Bull. Malays. Math. Sci. Soc. (2019) 42:1921–1935
https://doi.org/10.1007/s40840-017-0580-y

BULLETIN of the MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

# Bi-Integrable Couplings Associated with so(3, $\mathbb{R}$)

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Received: 10 April 2017 / Revised: 10 November 2017 / Published online: 29 November 2017


**Abstract** By a class of zero curvature equations over a non-semisimple matrix loop algebra, we generate a new hierarchy of bi-integrable couplings for a soliton hierarchy associated with so(3, $\mathbb{R}$). The bi-Hamiltonian structures are found by the associated variational identity, which imply that all the presented coupling systems possess infinitely many commuting symmetries and conserved functionals and, thus, are Liouville integrable.

**Keywords** Integrable coupling · Matrix loop algebra · Hamiltonian structure

Communicated by Ahmad Izani Md. Ismail.

This research was supported by the University of South Florida.

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**Mathematics Subject Classification** 37K05 · 37K10 · 35Q53

## 1 Introduction

The study of solitons in regard to integrable systems has facilitated a deeper understanding of mathematics and physics. Many well-known nonlinear partial differential equations have been found to have soliton solutions, for example, the Korteweg–de Vries equation and the sine-Gordon equation. It is known that zero curvature equations associated with simple Lie algebras generate classical integrable systems [1], and semisimple Lie algebras generate non-coupled systems of classical integrable systems. It is our business to further develop the study of non-semisimple Lie algebras in relation to integrable couplings. Soliton hierarchies, and specifically, integrable couplings and bi-integrable couplings, provide valuable new insights into the classification of multi-component integrable systems [2–6].

It is known that zero curvature equations on semidirect sums of matrix loop algebras generate integrable couplings [7,8], and the associated variational identity [9,10] is used to furnish Hamiltonian structures and bi-Hamiltonian structures of the resulting integrable couplings and bi-integrable couplings [11–17]. An important step in generating Hamiltonian structures is to search for non-degenerate, symmetric, and ad-invariant bilinear forms on the underlying loop algebras [13,18] as the trace identity proposed by Gui-Zhang Tu [18,19] is ineffective for non-semisimple Lie algebras which possess a degenerate Killing form. Semidirect sums of loop algebras bring various interesting integrable couplings and bi-integrable couplings [20–24], including higher-dimensional local bi-Hamiltonian integrable couplings [25–29], greatly enriching multi-component integrable systems. Recently, it has been of interest to study new integrable couplings and bi-integrable couplings generated from spectral problems associated with so$(3, \mathbb{R})$ [14].

Integrable couplings enlarge an original integrable system and often times retain its properties [2,4]. Bi-integrable couplings then take the integrable coupling system and enlarge that system. Again, the original properties frequently are maintained. An important feature is if a soliton hierarchy has infinitely many commuting symmetries and conserved densities, the integrable coupling and then bi-integrable coupling generally will too [14–17,30,31]. A bi-integrable coupling system is a natural way of extending a well-behaved integrable system. We show that the bi-integrable couplings of an original spectral problem associated with so$(3, \mathbb{R})$ will preserve bi-Hamiltonian structures, i.e., Liouville integrability, of the integrable couplings associated with the same spectral problem [32].

A zero curvature representation of a system of the form

$$u_t = K(u) = K(x, t, u, u_x, u_{xx}, \ldots), \tag{1}$$

where $u$ is a column vector of dependent variables and means there exists a Lax pair [33] $U = U(u, \lambda)$ and $V = V(u, \lambda)$ in a matrix loop algebra such that the zero curvature equation,

$$U_t - V_x + [U, V] = 0, \tag{2}$$

will generate system (1) [19]. The integrable coupling of system (1) is an integrable system of the form ([25,26] for definition):

$$\bar{u}_t = \bar{K}_1(\bar{u}) = \begin{bmatrix} K(u) \\ S(u, u_1) \end{bmatrix}, \qquad \bar{u} = \begin{bmatrix} u \\ u_1 \end{bmatrix}, \tag{3}$$

where $u_1$ is a new column vector of dependent variables. An integrable system of the form

$$\bar{u}_t = \bar{K}_1(\bar{u}) = \begin{bmatrix} K(u) \\ S_1(u, u_1) \\ S_2(u, u_1, u_2) \end{bmatrix}, \qquad \bar{u} = \begin{bmatrix} u \\ u_1 \\ u_2 \end{bmatrix}, \tag{4}$$

is called a bi-integrable coupling of (1). Note that in (4), $S_2$ depends on $u_2$, but $S_1$ does not. Now, we use zero curvature equations in order to generate bi-integrable couplings and associated Hamiltonian structures, through appropriate variational identities.

We will proceed with Sects. 2 through 6. In Sect. 2, we recall a soliton hierarchy presented in [32] for a matrix spectral problem in so(3, ℝ). In Sect. 3, we construct bi-integrable couplings from the results in Sect. 2 using an enlarged matrix loop algebra. We then use the corresponding variational identity to present the Hamiltonian structure of the bi-integrable coupling system in Sect. 4. In Sect. 5, infinitely many symmetries and conserved functionals are discussed. We finish the paper with a couple open questions.

## 2 A Soliton Hierarchy Associated with so(3, ℝ)

Let us recall the a soliton hierarchy [32] given by the spectral problem

$$\phi_x = U\phi\ , \quad U = U(u, \lambda) = \begin{bmatrix} 0 & q & \lambda \\ -q & 0 & -p \\ -\lambda & p & 0 \end{bmatrix} \in \bar{so}(3), \tag{5}$$

where

$$u = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix},$$

$\lambda$ is a spectral parameter, $p = p(x, t)$, $q = q(x, t)$, and $\bar{so}(3)$ is the special matrix loop algebra, i.e.,

$$\bar{\mathfrak{g}} = \bar{so}(3) = \{A \in so(3) | \text{entries of } A \text{ are Laurent series in } \lambda\}\,. \tag{6}$$

Under the assumption that W is of the form

$$W = \begin{bmatrix} 0 & c & a \\ -c & 0 & -b \\ -a & b & 0 \end{bmatrix} = \sum_{i\geq 0} \begin{bmatrix} 0 & c_i & a_i \\ -c_i & 0 & -b_i \\ -a_i & b_i & 0 \end{bmatrix} \lambda^{-i} = \sum_{i\geq 0} W_i \lambda^{-i}, \tag{7}$$

then the stationary zero curvature equation,

$$W_x = [U, W], \tag{8}$$

determines the system of equations

$$\begin{cases} a_x = pc - qb, \\ b_x = -\lambda c + qa, \\ c_x = -pa + \lambda b. \end{cases} \tag{9}$$

After setting $a, b, c$ to appropriate Laurent expansions, system (9) equivalently generates

$$\begin{cases} b_{i+1} = pa_i + c_{i,x}, \\ c_{i+1} = -b_{i,x} + qa_i, \qquad i \geq 0. \\ a_{i+1,x} = pc_{i+1} - qb_{i+1}, \end{cases} \tag{10}$$

Next, we set the initial conditions as $\{a_0 = -1,\ b_0 = 0 = c_0\}$ and take all constants of integration to be zero. We can present for $1 \leq i \leq 4$:

$$\begin{aligned}
&a_1 = 0, \quad c_1 = -q, \quad b_1 = -p, \\
&a_2 = \frac{1}{2}(p^2 + q^2), \quad c_2 = p_x, \quad b_2 = -q_x, \\
&a_3 = pq_x - p_x q, \quad c_3 = q_{xx} + \frac{1}{2}p^2 q + \frac{1}{2}q^3, \quad b_3 = p_{xx} + \frac{1}{2}p^3 + \frac{1}{2}pq^2, \\
&a_4 = -\frac{3}{4}p^2q^2 - \frac{3}{8}p^4 + \frac{1}{2}p_x^2 - pp_{xx} - \frac{3}{8}q^4 + \frac{1}{2}q_x^2 - qq_{xx}, \\
&b_4 = q_{xxx} + \frac{1}{2}(3p^2 + 3q^2)q_x, \quad c_4 = -p_{xxx} - \frac{1}{2}(3p^2 + 3q^2)p_x.
\end{aligned}$$

All functions $\{a_i, b_i, c_i | i \geq 0\}$ are differential polynomials of $u$ with respect to $x$.

The zero curvature equations are

$$U_{t_m} - V_x^{[m]} + \left[U, V^{[m]}\right] = 0 \quad \text{with} \quad V^{[m]} = (\lambda^m W)_+, \tag{11}$$

where $m \geq 0$, and, therefore, provide a hierarchy of soliton equations, i.e.,

$$u_{t_m} = K_m = \begin{bmatrix} -c_{m+1} \\ b_{m+1} \end{bmatrix} = \Phi^m \begin{bmatrix} q \\ -p \end{bmatrix} = J\frac{\delta \mathcal{H}_m}{\delta u}, \tag{12}$$

where $m \geq 0$. The Hamiltonian operator $J$, the hereditary recursion operator $\Phi$, and the Hamiltonian functions are defined as follows:

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \Phi = \begin{bmatrix} q\partial^{-1}p & \partial + q\partial^{-1}q \\ -\partial - p\partial^{-1}p & -p\partial^{-1}q \end{bmatrix}, \quad \mathcal{H}_m = \int -\frac{a_{m+2}}{m+1}\,\mathrm{d}x, \tag{13}$$

in which $m \geq 0$ and $\partial = \frac{\partial}{\partial x}$. The first nonlinear example is

$$u_{t_2} = K_2 = \begin{bmatrix} -q_{xx} - \frac{1}{2}p^2 q - \frac{1}{2}q^3 \\ p_{xx} + \frac{1}{2}p^3 + \frac{1}{2}pq^2 \end{bmatrix} = J \begin{bmatrix} p_{xx} + \frac{1}{2}p^3 + \frac{1}{2}pq^2 \\ q_{xx} + \frac{1}{2}p^2 q + \frac{1}{2}q^3 \end{bmatrix} = J\frac{\delta \mathcal{H}_2}{\delta u}. \quad (14)$$

## 3 Bi-Integrable Couplings

We construct Hamiltonian bi-integrable couplings for the soliton hierarchy by using a matrix loop Lie algebra. Define a triangular block matrix

$$M(A_1, A_2, A_3) = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & A_1 & \alpha A_2 \\ 0 & 0 & A_1 \end{bmatrix}. \quad (15)$$

It is known that block matrices of this form are closed under multiplication, i.e., constitute a Lie algebra [34]. The associated loop matrix Lie algebra $\tilde{\mathfrak{g}}(\lambda)$ is formed by all block matrices of the type

$$\tilde{\mathfrak{g}}(\lambda) = \{M(A_1, A_2, A_3) | M \text{ defined by (15), entries of A are Laurent series in } \lambda\}. \quad (16)$$

A spectral matrix is chosen from $\tilde{\mathfrak{g}}(\lambda)$ as

$$\bar{U} = \bar{U}(\bar{u}, \lambda) = M(U, U_1, U_2), \qquad \bar{u} = (p, q, r, s, v, w)^{\mathrm{T}}, \quad (17)$$

where $U$ is defined as in (5) and the supplementary spectral matrices $U_1$ and $U_2$ are

$$U_1 = U_1(u_1) = \begin{bmatrix} 0 & s & 0 \\ -s & 0 & -r \\ 0 & r & 0 \end{bmatrix}, \qquad u_1 = \begin{bmatrix} r \\ s \end{bmatrix}, \quad (18)$$

$$U_2 = U_2(u_2) = \begin{bmatrix} 0 & w & 0 \\ -w & 0 & -v \\ 0 & v & 0 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} v \\ w \end{bmatrix}. \quad (19)$$

In order to solve the enlarged stationary zero curvature equation,

$$\bar{W}_x = [\bar{U}, \bar{W}], \quad (20)$$

we take the solution to be of the following form:

$$\bar{W} = \bar{W}(\bar{u}, \lambda) = M(W, W_1, W_2) \in \tilde{\mathfrak{g}}(\lambda), \quad (21)$$

where $W$ is defined by (7) and solves $W_x = [U, W]$, and $W_1$ and $W_2$ are assumed to be

$$W_1 = W_1(u, u_1, \lambda) = \begin{bmatrix} 0 & g & e \\ -g & 0 & -f \\ -e & f & 0 \end{bmatrix} \in \bar{so}(3), \quad (22)$$

and

$$W_2 = W_2(u, u_1, u_2, \lambda) = \begin{bmatrix} 0 & g' & e' \\ -g' & 0 & -f' \\ -e' & f' & 0 \end{bmatrix} \in \bar{so}(3). \tag{23}$$

Equation (20) is equivalent to satisfying the following matrix equations:

$$\begin{cases} W_x = [U, W], \\ W_{1,x} = [U, W_1] + [U_1, W], \\ W_{2,x} = [U, W_2] + [U_2, W] + \alpha[U_1, W_1]. \end{cases} \tag{24}$$

The second and third equations in (24) generate

$$\begin{cases} e_x = pg - qf + rc - sb, \\ f_x = -\lambda g + qe + sa, \\ g_x = -pe + \lambda f - ra, \end{cases} \tag{25}$$

and

$$\begin{cases} e'_x = -fs\alpha + gr\alpha - qf' + pg' - wb + vc, \\ f'_x = qe' - \lambda g' + wa + se\alpha, \\ g'_x = -pe' + \lambda f' - re\alpha - va, \end{cases} \tag{26}$$

respectively. Plugging into recursion relations (25) and (26) into the Laurent expansions,

$$\begin{cases} e = \sum_{i\geq 0} e_i\lambda^{-i}, \quad f = \sum_{i\geq 0} f_i\lambda^{-i}, \quad g = \sum_{i\geq 0} g_i\lambda^{-i}, \\ e' = \sum_{i\geq 0} e'_i\lambda^{-i}, \quad f' = \sum_{i\geq 0} f'_i\lambda^{-i}, \quad g' = \sum_{i\geq 0} g'_i\lambda^{-i}, \end{cases} \tag{27}$$

we have

$$\begin{cases} f_{i+1} = g_{i,x} + pe_i + ra_i, \\ g_{i+1} = -f_{i,x} + qe_i + sa_i, \\ e_{i+1,x} = pg_{i+1} - qf_{i+1} + rc_{i+1} - sb_{i+1}, \\ f'_{i+1} = g'_{i,x} + pe'_i + va_i + \alpha rc_i, \\ g'_{i+1} = -f'_{i,x} + qe'_i + wa_i + \alpha sc_i, \\ e'_{i+1,x} = pg'_{i+1} - qf'_{i+1} - \alpha sf_{i+1} + \alpha rg_{i+1} - wb_{i+1} + vc_{i+1}, \end{cases} \tag{28}$$

where $i \geq 0$. We take the initial data as $\{e_0 = -1,\ f_0 = g_0 = 0;\ e'_0 = -1,\ f'_0 = g'_0 = 0\}$ and suppose that the integration constants are zero. Then, recursion relation (28) uniquely generates $\{e_i, f_i, g_i, e'_i, f'_i, g'_i | i \geq 1\}$. We obtain

$$
\begin{cases}
e_1 = 0, \\
f_1 = -p - r, \\
g_1 = -q - s; \\
e_2 = \frac{1}{2}p^2 + \frac{1}{2}q^2 + rp + sq, \\
f_2 = -q_x - s_x, \\
g_2 = p_x + r_x; \\
e_3 = q_x p - qp_x - sp_x + rq_x + s_x p - r_x q, \\
f_3 = p_{xx} + \frac{1}{2}p^3 + \frac{1}{2}pq^2 + \frac{3}{2}rp^2 + psq + \frac{1}{2}rq^2 + r_{xx}, \\
g_3 = q_{xx} + \frac{1}{2}q^3 + \frac{1}{2}qp^2 + \frac{3}{2}sq^2 + qrp + \frac{1}{2}sp^2 + s_{xx}; \\
e_4 = (-p - r)p_{xx} + (-q - s)q_{xx} - pr_{xx} - qsxx\frac{1}{2}p_x^2 + p_x r_x + \frac{1}{2}q_x^2 + q_x s_x \\
\quad -\frac{3}{8}(p^2 + q^2)(p^2 + 4pr + q(q + 4s)), \\
f_4 = q_{xxx} + s_{xxx} + \frac{1}{2}(3p^2 + 6pr + 3q^2 + 6qs)q_x + \frac{1}{2}(3p^2 + 3q^2)s_x, \\
g_4 = -p_{xxx} - r_{xxx} + \frac{1}{2}(-3p^2 - 6pr - 3q^2 - 6qs)p_x + \frac{1}{2}(-3p^2 - 3q^2)r_x;
\end{cases}
$$

and

$$
\begin{cases}
e_1' = 0, \\
f_1' = -p - \alpha r - v, \\
g_1' = -q - \alpha s - w; \\
e_2' = \frac{1}{2}p^2 + \frac{1}{2}q^2 + \alpha rp + \alpha sq + vp + wq + \frac{1}{2}\alpha s^2 + \frac{1}{2}\alpha r^2, \\
f_2' = -q_x - \alpha s_x - w_x, \\
g_2' = p_x + \alpha r_x + v_x; \\
e_3' = q_x p - qp_x - \alpha sp_x + \alpha rq_x + \alpha s_x p - r_x q - wp_x + vq_x + w_x p - \alpha qr_x \\
\quad - v_x q + \alpha s_x r - \alpha sr_x, \\
f_3' = p_{xx} + \frac{1}{2}p^3 + \frac{1}{2}pq^2 + \alpha\frac{3}{2}rp^2 + \alpha psq + \alpha\frac{1}{2}rq^2 + r_{xx} + \alpha\frac{3}{2}pr^2 + \alpha rsq \\
\quad + pqw + \frac{3}{2}vp^2 + \frac{1}{2}vq^2 + \frac{1}{2}\alpha ps^2 + \alpha r_{xx} + v_{xx}, \\
g_3' = q_{xx} + \frac{1}{2}q^3 + \frac{1}{2}qp^2 + \alpha\frac{3}{2}sq^2 + \alpha qrp + \alpha\frac{1}{2}sp^2 + s_{xx} + \alpha\frac{3}{2}qs^2 + \alpha srp \\
\quad + pvq + \frac{3}{2}wq^2 + \frac{1}{2}wp^2 + \frac{1}{2}\alpha pr^2 + \alpha s_{xx} + w_{xx}; \\
e_4' = (-\alpha r - p - v)p_{xx} + (-\alpha s - q - w)q_{xx} - \alpha(p + r)r_{xx} - \alpha(q + s)s_{xx} \\
\quad - v_{xx}p - w_{xx}q + \frac{1}{2}p_x^2 + (\alpha r_x + v_x)p_x \\
\quad + \frac{1}{2}q_x^2 + (\alpha s_x + w_x)q_x + \frac{1}{2}\alpha r_x^2 + \frac{1}{2}\alpha s_x^2 - \frac{3}{8}p^4 + \frac{3}{2}(-\alpha r - v)p^3 \\
\quad + \frac{1}{8}(-6q^2 + (-12\alpha s - 12w)q - 18\alpha r^2 - 6\alpha s^2)p^2 \\
\quad - \frac{3}{2}q((\alpha r + v)q + 2\alpha rs)p - \frac{3}{2}q^2\left(\frac{1}{4}q^2 + (\alpha s + w)q + \frac{1}{2}\alpha(r^2 + 3s^2)\right), \\
f_4' = q_{xxx} + \alpha s_{xxx} + w_{xxx} + \frac{1}{2}(3p^2 + (6\alpha r + 6v)p + 3q^2 + (6\alpha s + 6w)q \\
\quad + 3\alpha r^2 + 3\alpha s^2)q_x + \frac{1}{2}(3\alpha p^2 + 6\alpha pr + 3\alpha q^2 + 6\alpha qs)s_x \\
\quad + \frac{1}{2}(3p^2 + 3q^2)w_x, \\
g_4' = -p_{xxx} - \alpha r_{xxx} - v_{xxx} + \frac{1}{2}(-3p^2 - (6\alpha r + 6v)p - 3q^2 - (6\alpha s + 6w)q \\
\quad - 3\alpha r^2 - 3\alpha s^2)p_x + \frac{1}{2}(-3\alpha p^2 - 6\alpha pr - 3\alpha q^2 - 6\alpha qs)r_x \\
\quad - \frac{1}{2}(3p^2 + 3q^2)v_x.
\end{cases}
$$

These functions are differential polynomials in the variables $p$, $q$, $r$, $s$, $v$, and $w$.

Similar to [35], for each integer $m \geq 0$, we further introduce an enlarged Lax matrix

$$\bar{V}^{[m]} = (\lambda^m \bar{W})_+ = M\left(V^{[m]}, V_1^{[m]}, V_2^{[m]}\right) \in \tilde{\mathfrak{g}}(\lambda), \tag{29}$$

where $V^{[m]}$ is defined by (11) and $V_i^{[m]} = (\lambda^m W_i)_+, i = 1, 2$. The enlarged zero curvature equation,

$$\bar{U}_{t_m} - \bar{V}_x^{[m]} + \left[\bar{U}, \bar{V}^{[m]}\right] = 0, \tag{30}$$

gives the following matrix equations:

$$\begin{cases} U_{1,t_m} - V_{1,x}^{[m]} + \left[U, V_1^{[m]}\right] + \left[U_1, V^{[m]}\right] = 0, \\ U_{2,t_m} - V_{2,x}^{[m]} + \left[U, V_2^{[m]}\right] + \left[U_2, V^{[m]}\right] + \alpha\left[U_1, V_1^{[m]}\right] = 0, \end{cases} \tag{31}$$

along with the system in (11). The above equations then present the additional systems

$$\bar{v}_{t_m} = S_m = S_m(\bar{v}) = \begin{bmatrix} S_{1,m}(u, u_1) \\ S_{2,m}(u, u_1, u_2) \end{bmatrix}, \quad m \geq 0, \tag{32}$$

where $\bar{v} = (r, s, v, w)^{\mathrm{T}}$ and

$$S_{1,m}(u, u_1) = \begin{bmatrix} -g_{m+1} \\ f_{m+1} \end{bmatrix},$$

and

$$S_{2,m}(u, u_1, u_2) = \begin{bmatrix} -g'_{m+1} \\ f'_{m+1} \end{bmatrix}.$$

Then the enlarged zero curvature equation generates a hierarchy of bi-integrable couplings,

$$\bar{u}_{t_m} = \begin{bmatrix} p \\ q \\ r \\ s \\ v \\ w \end{bmatrix}_{t_m} = \begin{bmatrix} -c_{m+1} \\ b_{m+1} \\ -g_{m+1} \\ f_{m+1} \\ -g'_{m+1} \\ f'_{m+1} \end{bmatrix} = \bar{K}_m(\bar{u}), \quad m \geq 0, \tag{33}$$

for soliton hierarchy (12).

In particular, when $m = 2$, we have $u_{t_2} = \bar{K}_2$, i.e.,

$$\begin{bmatrix} p \\ q \\ r \\ s \\ v \\ \\ w \\ \\ \end{bmatrix}_{t_2} = \begin{bmatrix} -q_{xx} - \frac{1}{2}p^2q - \frac{1}{2}q^3 \\ p_{xx} + \frac{1}{2}p^3 + \frac{1}{2}pq^2 \\ -q_{xx} - \frac{1}{2}q^3 - \frac{1}{2}qp^2 - \frac{3}{2}sq^2 - qrp - \frac{1}{2}sp^2 - s_{xx} \\ p_{xx} + \frac{1}{2}p^3 + \frac{1}{2}pq^2 + \frac{3}{2}rp^2 + psq + \frac{1}{2}rq^2 + r_{xx} \\ -q_{xx} - \frac{1}{2}q^3 - \frac{1}{2}qp^2 - \alpha\frac{3}{2}sq^2 - \alpha qrp - \alpha\frac{1}{2}sp^2 - s_{xx} - \alpha\frac{3}{2}qs^2 - \alpha srp \\ -pvq + \frac{3}{2}wq^2 - \frac{1}{2}wp^2 - \frac{1}{2}\alpha ps^2 - \alpha s_{xx} - w_{xx} \\ p_{xx} + \frac{1}{2}p^3 + \frac{1}{2}pq^2 + \alpha\frac{3}{2}rp^2 + \alpha psq + \alpha\frac{1}{2}rq^2 + r_{xx} + \alpha\frac{3}{2}pr^2 + \alpha rsq \\ +pqw + \frac{3}{2}vp^2 + \frac{1}{2}vq^2 + \frac{1}{2}\alpha ps^2 + \alpha r_{xx} + v_{xx} \end{bmatrix}. \tag{34}$$

## 4 Hamiltonian Structures

We have a systematic approach for generating Hamiltonian structures for the bi-integrable coupling in (33) using the variational identity over the enlarged matrix loop algebra $\tilde{\mathfrak{g}}(\lambda)$ [13,18]. The variational identity is as follows:

$$\frac{\delta}{\delta \bar{u}} \int \langle \bar{W}, \bar{U}_\lambda \rangle \mathrm{d}x = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \langle \bar{W}, \bar{U}_{\bar{u}} \rangle, \quad \gamma = \text{constant}. \tag{35}$$

As seen in [35], there is a convenient method to constructing a symmetric and ad-invariant bilinear form on $\tilde{\mathfrak{g}}(\lambda)$ by rewriting the semidirect sum $\tilde{\mathfrak{g}}(\lambda)$ into a vector form. First, we define a mapping

$$\sigma : \tilde{\mathfrak{g}}(\lambda) \mapsto \mathbb{R}^9, A \mapsto (a_1, \ldots, a_9)^{\mathrm{T}}, \tag{36}$$

where

$$A = M(A_1, A_2, A_3) \in \tilde{\mathfrak{g}}(\lambda), \quad A_i = \begin{bmatrix} 0 & a_{3i} & a_{3i-2} \\ -a_{3i} & 0 & -a_{3i-1} \\ -a_{3i-2} & a_{3i-1} & 0 \end{bmatrix}, \quad 1 \le i \le 3. \tag{37}$$

The map $\sigma$ induces a Lie algebra structure on $\mathbb{R}^9$ isomorphic to the enlarged matrix loop algebra $\tilde{\mathfrak{g}}(\lambda)$. Thus, the corresponding Lie bracket $[\cdot, \cdot]$ on $\mathbb{R}^9$ is generated by letting

$$[a, b]^{\mathrm{T}} = a^{\mathrm{T}} R(b), \tag{38}$$

where $a = (a_1, \ldots, a_9)^{\mathrm{T}}, b = (b_1, \ldots, b_9)^{\mathrm{T}} \in \mathbb{R}^9$ and

$$R(b) = M(R_1, R_2, R_3), \tag{39}$$

with

$$R_i = \begin{bmatrix} 0 & -b_{3i} & b_{3i-1} \\ b_{3i} & 0 & -b_{3i-2} \\ -b_{3i-1} & b_{3i-2} & 0 \end{bmatrix}, \quad 1 \le i \le 3. \tag{40}$$

There is an Lie isomorphism, $\sigma$, between the Lie algebra $(\mathbb{R}^9, [\cdot, \cdot])$ with the enlarged matrix loop algebra $\tilde{\mathfrak{g}}(\lambda)$.

We may find a bilinear form on $\mathbb{R}^9$ by

$$\langle a, b \rangle = a^{\mathrm{T}} F b, \tag{41}$$

where $F$ is a constant matrix and the symmetric property of $\langle \cdot, \cdot \rangle$ requires that

$$F^{\mathrm{T}} = F. \tag{42}$$

The symmetric condition along with the ad-invariance property

$$\langle a, [b, c] \rangle = \langle [a, b], c \rangle,$$

provides the condition

$$F(R(b))^{\mathrm{T}} = -R(b) F, \quad b \in \mathbb{R}^9. \tag{43}$$

Upon solving the derived system of equations from (43) for an arbitrary vector $b \in \mathbb{R}^9$, we find

$$F = \begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \\ \eta_2 & \alpha\eta_3 & 0 \\ \eta_3 & 0 & 0 \end{bmatrix} \otimes F_0, \tag{44}$$

where

$$F_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{45}$$

and $\eta_i$, $1 \leq i \leq 3$, are arbitrary constants. Thus, the bilinear form on the semidirect sum $\tilde{\mathfrak{g}}(\lambda)$ of the two Lie subalgebras $\tilde{g}$ and $\tilde{g}_c$ is defined as

$$\begin{aligned}\langle A, B\rangle_{\tilde{\mathfrak{g}}(\lambda)} &= \langle\sigma(A), \sigma(B)\rangle_{\mathbb{R}^9}\\ &= (a_1, \ldots, a_9)F(b_1, \ldots, b_9)^{\mathrm{T}}\\ &= (a_1b_1 + a_2b_2 + a_3b_3)\eta_1 + (a_1b_4 + a_2b_5 + a_3b_6 + a_4b_1 + a_5b_2\\ &\quad + a_6b_3)\eta_2 + (\alpha a_4b_4 + \alpha a_5b_5 + \alpha a_6b_6 + a_1b_7 + a_2b_8 + a_3b_9\\ &\quad + a_7b_1 + a_8b_2 + a_9b_3)\eta_3, \end{aligned} \tag{46}$$

where $A$ and $B$ are two matrices in $\tilde{\mathfrak{g}}(\lambda)$ presented by

$$\begin{cases} A = \sigma^{-1}((a_1, \ldots, a_9)^{\mathrm{T}}) \in \tilde{\mathfrak{g}}(\lambda),\\ B = \sigma^{-1}((b_1, \ldots, b_9)^{\mathrm{T}}) \in \tilde{\mathfrak{g}}(\lambda). \end{cases} \tag{47}$$

Bilinear form (46) is symmetric and ad-invariant due to the isomorphism $\sigma$. A bilinear form, defined by (46), is non-degenerate iff the determinant of F is not zero, i.e.,

$$\det(F) = -\eta_3^9\alpha^3 \neq 0. \tag{48}$$

Therefore, we choose $\eta_3 \neq 0$ to obtain a non-degenerate, symmetric, and ad-invariant bilinear form over the enlarged matrix loop algebra $\tilde{\mathfrak{g}}(\lambda)$.

Now, we compute

$$\langle\bar{W}, \bar{U}_\lambda\rangle_{\tilde{\mathfrak{g}}(\lambda)} = a\eta_1 + e\eta_2 + e'\eta_3 \tag{49}$$

and

$$\langle\bar{W}, \bar{U}_{\bar{u}}\rangle_{\tilde{\mathfrak{g}}(\lambda)} = \begin{bmatrix} b\eta_1 + f\eta_2 + f'\eta_3\\ c\eta_1 + g\eta_2 + g'\eta_3\\ b\eta_2 + \alpha f\eta_3\\ c\eta_2 + \alpha g\eta_3\\ b\eta_3\\ c\eta_3 \end{bmatrix}. \tag{50}$$

In addition, the formula $\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\mathrm{tr}(W^2)|$ [19] yields that the constant $\gamma = 0$, and thus, the corresponding variational identity is

$$\frac{\delta}{\delta\bar{u}}\int\frac{-a_{m+1}\eta_1 - e_{m+1}\eta_2 - e'_{m+1}\eta_3}{m}\mathrm{d}x = \begin{bmatrix} b_m\eta_1 + f_m\eta_2 + f'_m\eta_3\\ c_m\eta_1 + g_m\eta_2 + g'_m\eta_3\\ b_m\eta_2 + \alpha f_m\eta_3\\ c_m\eta_2 + \alpha g_m\eta_3\\ b_m\eta_3\\ c_m\eta_3 \end{bmatrix}, \quad m \geq 1. \tag{51}$$

We consequently obtain a Hamiltonian structure for hierarchy (33) of bi-integrable couplings,

$$\bar{u}_{t_m} = \bar{J}\frac{\delta\bar{\mathcal{H}}_m}{\delta\bar{u}}, \quad m \geq 0, \tag{52}$$

with the Hamiltonian functionals,

$$\bar{\mathcal{H}}_m = \int \frac{-a_{m+2}\eta_1 - e_{m+2}\eta_2 - e'_{m+2}\eta_3}{m+1} \mathrm{d}x, \tag{53}$$

and the Hamiltonian operator,

$$\bar{J} = \begin{bmatrix} 0 & \eta_1 & 0 & \eta_2 & 0 & \eta_3 \\ -\eta_1 & 0 & -\eta_2 & 0 & -\eta_3 & 0 \\ 0 & \eta_2 & 0 & \alpha\eta_3 & 0 & 0 \\ -\eta_2 & 0 & -\alpha\eta_3 & 0 & 0 & 0 \\ 0 & \eta_3 & 0 & 0 & 0 & 0 \\ -\eta_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{-1}, \tag{54}$$

and note that $\det(\bar{J}) \neq 0$. In particular, when $m = 2$, the Hamiltonian functional is

$$\bar{\mathcal{H}}_2 = \int \frac{1}{3}\left(-a_4\eta_1 - e_4\eta_2 - e'_4\eta_3\right) \mathrm{d}x, \tag{55}$$

where

$$\begin{aligned} -a_4\eta_1 - e_4\eta_2 - e'_4\eta_3 = {} & ((\eta_1+\eta_2+\eta_3)p + (\alpha\eta_3+\eta_2)r + \eta_3 v)\, p_{xx} \\ & + ((\eta_1+\eta_2+\eta_3)q + (\alpha\eta_3+\eta_2)s\eta_3 w)\, q_{xx} \\ & + ((\alpha\eta_3+\eta_2)p + \eta_3\alpha r)\, r_{xx}\, ((\alpha\eta_3+\eta_2)q + \eta_3\alpha s)\, s_{xx} \\ & + \eta_3 p v_{xx}\eta_3 q w_{xx} - \frac{1}{2}(\eta_1+\eta_2+\eta_3)p_x^2 \\ & + ((-\alpha\eta_3-\eta_2)r_x - \eta_3 v_x)\, p_x - (\eta_1+\eta_2+\eta_3)q_x^2 \\ & + ((-\alpha\eta_3-\eta_2)s_x - \eta_3 w_x)\, q_x - \frac{1}{2}\eta_3\alpha r_x^2 - \frac{1}{2}\eta_3\alpha s_x^2 \\ & + \frac{3}{8}(\eta_1+\eta_2+\eta_3)p^4\frac{3}{2}\,((\alpha\eta_3+\eta_2)r + \eta_3 v)\, p^3 \\ & + \frac{1}{8}\Bigg(6\,(\eta_1+\eta_2+\eta_3)\,q^2 + ((12\alpha\eta_3 + 12\eta_2)s + 12\eta_3 w)\, q \\ & + 18\eta_3\left(r^2 + \frac{1}{3}s^2\right)\alpha\Bigg)p^2 + \frac{3}{2}(((\alpha\eta_3+\eta_2)r + \eta_3 v)q \\ & + 2\eta_3\alpha r s)pq + \frac{3}{8}q^2((\eta_1+\eta_2+\eta_3)q^2 + ((4\alpha\eta_3 \\ & + 4\eta_2)s + 4\eta_3 w)q + 2\alpha\eta_3(r^2+3s^2)). \end{aligned} \tag{56}$$

## 5 Symmetries and Conserved Functionals

We may solve the recursion relation of symmetries

$$\bar{K}_m = \bar{\Phi}\bar{K}_{m-1}, \quad m \geq 0, \tag{57}$$

for a recursion operator, $\bar{\Phi}$, to obtain

$$\bar{\Phi} = \begin{bmatrix} \Phi & 0 & 0 \\ \Phi_1 & \Phi & 0 \\ \Phi_2 & \alpha\Phi_1 & \Phi \end{bmatrix}, \tag{58}$$

where $\Phi$ is given by (13) and

$$\Phi_1 = \begin{bmatrix} q\partial^{-1}r + s\partial^{-1}p & q\partial^{-1}s + s\partial^{-1}q \\ -p\partial^{-1}r - r\partial^{-1}p & -p\partial^{-1}s - r\partial^{-1}q \end{bmatrix}, \tag{59}$$

and

$$\Phi_2 = \begin{bmatrix} q\partial^{-1}v + w\partial^{-1}p + \alpha s\partial^{-1}r & q\partial^{-1}w + w\partial^{-1}q + \alpha s\partial^{-1}s \\ -p\partial^{-1}v - v\partial^{-1}p - \alpha r\partial^{-1}r & -p\partial^{-1}w - v\partial^{-1}q - \alpha r\partial^{-1}s \end{bmatrix}. \tag{60}$$

It can be shown by a symbolic computation that $\bar{\Phi}$ is a hereditary operator [36,37]. Therefore,

$$\bar{\Phi}'(\bar{u})[\bar{\Phi}\bar{T}_1]\bar{T}_2 - \bar{\Phi}\bar{\Phi}'(\bar{u})[\bar{T}_1]\bar{T}_2$$

is symmetric with respect to $\bar{T}_1$ and $\bar{T}_2$, and the two operators $\bar{J}$ and $\bar{M} = \bar{\Phi}\bar{J}$ make a Hamiltonian pair [38], i.e., $\bar{J}$, $\bar{M}$, and $\bar{J} + \bar{M}$ are all Hamiltonian operators. Thus, the hierarchy (33) of bi-integrable couplings possesses a bi-Hamiltonian structure [38,39] and is Liouville integrable. It follows that there are infinitely many symmetries and conserved functionals:

$$[\bar{K}_m, \bar{K}_n] = 0, \quad m, n \geq 0, \tag{61}$$

and

$$\{\bar{\mathcal{H}}_m, \bar{\mathcal{H}}_n\}_{\bar{J}} = \{\bar{\mathcal{H}}_m, \bar{\mathcal{H}}_n\}_{\bar{M}} = 0, \quad m, n \geq 0. \tag{62}$$

## 6 Concluding Remarks

We have obtained a new class of bi-integrable couplings (33) for the soliton hierarchy (12) using on non-semisimple Lie algebra (16). We showed the resulting hierarchy of bi-integrable couplings possesses a bi-Hamiltonian structure and is Liouville integrable. It remains an open question how to generate a Hamiltonian structure for matrix loop algebra (15) when $\alpha = 0$ as the bilinear form presented in Sect. 4 is degenerate.

Some enlarged matrix loop algebras do not possess any non-degenerate, symmetric, and ad-invariant bilinear forms required in the variational identity. In the following example of a bi-integrable coupling,

$$\begin{cases} u_t = K(u) \\ v_t = K'(u)[v] \\ w_t = K'(u)[w]. \end{cases} \tag{63}$$

where $K'(u)$ denotes the Gateaux derivative, is there any Hamiltonian structure for this specific bi-integrable coupling?

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