-degree 0, called the *root*, such that every vertex different from the root has in-degree 1. The *height* of a rooted tree T, denoted by $h(T)$, is $\max\{d(r, v) : v \in V(T)\}$, where r is the root of T. A digraph D is *contrafunctional* if each vertex of D has in-degree 1. The *complement* of a digraph D is the digraph $\overline{D}$, where $V(\overline{D}) = V(D)$ and $(u, v) \in A(\overline{D})$ if and only if $(u, v) \notin A(D)$.

A *k-dominating set* of a digraph D is a subset S of the vertex set of D such that every vertex not in S has at least k in-neighbors in S. The minimum cardinality of a k-dominating set of a digraph D is called the *k-domination number* of D and is denoted by $\gamma_k(D)$. A k-dominating set of D of cardinality $\gamma_k(D)$ is called a $\gamma_k(D)$-*set*. If $k = 1$, then the k-dominating set is exactly the dominating set and we simply write $\gamma(D)$ for $\gamma_1(D)$, which was introduced by Fu [7] and now has been studied extensively (see, for example, [4,8,9]).

A *signed dominating function* (abbreviated SDF) on D is a function $f : V(D) \to \{-1, 1\}$ such that $\sum_{x\in N^-[v]} f(x) \geq 1$ for each vertex $v \in V(D)$. The *weight* of an SDF f is $\omega(f) = \sum_{v\in V(D)} f(v)$. The *signed domination number* $\gamma_S(D)$ of a digraph

D is the minimum weight of an SDF on D. An SDF on D with weight $\gamma_S(D)$ is called a $\gamma_S(D)$-*function*. The signed domination number of a digraph was introduced by Zelinka [16] and has been studied by several authors, for example, in Karami et al. [12], Volkmann [15] and elsewhere.

A *Roman dominating function* (abbreviated RDF) on a digraph D is a function $f : V(D) \to \{0, 1, 2\}$ satisfying the condition that every vertex v with $f(v) = 0$ has an in-neighbor u with $f(u) = 2$. The *weight* of an RDF f is the sum $\omega(f) = \sum_{v \in V(D)} f(v)$. The *Roman domination number* of a digraph D, denoted by $\gamma_R(D)$, is the minimum weight of an RDF on D. A Roman dominating function on D with weight $\gamma_R(D)$ is called a $\gamma_R(D)$-*function*. An RDF f on D can be represented by the ordered partition (V_0, V_1, V_2), where $V_i = \{v \in V(D) : f(v) = i\}$ for $i \in \{0, 1, 2\}$. The Roman domination of a digraph has been studied by Sheikholeslami and Volkmann [14].

Let G be a finite, simple and undirected graph with vertex set $V(G)$. A *double Roman dominating function* (abbreviated DRDF) on a graph G is defined in [3] as a function $f : V(G) \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then the vertex v must be adjacent to at least two vertices assigned 2 under f or one vertex assigned 3, while if $f(v) = 1$, then the vertex v must be adjacent to at least one vertex assigned 2 or 3. The *weight* of a DRDF f is $\omega(f) = \sum_{v \in V(G)} f(v)$. The *double Roman domination number* $\gamma_{dR}(G)$ of a graph G is the minimum weight of a DRDF on G.

In this paper, motivated by the work in [3,14], we initiate the study of the double Roman domination number of digraphs. A *double Roman dominating function* (abbreviated DRDF) on a digraph D is a function $f : V(D) \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then the vertex v must have at least two in-neighbors assigned 2 under f or one in-neighbor assigned 3, while if $f(v) = 1$, then the vertex v must have at least one in-neighbor assigned 2 or 3. The *weight* of a DRDF f is $\omega(f) = \sum_{v \in V(D)} f(v)$. The *double Roman domination number* $\gamma_{dR}(D)$ of a digraph D is the minimum weight of a DRDF on D. A $\gamma_{dR}(D)$-*function* is a DRDF on D with weight $\gamma_{dR}(D)$. A DRDF f on D can be represented by the ordered partition (V_0, V_1, V_2, V_3), where $V_i = \{v \in V(D) : f(v) = i\}$ for $i \in \{0, 1, 2, 3\}$.

The rest of the paper is organized as follows. In the next section, we give two simple but useful properties of the double Roman domination number of a digraph. We then relate the double Roman domination number of digraphs to other domination parameters such as Roman domination number, k-domination number and signed domination number in Sect. 3. In Sect. 4, we establish lower and upper bounds on the double Roman domination number of a digraph in terms of its order, maximum out-degree and minimum in-degree. Finally, in Sect. 5 we present a Nordhaus–Gaddum result for the double Roman domination number of a digraph.

2 Preliminaries

In this section, we shall give two simple properties of the double Roman domination number of a digraph that will be useful in the next sections.

Proposition 1 *For any digraph D, there exists a $\gamma_{dR}(D)$-function such that no vertex needs to be assigned the value* 1.

Proof Let f be a $\gamma_{dR}(D)$-function. Suppose that there exists some vertex v of D such that $f(v) = 1$. Then by the definition of $\gamma_{dR}(D)$-function, we have that there exists a vertex $u \in N^-(v)$ such that either $f(u) = 2$ or $f(u) = 3$. If $f(u) = 3$, then we define the function f' by $f'(v) = 0$ and $f'(x) = f(x)$ for each $x \in V(D)\backslash\{v\}$. Obviously, f' is a DRDF of weight $\gamma_{dR}(D) - 1$, a contradiction. If $f(u) = 2$, then we define the function f'' by $f''(v) = 0$, $f''(u) = 3$ and $f''(x) = f(x)$ for each $x \in V(D)\backslash\{u, v\}$. Clearly, f'' is a DRDF of weight $\gamma_{dR}(D)$, implying that f'' is a $\gamma_{dR}(D)$-function. □

By Proposition 1, it is reasonable to claim that $V_1 = \emptyset$ for all double Roman dominating functions under consideration. In this case, any double Roman dominating function f on D can be represented by the ordered partition (V_0, V_2, V_3), where $V_i = \{v \in V(D) : f(v) = i\}$ for $i \in \{0, 2, 3\}$.

Proposition 2 *Let D be a digraph and let $f = (V_0, V_1, V_2)$ be a $\gamma_R(D)$-function. Then $\gamma_{dR}(D) \leq 2|V_1| + 3|V_2|$.*

Proof We set $g(v) = 0$ for each $v \in V_0$, $g(v) = 2$ for each $v \in V_1$ and $g(v) = 3$ for each $v \in V_2$. Then it is easy to see that g is a DRDF on D and hence $\gamma_{dR}(D) \leq \omega(g) = 2|V_1| + 3|V_2|$. □

3 Relations to Other Domination Parameters

In this section, we shall relate the double Roman domination number of digraphs to other domination parameters such as Roman domination number, k-domination number and signed domination number.

We first give some relations between the double Roman domination number and Roman domination number of digraphs.

Theorem 1 *For any digraph D, $\gamma_{dR}(D) \leq 2\gamma_R(D)$ with equality if and only if D is empty.*

Proof Let $f = (V_0, V_1, V_2)$ be a $\gamma_R(D)$-function such that $|V_1|$ is minimal. Then by Proposition 2,

$$\gamma_{dR}(D) \leq 2|V_1| + 3|V_2| = 2\gamma_R(D) - |V_2| \leq 2\gamma_R(D), \tag{1}$$

establishing the desired result.

The sufficiency is trivial. To show the necessity, let $\gamma_{dR}(D) = 2\gamma_R(D)$. Then we have equality throughout the inequality chain (1). Therefore, $|V_2| = 0$ and hence by the definition of $\gamma_R(D)$-function, $|V_0| = 0$. This implies that $V_1 = V(D)$. If there exists some arc, say (u, v), of D, then we set $g(v) = 0$, $g(u) = 2$ and $g(x) = 1$ for each $x \in V(D)\backslash\{u, v\}$. It is easy to see that g is an RDF on D with $\omega(g) = \gamma_R(D)$. This implies that g is a $\gamma_R(D)$-function. Moreover, $|x \in V(D) : f(x) = 1\}| - |x \in V(D) : g(x) = 1\}| = 2$, contradicting the minimality of f. Therefore, D is empty. □

Theorem 2 *For any digraph D, $\gamma_{dR}(D) \geq \gamma_R(D) + 1$.*

Proof Let $f = (V_0, V_2, V_3)$ be a $\gamma_{dR}(D)$-function. If $V_3 \neq \emptyset$, then every vertex in V_3 can be reassigned the value 2 and the resulting function will be an RDF on D and hence

$$\gamma_{dR}(D) = 2(|V_2| + |V_3|) + |V_3| \geq \gamma_R(D) + |V_3| \geq \gamma_R(D) + 1.$$

Suppose next that $V_3 = \emptyset$. Then by the definition of $\gamma_{dR}(D)$-function, $V_2 \neq \emptyset$, for otherwise, $V_0 = V(D)$, a contradiction. Therefore, all vertices are assigned either the value 0 or the value 2, and all vertices in V_0 must have at least two in-neighbors in V_2. In this case one vertex in V_2 can be reassigned the value 1 and the resulting function will be an RDF on D and hence $\gamma_{dR}(D) = 2|V_2| \geq \gamma_R(D) + 1$. □

Combining Theorems 1 and 2, we may obtain the following result immediately.

Corollary 1 *For any nontrivial connected digraph D,*

$$\gamma_R(D) + 1 \leq \gamma_{dR}(D) \leq 2\gamma_R(D) - 1.$$

We now establish a relation between the double Roman domination number and the domination number of digraphs.

Theorem 3 *For any digraph D,*

$$2\gamma(D) \leq \gamma_{dR}(D) \leq 3\gamma(D).$$

Moreover,

(a) The left equality holds if and only if $\gamma(D) = \gamma_2(D)$.
(b) The right equality holds if and only if there exists a $\gamma_{dR}(D)$-function (V_0, V_2, V_3) such that $V_2 = \emptyset$.

Proof Let $f = (V_0, V_2, V_3)$ be a $\gamma_{dR}(D)$-function and let S be a $\gamma(D)$-set. We set $f'(v) = 3$ for each $v \in S$ and $f'(v) = 0$ otherwise. Then it is easy to see that f' is a DRDF on D and hence $\gamma_{dR}(D) \leq 3|S| = 3\gamma(D)$. On the other hand, by the definition of $\gamma_{dR}(D)$-function, $V_2 \cup V_3$ is a dominating set of D and hence $\gamma(D) \leq |V_2| + |V_3|$, implying that

$$\gamma_{dR}(D) = 2|V_2| + 3|V_3| \geq 2(|V_2| + |V_3|) \geq 2\gamma(D). \quad (2)$$

(a) Suppose that $\gamma_{dR}(D) = 2\gamma(D)$. Then we have equality throughout the inequality chain (2). This means that $V_3 = \emptyset$ and hence $\gamma_{dR}(D) = 2|V_2|$, implying that $\gamma(D) = |V_2|$. Therefore, by the definition of $\gamma_{dR}(D)$-function, we have that every vertex in $V(D)\backslash V_2$ must have at least two in-neighbors in V_2. Thus, $\gamma_2(D) \leq |V_2| = \gamma(D)$. On the other hand, clearly $\gamma(D) \leq \gamma_2(D)$. As a result, $\gamma(D) = \gamma_2(D)$.

Conversely, suppose that $\gamma(D) = \gamma_2(D)$. Let S' be a $\gamma_2(D)$-set. We set $g(v) = 2$ for each $v \in S'$ and $g(v) = 0$ otherwise. Then clearly g is a DRDF on D and hence $\gamma_{dR}(D) \leq 2|S'| = 2\gamma_2(D) = 2\gamma(D)$. As proved previously, $\gamma_{dR}(D) \geq 2\gamma(D)$. Thus, $\gamma_{dR}(D) = 2\gamma(D)$.

(b) Suppose that $\gamma_{dR}(D) = 3\gamma(D)$. Set $g(v) = 3$ for each $v \in S$ and $g(v) = 0$ otherwise. Then clearly $\omega(g) = 3|S| = 3\gamma(D) = \gamma_{dR}(D)$, implying that g is a $\gamma_{dR}(D)$-function.

Conversely, suppose that $V_2 = \emptyset$. Then V_3 is a dominating set of D and hence $|V_3| \geq \gamma(D)$. Thus, $\gamma_{dR}(D) = 3|V_3| \geq 3\gamma(D)$. As proved earlier, $\gamma_{dR}(D) \leq 3\gamma(D)$. Therefore, $\gamma_{dR}(D) = 3\gamma(D)$. □

Proposition 3 *If D is a digraph, then $\gamma_{dR}(D) \leq 2\gamma_2(D)$.*

Proof Let S be a $\gamma_2(D)$-set. Define the function $f(x) = 2$ for $x \in S$ and $f(x) = 0$ otherwise. Then it is easy to verify that f is a DRDF on D and hence $\gamma_{dR}(D) \leq 2|S| = 2\gamma_2(D)$. □

If K_n^* is the complete digraph of order $n \geq 2$, then $\gamma(K_n^*) = 1$, $\gamma_R(K_n^*) = 2$ and $\gamma_{dR}(K_n^*) = 3$. Thus, Corollary 1 and the upper bound in Theorem 3 are sharp. In addition, let $u, v, x_1, x_2, \ldots, x_{n-2}$ be the vertex set of the digraph H such that $(u, x_i), (v, x_i) \in A(H)$ for $1 \leq i \leq n-2$. Then $\gamma(H) = 2$, $\gamma_2(H) = 2$ and $\gamma_{dR}(H) = 4$. This example shows that the lower bound in Theorem 3 and Proposition 3 is sharp.

We end this section by relating the double Roman domination number to signed domination number of digraphs. To this end, we need a result due to Ahangar et al. [1].

Let G be a bipartite (undirected) graph with bipartition $(\mathcal{L}, \mathcal{R})$ (standing for "left" and "right"). A subset S of vertices in $\mathcal{R}$ is a *left dominating set* of G if every vertex of $\mathcal{L}$ is adjacent to a vertex in S. The *left domination number*, denoted by $\gamma_{\mathcal{L}}(G)$, is the minimum cardinality of a left dominating set of G. A left dominating set of G of cardinality $\gamma_{\mathcal{L}}(G)$ is called a $\gamma_{\mathcal{L}}(G)$-set. Let $\delta_{\mathcal{L}}(G)$ denote the minimum degree of a vertex of $\mathcal{L}$ in G. Ahangar et al. [1] established the following upper bound on the left domination number of a bipartite (undirected) graph in terms of its order.

Theorem 4 ([1]) *Let G be a bipartite (undirected) graph of order n with bipartition $(\mathcal{L}, \mathcal{R})$. If $\delta_{\mathcal{L}}(G) \geq 2$, then $\gamma_{\mathcal{L}}(G) \leq n/3$.*

Theorem 5 *For any digraph D of order n,*

$$\gamma_{dR}(D) \leq \gamma_S(D) + 4n/3.$$

Proof Let f be a $\gamma_S(D)$-function and let $\mathcal{L}$ and $\mathcal{R}$ denote the sets of those vertices in D which are assigned under f the values -1 and 1, respectively. Then $|\mathcal{L}| + |\mathcal{R}| = n$ and $\gamma_S(D) = \omega(f) = |\mathcal{R}| - |\mathcal{L}|$, implying that $2|\mathcal{R}| = n + \gamma_S(D)$.

If $\mathcal{L} = \emptyset$, that is, if $\mathcal{R} = V(D)$, then we set $g(x) = 2$ for each $x \in V(D)$. Then it is easy to see that g is a DRDF on D, implying that

$$\gamma_{dR}(D) \leq \omega(g) = 2n = 2|\mathcal{R}| = 2\gamma_S(D) < \gamma_S(D) + 4n/3.$$

Hence we may assume that $\mathcal{L} \neq \emptyset$. Let D' be the bipartite spanning subdigraph of D with bipartition $(\mathcal{L}, \mathcal{R})$, where $A(D') = \{(u, v) \in A(D) : u \in \mathcal{R} \text{ and } v \in \mathcal{L}\}$. Since f is a $\gamma_S(D)$-function, each vertex of $\mathcal{L}$ has at least 2 in-neighbors in $\mathcal{R}$ in D' and hence $\delta^-_{\mathcal{L}}(D') \geq 2$, where $\delta^-_{\mathcal{L}}(D') = \min\{d^-_{D'}(v) : v \in \mathcal{L}\}$. Let H be the (undirected) graph obtained from D' by replacing any arc with an edge and let $\mathcal{R}_2$ be a $\gamma_{\mathcal{L}}(H)$-set. Then $\delta_{\mathcal{L}}(H) = \delta^-_{\mathcal{L}}(D') \geq 2$ and hence by Theorem 4, $|\mathcal{R}_2| = \gamma_{\mathcal{L}}(H) \leq n/3$. Moreover, since $\mathcal{R}_2$ is a $\gamma_{\mathcal{L}}(H)$-set, any vertex in $\mathcal{L}$ is adjacent to some vertex in $\mathcal{R}_2$ in H and hence any vertex in $\mathcal{L}$ has at least one in-neighbor in $\mathcal{R}_2$ in D' and so in D. Let $\mathcal{R}_1 = \mathcal{R} \backslash \mathcal{R}_2$. Set

$$g'(x) = \begin{cases} 0, & \text{if } x \in \mathcal{L}, \\ 2, & \text{if } x \in \mathcal{R}_1, \\ 3, & \text{if } x \in \mathcal{R}_2. \end{cases}$$

Then g' is a DRDF on D and hence

$$\begin{aligned} \gamma_{dR}(D) &\leq \omega(g') = 2|\mathcal{R}_1| + 3|\mathcal{R}_2| \\ &= 2(|\mathcal{R}_1| + |\mathcal{R}_2|) + |\mathcal{R}_2| = 2|\mathcal{R}| + |\mathcal{R}_2| \\ &= n + \gamma_S(D) + |\mathcal{R}_2| \\ &\leq \gamma_S(D) + 4n/3, \end{aligned}$$

which completes the proof. □

4 Upper and lower bounds

Our aim in the section is to establish upper and lower bounds on the double Roman domination number of a digraph in term of its order, maximum out-degree and minimum in-degree.

We first present upper bounds on the double Roman domination number of digraphs.

Proposition 4 *If D is a digraph of order n, then $\gamma_{dR}(D) \leq 2n$ with equality if and only if D is empty.*

Proof Define the function f by $f(x) = 2$ for each $x \in V(D)$. Then f is a DRDF on D and hence $\gamma_{dR}(D) \leq 2n$. If D is empty, then $\gamma_{dR}(D) = 2n$. Now assume that $\gamma_{dR}(D) = 2n$, and suppose to the contrary that D contains an arc (u, v). Define the function $g(v) = 0$, $g(u) = 3$ and $g(x) = 2$ for each $x \in V(D) \setminus \{u, v\}$. Then g is a DRDF on D of weight $2n - 1$, a contradiction. □

Theorem 6 *Let D be a digraph of order $n \geq 2$ such that $|A(D)| \geq 1$. Then $\gamma_{dR}(D) \leq 2n - 1$ with equality if and only if D has exactly one nontrivial component of order 2 or one nontrivial component H of order 3 such that H is a directed path or a directed cycle.*

Proof Proposition 4 implies $\gamma_{dR}(D) \leq 2n - 1$. If D has exactly one nontrivial component of order 2 or one nontrivial component H of order 3 such that H is a directed path or a directed cycle, then it is easy to see that $\gamma_{dR}(D) = 2n - 1$. Conversely, assume that $\gamma_{dR}(D) = 2n - 1$. Suppose that D contains two arcs (u, v) and (w, z).

If $u \neq w, z$ and $v \neq w, z$, then define f by $f(v) = f(z) = 0$, $f(u) = f(w) = 3$ and $f(x) = 2$ for each $x \in V(D)\setminus\{u, v, w, z\}$. Then f is a DRDF on D of weight $2n - 2$, a contradiction. Therefore, D contains at most one nontrivial component H.

If $v = z$ and $u \neq w$, then define g by $g(v) = 0$ and $g(x) = 2$ otherwise. Then g is a DRDF on D of weight $2n - 2$, a contradiction.

If $u = w$ and $v \neq z$, then define h by $h(u) = 3$, $h(v) = h(z) = 0$ and $h(x) = 2$ otherwise. Then h is a DRDF on D of weight $2n - 3$, a contradiction.

Using these observations, we deduce that $2 \leq |V(H)| \leq 3$, and if $|V(H)| = 3$, then H is a directed path or a directed cycle of order 3. □

Proposition 5 *Let T be a rooted tree with $h(T) = 1$. Then $\gamma_{dR}(T) = 3$.*

Proof Let r be the root of T. We set $f(r) = 3$ and $f(u) = 0$ for each $u \in V(D)\setminus\{r\}$. Then it is easy to see that f is a $\gamma_{dR}(D)$-function and hence $\gamma_{dR}(T) = \omega(f) = 3$. □

Theorem 7 *Let $T \ncong \overrightarrow{P}_3$ be a rooted tree of order $n \geq 2$. Then*

$$\gamma_{dR}(T) \leq (5n - 1)/3.$$

Proof We proceed by induction on n. If $n = 2$, then by Proposition 5, $\gamma_{dR}(T) = 3 = (5n - 1)/3$. Hence we may assume that $n \geq 3$. If $h(T) = 1$, then again by Proposition 5, $\gamma_{dR}(T) = 3 \leq (5n - 1)/3$.

Suppose next that $h(T) \geq 2$. Let r be the root of T; let x be a vertex of T such that $d(r, x) = h(T) - 1$; let T_1 be the connected component of $T - x$ that contains the root r and let $T_2 = T - T_1$. Note that $h(T_2) = 1$. Therefore, by Proposition 5, $\gamma_{dR}(T_2) = 3 \leq (5|V(T_2)| - 1)/3$. If $|V(T_1)| = 1$, then clearly $|V(T_2)| \geq 3$ since $T \ncong \overrightarrow{P}_3$ and (V_0, V_2, V_3) is a DRDF on D, where $V_3 = \{x\}$, $V_2 = \{r\}$ and $V_0 = V(D)\setminus\{r, x\}$, and hence $\gamma_{dR}(T) \leq 3 + 2 \leq (5n - 1)/3$. Assume next that $|V(T_1)| \geq 2$. If $T_1 \ncong \overrightarrow{P}_3$, then by the induction hypothesis, $\gamma_{dR}(T_1) \leq (5|V(T_1)| - 1)/3$ and hence

$$\begin{aligned}\gamma_{dR}(T) &\leq \gamma_{dR}(T_1) + \gamma_{dR}(T_2)\\ &\leq (5|V(T_1)| - 1)/3 + (5|V(T_2)| - 1)/3\\ &< (5n - 1)/3.\end{aligned}$$

If $T_1 \cong \overrightarrow{P}_3$, then $\gamma_{dR}(T_1) = 5 = 5|V(T_1)|/3$ and hence

$$\begin{aligned}\gamma_{dR}(T) &\leq \gamma_{dR}(T_1) + \gamma_{dR}(T_2)\\ &\leq 5|V(T_1)|/3 + (5|V(T_2)| - 1)/3\\ &= (5n - 1)/3,\end{aligned}$$

which completes our proof. □

Harary et al. [10] showed that every connected contrafunctional digraph D has a unique directed cycle and the removal of any arc of the directed cycle results in a rooted tree T. Therefore, we have $\gamma_{dR}(D) \leq \gamma_{dR}(T)$, which, together with Theorem 7, would yield the following result directly.

Corollary 2 *Let D be a connected contrafunctional digraph of order n. Then $\gamma_{dR}(D) = 5$ if $D \cong \overrightarrow{C}_3$, and $\gamma_{dR}(D) \leq (5n-1)/3$ otherwise.*

For a special class of contrafunctional digraphs, we will improve Corollary 2 slightly. For this purpose, we define the *height* of a connected contrafunctional digraph D, denoted by $h(D)$, to be the maximum distance from its unique directed cycle C to all vertices of D, i.e., $h(D) = \max\{d(C, v) : v \in V(D)\}$. In particular, the height of a directed cycle is exactly equal to 0.

Theorem 8 *Let D be a connected contrafunctional digraph of order n with $h(D) = 1$. Then $\gamma_{dR}(D) \leq 3n/2$.*

Proof Let C be the unique directed cycle of D, v_{i_j} be the vertex set of C such that v_{i_j} has at least one out-neighbor not in C for $1 \leq j \leq t$, and let V' be the set of out-neighbors of v_{i_j} not in C for $1 \leq j \leq t$. Define the function f by $f(v_{i_j}) = 3$ for $1 \leq j \leq t$ and $f(x) = 0$ for each $x \in V'$. We observe that $D' = D \backslash (\{v_{i_1}, v_{i_2}, \ldots, v_{i_t}\} \cup V')$ is empty or consists of some directed paths. If $w_1 w_2 \ldots w_k$ is such a directed path of D', then for $1 \leq i \leq k$, we define $f(w_i) = 0$ if i is odd and $f(w_i) = 3$ if i is even. Altogether, it is easy to verify that f is a DRDF on D of weight $\omega(f) \leq 3n/2$. Therefore, $\gamma_{dR}(D) \leq \omega(f) \leq 3n/2$. □

Theorem 9 *Let $D \ncong \overrightarrow{C}_3$ be a connected digraph of order $n \geq 3$ with $\delta^-(D) \geq 1$. Then*

$$\gamma_{dR}(D) \leq (5n-1)/3.$$

Proof If $n = 3$, then it is easy to see that $\gamma_{dR}(D) = 3 \leq (5n-1)/3$ since $D \ncong \overrightarrow{C}_3$. Hence we may assume that $n \geq 4$. Since $\delta^-(D) \geq 1$, we can choose an arbitrary incoming arc of v for each vertex v of D. Then all such arcs induce a spanning subdigraph H of D consisting of some connected components, say $H_1, H_2, \ldots, H_l$. Moreover, H_i ($i \in \{1, 2, \ldots, l\}$) is a connected contrafunctional subdigraph of D since each vertex of H_i has in-degree 1.

Firstly, we consider the case that H is not the disjoint union of copies of $\overrightarrow{C}_3$. Without loss of generality, assume that $H_1 \ncong \overrightarrow{C}_3$. Then by Corollary 2, we have $\gamma_{dR}(H_1) \leq (5|V(H_1)| - 1)/3$ and $\gamma_{dR}(H_i) \leq 5|V(H_i)|/3$ for each $i \in \{2, 3, \ldots, l\}$. Therefore,

$$\begin{aligned}\gamma_{dR}(D) &\leq \gamma_{dR}(H) = \sum_{i=1}^{l} \gamma_{dR}(H_i) \\ &\leq (5|V(H_1)| - 1)/3 + \sum_{i=2}^{l} 5|V(H_i)|/3 \\ &= (5n-1)/3.\end{aligned}$$

Next, we consider the case that H is the disjoint union of copies of $\overrightarrow{C}_3$. This implies that $H_i \cong \overrightarrow{C}_3$ for $i \in \{1, 2, \ldots, l\}$. Note that $n \geq 4$. Therefore, $l \geq 2$. Since D is connected but H is not, the arc set $A(D)$ of D consists of $A(H)$ and some arcs not in H. In addition, if we add some arc in $A(D)\backslash A(H)$ to H, then it is easy to verify that the resulting digraph has a strictly smaller double Roman domination number than that of H. Therefore, by Corollary 2, we have

$$\begin{aligned}\gamma_{dR}(D) &\leq \gamma_{dR}(H) - 1\\ &= \sum_{i=1}^{l} \gamma_{dR}(H_i) - 1\\ &= \sum_{i=1}^{l} 5|V(H_i)|/3 - 1\\ &\leq (5n-1)/3,\end{aligned}$$

which completes our proof. □

Theorem 10 *For any digraph D of order n, $\gamma_{dR}(D) \leq 2(n - \Delta^+) + 1$.*

Proof Let v be a vertex of out-degree Δ^+. Then it is easy to see that $f = (V_0, V_2, V_3)$ is a DRDF on D, where $V_0 = N^+(v)$, $V_2 = V(D)\backslash N^+[v]$ and $V_3 = \{v\}$. Thus, $\gamma_{dR}(D) \leq \omega(f) = 3 + 2(n - d^+(v) - 1) = 2(n - \Delta^+) + 1$. □

Theorem 11 *For any digraph D of order n with $\delta^- \geq 1$,*

$$\gamma_{dR}(D) \leq n\left\{3 - 3\left(\frac{3}{2(1+\delta^-)}\right)^{\frac{1}{\delta^-}} + 2\left(\frac{3}{2(1+\delta^-)}\right)^{\frac{1+\delta^-}{\delta^-}}\right\}.$$

Proof Given a digraph D and a real number p with $0 \leq p \leq 1$, select a set X of vertices each of which is selected independently with probability p (with p to be defined later). Then the expected size of X is np since X admits the binomial distribution with parameters n and p. Let $Y = V(D)\backslash N_D^+[X]$. We set $f(v) = 3$ for any $v \in X$, $f(v) = 2$ for any $v \in Y$ and $f(v) = 0$ otherwise. Then it is easy to see that f is a DRDF on D. Note that

$$\begin{aligned}P(v \in Y) &= P(v \in V(D)\backslash N_D^+[X])\\ &= (1-p)^{1+d^-(v)}\\ &\leq (1-p)^{1+\delta^-}.\end{aligned}$$

Thus,

$$\mathbf{E}(\omega(f)) \leq 3np + 2n(1-p)^{1+\delta^-},$$

where $\mathbf{E}(\omega(f))$ is the expected weight of f. It is not difficult to verify that the upper bound for $\mathbf{E}(\omega(f))$ is minimum when $p = 1 - \left(\frac{3}{2(1+\delta^-)}\right)^{\frac{1}{\delta^-}}$ and hence

$$\mathbf{E}(\omega(f)) \leq n \left\{3 - 3\left(\frac{3}{2(1+\delta^-)}\right)^{\frac{1}{\delta^-}} + 2\left(\frac{3}{2(1+\delta^-)}\right)^{\frac{1+\delta^-}{\delta^-}}\right\}.$$

This implies that there must be some DRDF on D with at most the above bound as its weight, which completes our proof. □

We next give a lower bound on the double Roman domination number of a digraph.

Theorem 12 *For any connected digraph D of order $n \geq 4$,*

$$\gamma_{dR}(D) \geq \left\lceil \frac{6n+3}{2\Delta^+ + 3} \right\rceil.$$

Proof Let $f = (V_0, V_2, V_3)$ be a $\gamma_{dR}(D)$-function and let $n_0 = |V_0|$. If $V_3 = \emptyset$, then it is easy to see that $\gamma_{dR}(D) = 2|V_2| = 2(n - n_0)$. Moreover, by the definition of $\gamma_{dR}(D)$-function, each $v \in V_0$ must have at least two in-neighbors assigned 2 under f and hence $\sum_{u \in N^-(v)} f(u) \geq 4$. Thus, $\gamma_{dR}(D) = \omega(f) \geq n_0 \cdot \frac{4}{\Delta^+}$. Then it follows that

$$2\gamma_{dR}(D) = 4n - 4n_0 \geq 4n - \gamma_{dR}(D)\Delta^+$$

and hence

$$\gamma_{dR}(D)(\Delta^+ + 2) \geq 4n,$$

implying that $\gamma_{dR}(D) \geq \left\lceil \frac{4n}{\Delta^+ + 2} \right\rceil \geq \left\lceil \frac{6n+3}{2\Delta^+ + 3} \right\rceil$.

If $V_3 \neq \emptyset$, then it is easy to see that $\gamma_{dR}(D) = 2(|V_2| + |V_3|) + |V_3| \geq 2(n - n_0) + 1$. Moreover, by the definition of $\gamma_{dR}(D)$-function, each $v \in V_0$ must have at least two in-neighbors assigned 2 under f or one in-neighbor assigned 3, and hence $\sum_{u \in N^-(v)} f(u) \geq 3$. Thus, $\gamma_{dR}(D) = \omega(f) \geq n_0 \cdot \frac{3}{\Delta^+}$. Then it follows that

$$3\gamma_{dR}(D) \geq 6n - 6n_0 + 3 \geq 6n - 2\gamma_{dR}(D)\Delta^+ + 3$$

and hence

$$\gamma_{dR}(D)(2\Delta^+ + 3) \geq 6n + 3,$$

implying that $\gamma_{dR}(D) \geq \left\lceil \frac{6n+3}{2\Delta^+ + 3} \right\rceil$. □

The following result, derived from Theorem 12, shows that the upper bound in Theorem 10 and the lower bound in Theorem 12 are sharp.

Corollary 3 *Let D be a connected digraph of order $n \geq 3$. Then $\gamma_{dR}(D) = 3$ if and only if $\Delta^+ = n - 1$.*

Proof Clearly, by the definition, $\gamma_{dR}(D) \geq 3$. Now let $\Delta^+ = n - 1$, and let v be a vertex of out-degree Δ^+. Define the function f by $f(v) = 3$ and $f(x) = 0$ for $x \in V(D) \setminus \{v\}$. Then f is a DRDF on D of weight 3 and hence $\gamma_{dR}(D) \leq 3$ and thus $\gamma_{dR}(D) = 3$. Conversely, assume that $\gamma_{dR}(D) = 3$. If $\Delta^+ \leq n-2$, then Theorem 12 leads to the contradiction

$$\gamma_{dR}(D) \geq \left\lceil \frac{6n+3}{2\Delta^+ + 3} \right\rceil \geq \left\lceil \frac{6n+3}{2(n-2)+3} \right\rceil \geq 4,$$

which completes our proof. □

5 A Nordhaus–Gaddum type result

In this section, we derive a Nordhaus–Gaddum bound on the double Roman domination number of digraphs.

Theorem 13 *For any digraph D of order $n \geq 4$,*

$$\gamma_{dR}(D) + \gamma_{dR}(\overline{D}) \leq 2n + 3.$$

Proof It is easy to see that $d_D^+(v) + d_{\overline{D}}^+(v) = n - 1$ for any vertex $v \in V(D)$. This implies that $\Delta^+(\overline{D}) = n - 1 - \delta^+(D)$. Then by Theorem 10, we have

$$\begin{aligned}\gamma_{dR}(D) + \gamma_{dR}(\overline{D}) &\leq (2n - 2\Delta^+(D) + 1) + (2n - 2\Delta^+(\overline{D}) + 1)\\ &= 2n - 2\Delta^+(D) + 2\delta^+(D) + 4\\ &\leq 2n + 4. \end{aligned} \tag{3}$$

Suppose that $\gamma_{dR}(D) + \gamma_{dR}(\overline{D}) = 2n + 4$. Then we have equality throughout the inequality chain (3). This implies that $\Delta^+(D) = \delta^+(D)$. Let $k = \Delta^+(D) = \delta^+(D)$. Then $\Delta^+(\overline{D}) = \delta^+(\overline{D}) = n - 1 - k$. Without loss of generality, we may assume that $k \leq (n-1)/2$, since our argument is symmetric in D and $\overline{D}$. Since equality holds, $\gamma_{dR}(D) = 2(n-k) + 1$ and $\gamma_{dR}(\overline{D}) = 2k + 3$. Let $v \in V(D)$.

Claim 1 *All of the out-neighbors of every vertex not in $N_D^+[v]$ are in $N_D^+[v]$.*

Proof of Claim 1 If some vertex u outside $N_D^+[v]$ in D has at least one out-neighbor, say w, outside $N_D^+[v]$, then set $f(v) = 3$, $f(w) = 1$, $f(x) = 0$ for $x \in N_D^+(v)$ and $f(x) = 2$ otherwise. Clearly, f is a DRDF on D with weight $2(n-k)$, a contradiction to the fact that $\gamma_{dR}(D) = 2(n-k) + 1$. So, this claim is true.

Claim 2 *For any vertex u outside $N_D^+[v]$, $(u, v) \in A(D)$.*

Proof of Claim 2 Suppose, to the contrary, that there exists some vertex u outside $N_D^+[v]$ such that $(u, v) \notin A(D)$. Note that $\Delta^+(D) = \delta^+(D) = k$. Hence by Claim 1, $N_D^+(u) = N_D^+(v)$. We set $g(x) = 0$ for any $x \in N_D^+(v)$ and $g(x) = 2$ otherwise. Then g is a DRDF on D with weight $2(n-k)$, a contradiction. So, this claim is true.

Claim 3 *There exists at most one vertex outside* $N_D^+[v]$.

Proof of Claim 3 Suppose, to the contrary, that there exist at least two vertices, say u and w, outside $N_D^+[v]$. Then by Claim 2, we have that $(u, v), (w, v) \in A(D)$. Note that $\Delta^+(D) = \delta^+(D) = k$. Hence by Claim 1, $N_D^+(u) = N_D^+(w)$ or $|N_D^+(v) \setminus (N_D^+(u) \cap N_D^+(w))| = 2$. Let $N_D^+(v) = \{v_1, v_2, \ldots, v_k\}$.

If $N_D^+(u) = N_D^+(w)$, then we assume, without loss of generality, that $N_D^+(u) = N_D^+(w) = \{v, v_1, v_2, \ldots, v_{k-1}\}$. We set $h(x) = 0$ for any $x \in \{v, v_1, v_2, \ldots, v_{k-1}\}$ and $h(x) = 2$ otherwise. It is easy to see that h is a DRDF on D with weight $2(n-k)$, a contradiction. If $|N_D^+(v) \setminus (N_D^+(u) \cap N_D^+(w))| = 2$, then we assume, without loss of generality, that $N_D^+(u) = \{v, v_1, v_2, \ldots, v_{k-1}\}$ and $N_D^+(w) = \{v, v_2, v_3, \ldots, v_k\}$. Set $h'(v_1) = h'(v_k) = 1$, $h'(x) = 0$ for any $x \in \{v, v_2, v_3, \ldots, v_{k-1}\}$ and $h'(x) = 2$ otherwise. It is easy to see that h' is a DRDF on D with weight $2(n-k)$, a contradiction. So, this claim is true.

Thus, by Claim 3, we have $k = d_D^+(v) \geq n-2$ for any $v \in V(D)$. Together with our earlier assumptions, $k \leq (n-1)/2$. Therefore, $n - 2 \leq k \leq (n-1)/2$ and hence $n \leq 3$, a contradiction. This implies that $\gamma_{dR}(D) + \gamma_{dR}(\overline{D}) \leq 2n + 3$, which completes our proof. □

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