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On a Special Quotient of the Generating Graph of a Finite Group

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Abstract Suppose G is a finite group with identity element 1. The generating graph $\Gamma(G)$ is defined as a graph with vertex set G in such a way that two distinct vertices are connected by an edge if and only if they generate G and the Q-generating graph $\Omega(G)$ is defined as the quotient graph $\frac{\Gamma(G)\backslash\{1\}}{\mathcal{C}^{\star}(G)}$, where $\mathcal{C}^{\star}(G)$ is the set of all non-identity conjugacy classes of G and $\Gamma(G)\backslash\{1\}$ is a graph obtained from $\Gamma(G)$ by removing the vertex 1. In this paper, some structural properties of this graph are investigated. The structure of Q-generating graphs of dihedral, semidihedral, dicyclic and all sporadic groups other than M, B and Fi_{24}' is also presented.

Keywords Generating graph · Q-generating graph · Sporadic group

Mathematics Subject Classification 20C40

1 Introduction

Throughout this paper, group means finite group. Suppose G is a finite group and $P(G) = \frac{|\{(x,y)\in G\times G \mid \langle x,y\rangle = G\}|}{|G|^2}$. It is clear that $0 \leq P(G) \leq 1$ and this quantity is the probability that G can be generated by two elements. This quantity is the source of several research works in computational group theory [23]. It is well known that $P(G) \neq 0$, when G is a non-abelian finite simple group. By motivation of this prob-

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ability function, the **generating graph** $\Gamma(G)$ is defined as a graph with vertex set $V(\Gamma(G)) = G$ in such a way that two distinct vertices are connected by an edge if and only if they generate G [32,33].

Suppose X is a graph and θ is a partition on $V(X)$. The quotient graph $\widehat{X}$ is a graph with vertex set θ in such a way that two elements $A, B \in \theta$ are adjacent if and only if there are $a \in A$ and $b \in B$ such that a and b are adjacent in X. For simplicity of our argument, we introduce another graph $\Gamma^\star(G) = \Gamma(G) - \{1_G\}$, where 1_G is the identity element of group G.

The Q-**generating graph** $\Omega(G)$ is the quotient graph $\widehat{\Gamma^\star(G)}$ over the partition θ of all non-identity conjugacy classes of G.

Let G be a finite group with at least one non-identity conjugacy class of elements. The Q-generating graph of G has all non-identity conjugacy classes as the vertices and two different classes are adjacent when there is at least one element in each class which make a pair of generators of G. It is well known that for any involution a in a finite simple group G, there exists $b \in G$ with $G = \langle a, b\rangle$. So, the condition of choosing a pair of elements from two different classes of a group makes the Q-generating graph to be simple without loop.

Following Woldar [39], the group G is said to be nX-**complementary generated** if for an arbitrary non-identity element $x \in G$, there exists a $y \in nX$ such that $G =< x, y >$. If there exist $x \in lX$, $y \in mY$ and $z \in nZ$ such that $xy = z$ and $G =< x, y >$, then the group G is said to be (lX, mY, nZ)-generated. A group G is called (l, m, n)-**generated**, if these exist three conjugacy classes lX, mY and nZ in G such that G is a (lX, mY, nZ)-generated. If G is (l, m, n)-generated, then we can see that for any permutation π of S_3, the group G is also $((l)\pi, (m)\pi, (n)\pi)$-generated. Therefore, without loss of generality, we may assume that $l \le m \le n$.

Suppose G is a non-abelian simple group. By [13], if G is (l, m, n)-generated, then either $G \cong A_5$ or $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$. Hence, if G is a non-abelian finite simple group and l, m, n are divisors of $|G|$ such that $\frac{1}{l} + \frac{1}{m} + \frac{1}{n} < 1$, then it is natural to ask whether or not G is (l, m, n)-generated. The motivation for this question came from the calculation of the genus of finite simple groups [37,40]. The problem of finding all triples (l, m, n) such that G is (l, m, n)-generated was presented many years ago by Moori [35]. Ganief and Moori [25,29] computed all 2-generations of the Janko groups J_1, J_2, J_3 and J_4. We refer the interested readers to [24] and references therein for the motivation of this study and for more information on this topic. Interest in studying the $(2, m, n)$-generated groups has a geometrical motivation which is related to the study of regular maps on surfaces and their automorphisms. Brahana [11] proved a necessary and sufficient condition for a group G to be the automorphism group of a regular map on a surface is that G can be generated by an involution and another element of order greater than 2.

Suppose lX, mY and nZ are conjugacy classes of a finite group G. The cardinality of the set

$$\Lambda = \{(x, y) \mid x \in lX,\ \ y \in mY \ \&\ xy = z \in nZ \text{ is a fixed element}\}$$

is denoted by $\Delta_G = \Delta_G(lX, mY, nZ)$. This number is called the **structure constant** of G in the classes lX, mY and nZ. The quantities $\Delta^\star_G = \Delta^\star_G(lX, mY, nZ)$ and

$\Sigma(H_1 \cup H_2 \cup \cdots \cup H_r)$ are also defined as the number of pairs $(x, y) \in \Lambda$ such that $G = \langle x, y\rangle$ and $\langle x, y\rangle \subseteq H_i$, for some $1 \leq i \leq r$, respectively. The number of pairs $(x, y) \in \Lambda$ generating a subgroup H of G will be given by $\Sigma^{\star}(H)$, and the centralizer of a representative of lX will be denoted by $C_G(lX)$. A general conjugacy class of a subgroup H of G with elements of order n will be denoted by nX. It is clear that if $\Delta^{\star}(G) > 0$, then G is (lX, mY, nZ)-generated. In this case, the triple (lX, mY, nZ) is called a **generating triple** for G and we will use **GAP** [36] for the computations in order to compute the generating triples of G. For the sake of completeness, we mention here some useful results in resolving generation-type questions for finite groups.

Lemma 1.1 [39] *The group G is nX-complementary generated if and only if for each conjugacy class pY in G, there is a conjugacy class t_pZ, where t_p is a divisor of $|G|$ related to the prime p, such that G is (pY, nX, t_pZ)-generated.*

Lemma 1.2 [39] *Let G be a finite simple group with a conjugacy class pX where p is a greatest prime divisor $|G|$, then G is a pX-complementary generated.*

Theorem 1.3 [25] *Let G be a finite centerless group and suppose lX, mY and nZ are G-conjugacy classes for which $\Delta^{\star}(G) = \Delta^{\star}_G(lX, mY, nZ) < |C_G(z)|, z \in nZ$. Then, $\Delta^{\star}(G) = 0$ and therefore G is not (lX, mY, nZ)-generated.*

Throughout this paper, our notation is standard and taken mainly from [3,14,31]. The complete and star graph with exactly n vertices are denoted by K_n and $Star_n$, respectively.

2 Examples

In this section, we aim to construct and mention some of the properties of the Q-generating graphs of some finite groups as dihedral, semidihedral, dicyclic, V_{8n} and U_{6n}. The conjugacy classes of D_{2n}, T_{4n}, V_{8n} and U_{6n}, when n is odd, are calculated in the famous book of James and Liebeck [31]. The conjugacy classes of V_{8n}, with n even and SD_{8n} of order $8n$ are computed in [22,30], respectively.

Example 2.1 $D_{2n} = \langle a, b \mid a^n = b^2 = 1,\ bab^{-1} = a^{-1}\rangle$. For an even $n = 2m$, the Q-generating graph $\Omega(D_{2n})$ is a disconnected graph with $m + 2$ vertices and $2\phi(m-1) + 1$ edges, where ϕ denotes the Euler's totient function. The set of vertices is:

$$V(\Omega(D_{2(2m)})) = \Big\{\{a^m\}, \{a^{-1}, a\}, \{a^2, a^{-2}\}, \ldots, \{a^{m-1}, a^{-m+1}\},$$
$$\{b, a^2b, a^4b, \ldots, a^{n-2}b\}, \{ab, a^3b, a^5b, \ldots, a^{n-1}b\}\Big\}.$$

In this case, $\Omega(D_{2n})$ contains the triangles which have an edge $\{b^{D_{2n}} - ab^{D_{2n}}\}$ in common and the isolated vertices corresponding to the representatives a^r, $1 \leq r \leq$

$m-1$ and $(r,n)\neq 1$. For an odd number n, $\Omega(D_{2n})$ is disconnected with $\frac{1}{2}(n+3)-1$ vertices

$$V(\Omega(D_{2n})) = \left\{\{a,a^{-1}\},\{a^{-2},a^{2}\},\ldots,\{a^{(n-1)/2},a^{-(n-1)/2}\},\{b,ab,\ldots,a^{n-1}b\}\right\}.$$

and $\phi((n-1)/2)$ edges. We can see that in this case $\Omega(D_{2n})$ contains the star $Star_{\phi(n-1)/2}$ with the vertex $b^{D_{2n}}$ in the center and the isolated vertices corresponding to the representatives a^r, $1\leq r\leq (n-1)/2$ and $(r,n)\neq 1$.

Example 2.2 $SD_{8n}=\langle a,b \mid a^{4n}=b^2=1,\ bab=a^{2n-1}\rangle$. For an even number $n\geq 2$, $\Omega(SD_{8n})$ has $2n+2$ non-identity conjugacy classes as the vertices

$$V(\Omega(SD_{8n})) = \Big\{\{a^{2n}\},\{a^{r},a^{(2n-1)r}\},$$
$$r\in\{1,3,\ldots,n-1,2,4,\ldots,2n-2,2n+1,2n+3,2n+5,\ldots,3n-1\},$$
$$\{ba^{2t},0\leq t\leq 2n-1\},\{ba^{2t+1},0\leq t\leq 2n-1\Big\}.$$

It is clear that $b^{SD_{8n}}$ and $(ab)^{SD_{8n}}$ are adjacent and $a^{SD_{8n}}-ab^{SD_{8n}}\in E(\Omega(SD_{8n}))$. The vertices $\{a^{2n}\}$ and $\{a^{2r},a^{(2n-1)r}\}$ are isolated, where r is even. Then, the graph is a union of isolated vertices and triangles sharing a common edge. For an odd number n, $\Omega(SD_{8n})$ has $2n+5$ vertices as follows,

$$V(\Omega(SD_{8n})) = \{\{a^{n}\},\{a^{2n}\},\{a^{3n}\},\{a^{r},a^{(2n-1)r}\},$$
$$r\in\{1,3,\ldots,n-2,2,4,\ldots,2n-2,2n+1,2n+3,2n+5,\ldots,3n-1\},$$
$$\{ba^{4t},0\leq t\leq n-1\},\{ba^{4t+1},0\leq t\leq n-1\},$$
$$\{ba^{4t+2},0\leq t\leq n-1\},\{ba^{4t+3},0\leq t\leq n-1\}.$$

The vertex with the representative b is adjacent to the vertices with representatives ba and ba^3. Also the vertices with representatives a^r, $(r,n)=1$ and r is not even, are linked to the vertices with the representatives b, ba, ba^2 and ba^3. Moreover, there are no edges between the vertices with representative a^r. In this graph, $\{a^n\}$, $\{a^{2n}\}$, $\{a^{3n}\}$, $\{a^r,a^{(2n-1)r}\}$, where r is even number in the mentioned set of vertices, are the isolated vertices of $\Omega(SD_{8n})$.

Example 2.3 $T_{4n}=\langle a,b \mid a^{2n}=1, a^n=b^2, b^{-1}ab=a^{-1}\rangle$. The structure of $\Omega(T_{4n})$ is similar to $\Omega(D_{2(2m)})$, with $n+2$ vertices of

$$V(\Omega(T_{4n})) = \Big\{\{a^{n}\},\{a^{r},a^{-r}\},1\leq r\leq n-1,\{a^{2s}b,0\leq s\leq n-1\},$$
$$\{a^{2s+1}b,0\leq s\leq n-1\}\Big\}$$

and $2\phi(n)+1$ edges. This graph is also disconnected with $\phi(n)$ triangles which have a common edge and $n-\phi(n)$ isolated vertices.

Example 2.4 $V_{8n}=\langle a,b \mid a^{2n}=b^4=1,\ aba=b^{-1},\ ab^{-1}a=b\rangle$. When n is even, the vertices of $\Omega(V_{8n})$ are listed as follows:

$$\begin{aligned}V(\Omega(V_{8n})) = \Big\{ & \{b^2\}, \{a^n\}, \{a^n b^2\}, \{a^{2r+1}, a^{-2r-1}b^2\}, 0 \le r \le n-1,\\ & \{a^{2s}, a^{-2s}\}, 1 \le s \le \frac{n}{2}-1, \{a^{2s}b^2, a^{-2s}b^2\}, 1 \le s \le \frac{n}{2}-1,\\ & \{a^{2k}b^{(-1)^k} | 0 \le k \le n-1\}, \{a^{2k}b^{(-1)^{k+1}} | 0 \le k \le n-1\},\\ & \{a^{2k+1}b^{(-1)^k} | 0 \le k \le n-1\}, \{a^{2k+1}b^{(-1)^{k+1}} | 0 \le k \le n-1\}\Big\}.\end{aligned}$$

The Q-generating graph $\Omega(V_{8n})$ is a disconnected graph in which $\{b^2\}$, $\{a^n\}$, $\{a^n b^2\}$, $(a^{2s})^{V_{8n}}$, $(a^{2s}b^2)^{V_{8n}}$ and $(a^{2r+1})^{V_{8n}}$ are the isolated vertices, where $1 \le r \le n-1$, $(2r+1,n)=1$ and $1 \le s \le \frac{n}{2}$. Among the vertices with representatives b, b^{-1}, ab and ab^{-1}, we can see that b is adjacent to the vertex $\{a^j b \mid j \text{ is odd}, k=1,3\}$ and also ab is adjacent to the vertex $\{a^j b \mid j \text{ is even}, k=1,3\}$. These four vertices are also adjacent to the vertices of $\{a^{2r+1}, a^{-2r-1}b^2\}$, where $(2r+1,n)=1$.

When n is odd, the graph $\Omega(V_{8n})$ is disconnected with the vertex set

$$\begin{aligned}V(\Omega(V_{8n})) = \Big\{ & \{b^2\}, \{a^{2r+1}, a^{-2r-1}b^2\}, 0 \le r \le \frac{n-1}{2},\\ & \{a^{2s}, a^{-2s}\}, 1 \le s \le \frac{n-1}{2}, \{a^{2s}b^2, a^{-2s}b^2\}, 1 \le s \le \frac{n-1}{2},\\ & \{a^j b^k | j \text{ even}, k=1,3\}, \{a^j b^k | j \text{ odd }, k=1,3\}\Big\}.\end{aligned}$$

When $n=p>2$ is prime, then the graph $\Omega(V_{8p})$ is a union of the triangles which share an edge and the vertices corresponding to the class representatives b^2, a^{2s}, $a^{2s}b^2$ and a^p, are the isolated vertices. When n is odd but not prime, then $\Omega(V_{8n})$ is again disconnected and the vertices $\{b^2\}$, $(a^{2s})^{V_{8n}}$, $(a^{2s}b^2)^{V_{8n}}$, $a^{n\,V_{8n}}$ and $\{a^{2r+1}, a^{-2r-1}b^2\}$, $0 \le r \le n-1$ are the isolated vertices, where $(2r+1,n) \ne 1$ or $(-2r-1,n) \ne 1$. In this case, again the graph is the union of triangles with a common edge and some isolated vertices.

Example 2.5 $U_{6n}=\langle a,b \mid a^{2n}=b^3=1,\ bab=a\rangle$. The Q-generating graph $\Omega(U_{6n})$ is also disconnected with the vertex set

$$V(\Omega(U_{6n})) = \{\{a^{2r}\}, \{a^{2r}b, a^{2r}b^2\}, \{a^{2r+1}b, a^{2r+1}, a^{2r+1}b^2\}, 0 \le r \le n-1\}.$$

Among the vertices, the vertices $(a^i)^{U_{6n}}$ are isolated, when i is even. The vertices $(a^j)^{U_{6n}}$, $j \ne i$ and $(j,n)=1$ are joined to all other vertices, because they generate U_{6n}.

It is merit to mention here that Breuer et al. [12], proved that if G is a non-abelian finite simple group, then for every pair of non-identity elements x_1 and x_2 in G there exists an element y in G such that $\langle x_1, y\rangle = \langle x_2, y\rangle = G$. So $\Gamma^\star(G)$ is a connected graph of diameter 2. Notice that if the Q-generating graph $\Omega(G)$ is connected, then it has at most diameter 2.

3 Finite Groups with Complete Q-Generating Graph

In this section, we first present a characterization of finite solvable groups with complete Q-generating graph. The sporadic groups with complete Q-generating graphs are also classified.

Theorem 3.1 *Let G be a finite solvable group such that the Q-generating graph $\Omega(G)$ is complete. Then, G is isomorphic to one of the following groups:*

1. *G is isomorphic to $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_p$,*
2. *$G \cong S_3$ or G is a non-abelian group of order $2^r(2^r - 1)$, r is prime, and $2^r - 1$ is a Mersenne prime. Moreover, the Sylow 2-subgroup of G is normal and elementary abelian.*

Proof Suppose the Q-generating graph of a solvable group G is complete. Suppose $1 < N < G$ and that N is normal. (If no such N exists, then G is simple, and since G is solvable, it has prime order and we are done.) Then, no two distinct classes of G contained in N can be connected, so all non-trivial elements of N are conjugate in G, and it follows that N is an elementary abelian p-group for some prime p.

Also, there cannot exist a normal subgroup M of G with $N < M < G$ because by the same reasoning, all non-trivial elements of M would have to be conjugate in G and this is not the case since elements of M not in N cannot be conjugate to elements of N. Since no such subgroup M can exist, $\frac{G}{N}$ is simple, and since G is solvable, $\frac{G}{N}$ has prime order q for some prime q.

If $|N| = 2$, then N is central and $|G| = 2q$. Then, G is abelian and every minimal normal subgroup of G has order 2, so $q = 2$ and $|G| = 4$. We can now assume $|N| > 2$, and since the non-identity elements of N are conjugate in G, we see that G is not abelian. Then, N is the centralizer of each of its non-identity elements, so the class of each of these elements has size q. It follows that $|N| = q + 1$, and this is a power of p. If $q = 2$, then $|N| = 3$ and $G = S_3$. Otherwise, q is odd, so $|N|$ is even and $p = 2$. If $|N| = 2^r$, we have $2^r - 1 = q$, so q is Mersenne. □

Lemma 3.2 *The Q-generating graphs of sporadic groups are connected.*

Proof By a result of Woldar [38], if pX is a conjugacy class of a group G such that p is the greatest prime divisor of $|G|$, then the group G is pX-complementary generated. Hence, the vertex pX of the corresponding Q-generating graph $\Omega(G)$ is joined to all other vertices. As a consequence, $\Omega(G)$ is connected. □

Lemma 3.3 *The following are hold:*

1. *J_1 and J_3 are nX-complementary generated if and only if $n > 2$.*

2. *The group $O'N$ is nX-complementary generated for $n \in \{4, 6, 8, 10, 12, 14, 15, 16, 20, 28\}$ or $n = p$, an odd prime divisor of $|O'N|$.*
3. *The group Th is nX-complementary generated if and only if $n > 2$ and also it is (p, q, r)-generated where p, q, r are the distinct prime divisors of $|Th|$, with $p < q < r$, except when $(p, q, r) = (2, 3, 5)$.*

Proof The proof of (1) follows from [24], proof of (2) follows from [19], and proof of (3) follows from [5]. □

Theorem 3.4 *Suppose G is a sporadic group. Then, $\Omega(G)$ is complete if and only if $G \cong J_1, J_3, O'N$ or Th. Moreover, $\Omega(J_1) \cong K_{14}$, $\Omega(J_3) \cong K_{20}$, $\Omega(O'N) \cong K_{29}$ and $\Omega(Th) \cong K_{46}$.*

Proof The first Janko group J_1 has 14 non-identity conjugacy classes as the vertices

$$2A, 3A, 5A, 5B, 6A, 7A, 10A, 10B, 11A, 15A, 15B, 19A, 19B, 19C,$$

in which there is only one class of involutions. By Lemma 3.3(1), J_1 is nX-complementary generated for each divisor n of $|J_1|$, $n > 2$. Hence, all vertices are connected and $\Omega(J_1) \cong K_{14}$.

The group J_3 has 20 non-identity conjugacy classes which are the vertices as $V(\Omega(J_3)) = \{2A, 3A, 3B, 4A, 5A, 5B, 6A, 8A, 9A, 9B, 9C, 10A, 10B, 12A, 15A, 15B, 17A, 17B, 19A, 19B\}$. This group is nX-complementary generated for $n > 2$, and it has only one conjugacy class of involutions. It means that $\Omega(J_3)$ is complete and isomorphic to K_{20}.

We now prove that the group $O'N$ with 29 non-identity conjugacy classes has a complete Q-generating graph isomorphic to K_{29}. The vertices are

2A	3A	4A	4B	5A	6A	7A	7B	8A	8B	10A
11A	12A	14A	15A	15B	16A	16B	16C	16D	19A	19B
19C	20A	20B	28A	28B	31A	31B				

Based on Lemma 3.3(2), since $O'N$ is pX-complementary generated for each prime divisor $p > 2$ of its order, and it has only one conjugacy class of involutions, we can conclude that each pX is connected to other vertices. Also the other vertices nX, n is not prime, are adjacent to other vertices, again because $O'N$ is nX-complementary generated. Then, $\Omega(O'N) \cong K_{29}$.

The Q-generating graph $\Omega(Th)$ has 46 vertices as follows.

2A	3A	3B	3C	4A	4B	5A	6A	6B	6C	7A
8A	8B	9A	9B	9C	10A	12A	12B	12C	12D	13A
14A	15A	15B	18A	18B	19A	20A	21A	24A	24B	24C
27A	27B	27C	28A	30A	30B	31A	31B	36A	36B	36C
39A	39B									

Based on Lemma 3.3(3), Th has only one class of involutions and for $n > 2$ it is nX-complementary generated, so all vertices are joined and $\Omega(Th) \cong K_{46}$.

By Atlas of finite groups [14], we can see that the Monster group M, the baby monster group B and the Fischer group Fi'_{24} have more than one conjugacy classes of involutions. Suppose G is one of these groups and A and B are two G-conjugacy classes of involutions. Choose $x \in A$ and $y \in B$. Since $\langle x, y\rangle$ is a dihedral group of order $2o(xy)$ and G is simple, $\{x, y\}$ is a not a generating set for G. This proves that the conjugacy classes A and B are not adjacent in the Q-generating graph of G. Therefore, G is not complete. The graphs of some other sporadic groups are not complete, and their proofs will be given Sect. 4. □

4 Sporadic Groups with Non-complete Q-Generating Graphs

Now we bring one of the main results of this paper, which is related to the structure of Q-generating graphs of some other sporadic groups. In an exact phrase, the Q-generating graphs of M_{11}, M_{22}, M_{23}, McL, J_2, J_4, Fi_{22}, Fi_{23}, Ly, He, Co_1, Co_2, Co_3, Suz and HS are calculated. To do this, we need the following crucial lemmas:

Lemma 4.1 *The (l, m, n)-generating triple of some sporadic groups are as follows:*

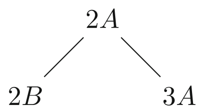
1. *Every sporadic group G is $(2, m, n)$-generated, for some integers m and n the divisors of $|G|$.*
2. *The sporadic groups except M_{11}, M_{22}, M_{23} and McL are $(2, 3, n)$-generated, for some n.*
3. *The groups J_3, Fi_{22}, Ly, He, HN, Co_3 and Ru are (p, q, r)-generated where p, q, r are the distinct prime divisors of $|J_3|$, $|Fi_{22}|$, $|Ly|$, $|He|$, $|HN|$, $|Co_3|$ and $|Ru|$, respectively, with $p < q < r$, except when $(p, q, r) = (2, 3, 5)$.*
4. *The Conway group Co_1 is (p, q, r)-generated for each prime p and $q \in \{7, 11, 13, 23\}$. This group is also $(pX, 5Y, t_pZ)$-generated for each prime class pX and $Y \in \{B, C\}$. Besides Co_1 is $(2, p, q)$-generated for all $p, q \in \{3, 5, 7, 11, 13, 23\}$ with $p < q$, except when $(p, q) = (3, 5)$ or $(3, 7)$.*
5. *Co_2 is (p, q, r)-generated for all p, q, r in $\{2, 3, 5, 7, 11, 23\}$ with $p < q < r$, except with $(p, q, r) = (2, 3, 5)$ or $(2, 3, 7)$.*
6. *The Suzuki's sporadic simple group Suz is $(2, 3, t)$-generated, where t is an odd divisor of $|Suz|$ except $t = 7$.*
7. *The Higman–Sims group HS is (p, q, r)-generated for all p, q, r in $\{2, 3, 5, 7, 11\}$ with $p < q < r$, except with $(p, q, r) = (2, 3, 5)$ or $(2, 3, 7)$.*

Proof We refer to the papers [1,2,4,7,15–18,20,24–28,34,39] for a complete proof for different parts of this result. □

Lemma 4.2 *The nX-complementary generations of the sporadic groups are as follows:*

1. *McL is nX-complementary generated if and only if $n \geq 4$.*
2. *J_4 and Ru are nX-complementary generated if and only if $n > 2$.*
3. *J_2 is nX-complementary generated if and only if $nX \in \{5C, 5D\}$ or $n \geq 6$.*

Fig. 1 The graph H_1



4. *He is nX-complementary generated if and only if $n \geq 4$ or $nX = 3B$.*
5. *Co_1 is nX-complementary generated if and only if $n \geq 4$ and $nX \notin \{4A, 4B, 4C, 4D, 5A, 6A\}$.*
6. *Co_2 is nX-complementary generated if and only if $n \geq 7$ or $nX \in \{4G, 5A, 5B, 6A, 6B, 6E, 6F\}$.*
7. *Ly is nX-complementary generated if and only if $n \geq 3$ and $nX \neq 3A$.*
8. *Fi_{22} is nX-complementary generated if and only if $nX \in \{6K, 8C, 8D, 9C, 12E, \ldots, 12K\}$ or $n \in \{7, 10, 11, 13, \ldots, 30\}$.*
9. *Fi_{23} is nX-complementary generated if and only if $n > 12$ or $n \in \{7, 8, 10, 11\}$ or*

$$nX \in \{6N, 6O, 9D, 9E, 12C, 12D, \ldots, 12O\}.$$

10. *Suz is nX-complementary generated if and only if $nX = 3C$ or $n \geq 4$ and $nX \neq 4A, 6A$.*
11. *HN is nX-complementary generated if and only if $nX \notin \{2A, 2B, 3A, 5A, 5B\}$.*
12. *HS is nX-complementary generated if and only if $nX = 4C$ or $n \geq 5$.*

Proof The proofs of (1) and (12) follow from [26], proofs of (2), (3) and (8) follow from the main results of [8,24] and proofs of (4), (5), (6), (7), (9), (10) and (11) follow from [6,9,10,18,20,21,28], respectively. □

In the following two results, the Q-generating graph of some sporadic groups is obtained.

Theorem 4.3 *The Q-generating graph of the sporadic groups He, J_2, M_{12}, M_{24}, McL, HN, Fi_{22}, Fi_{23}, Co_1, Co_2, Co_3, Suz and HS can be computed as follows:*

1. $\Omega(He) \cong K_{32} - H_1$,	2. $\Omega(J_2) \cong K_{20} - H_2$,
3. $\Omega(M_{12}) \cong K_{14} - H_3$,	4. $\Omega(M_{24}) \cong K_{25} - H_4$,
5. $\Omega(McL) \cong K_{23} - H_5$,	6. $\Omega(HN) \cong K_{53} - H_6$,
7. $\Omega(Fi_{22}) \cong K_{64} - H_7$,	8. $\Omega(Fi_{23}) \cong K_{97} - H_8$,
9. $\Omega(Co_1) \cong K_{100} - H_9$,	10. $\Omega(Co_2) \cong K_{59} - H_{10}$,
11. $\Omega(Co_3) \cong K_{41} - H_{11}$,	12. $\Omega(Suz) \cong K_{42} - H_{12}$
13. $\Omega(HS) \cong K_{23} - H_{13}$,	

in which the graph H_i, $1 \leq i \leq 13$, are depicted in Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 and 13.

Proof Our main proof will consider some separete cases as follows:

$2A$	$2B$	$3A$	$3B$	$4A$	$4B$	$4C$	$5A$	$6A$	$6B$	$7A$
$7B$	$7C$	$7D$	$7E$	$8A$	$10A$	$12A$	$12B$	$14A$	$14B$	$14C$
$14D$	$15A$	$17A$	$17B$	$21A$	$21B$	21	$21D$	$28A$	$28B$	

1. In the following, there are 32 non-identity conjugacy classes of Held group He, Based on Lemma 4.2(4), in the Q-generating graph of He all conjugacy classes nX, $n \geq 4$, are adjacent. On the other hand, the vertex $3B$ and all other vertices

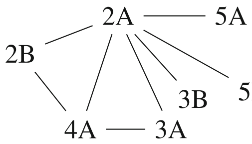


Fig. 2 The graph H_2

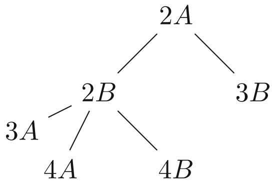


Fig. 3 The graph H_3

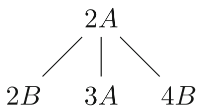


Fig. 4 The graph H_4

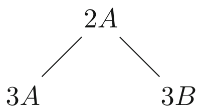


Fig. 5 The graph H_5

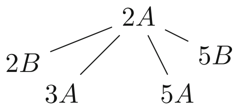


Fig. 6 The graph H_6

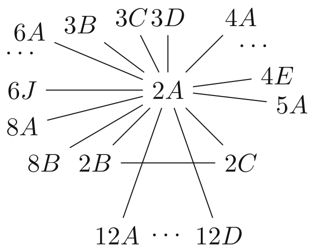


Fig. 7 The graph H_7

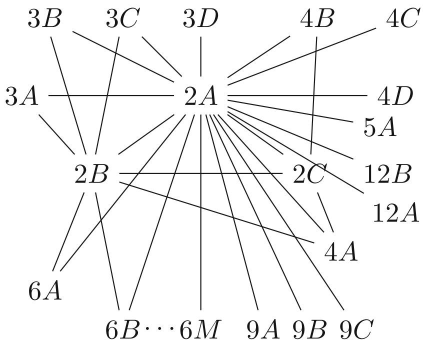


Fig. 8 The graph H_8

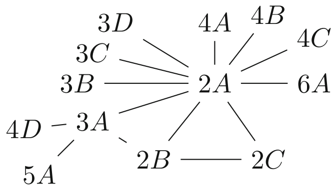


Fig. 9 The graph H_9

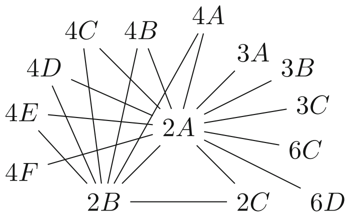


Fig. 10 The graph H_{10}

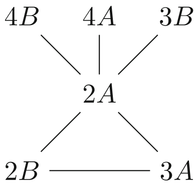


Fig. 11 The graph H_{11}

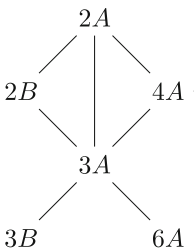
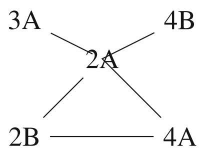


Fig. 12 The graph H_{12}

Fig. 13 The graph H_{13}



are adjacent and since He is simple, there is no edge between $2A$ and $2B$. Also for each vertex tZ, $t \neq 2$, we have $\Delta_{He}(2A, 3A, tZ) < |C_{He}(tZ)|$. Then by Theorem 1.3, $\Delta^*(2A, 3A, tZ) = 0$ and so $2A - 3A \notin E(\Omega(He))$. Therefore, $\Omega(He) \cong K_{32} - \{2A - 2B, 2A - 3A\}$.

2. The Q-generating graph of J_2 with 20 vertices of is not a complete graph.

$2A$	$2B$	$3A$	$3B$	$4A$	$5A$	$5B$	$5C$	$5D$	$6A$
$6B$	$7A$	$8A$	$10A$	$10B$	$10C$	$10D$	$12A$	$15A$	$15B$

By Lemma 4.2(3), J_2 is nX-complementary generated if and only if $n \geq 6$ or $nX \in \{5C, 5D\}$ and it is not $(2A, mY, tZ)$-generated where tZ is an arbitrary conjugacy class and $mY \in \{3A, 3B, 4A, 5A, 5B\}$. Besides, our calculations show that $\Delta(2B, 3A, tZ) < |C_{J_2}(tZ)|$ for each vertex tZ and so $\Delta^*(2B, 3A, tZ) = 0$. Also for $tZ \neq 7A$, we have $\Delta_{J_2}(3A, 4A, tZ) < |C_{J_2}(tZ)|$ and hence $\Delta^*(3A, 4A, tZ) = 0$. But $\Delta_{J_2}(3A, 4A, 7A) = 14 > |C_{J_2}(7A)| = 7$ and $(3A, 4A, 7A)$ is a generating triple of the maximal subgroup $U_3(3)$. Our computations show that the elements of $7A$ are located in two different conjugacy classes $7a$ and $7b$ of $U_3(3)$ which have non-empty intersection with $3a$ and $4c$, then $\Sigma_{U_3(3)}((3a, 4c, 7a) + (3a, 4c, 7b) = 7 + 7 = 14$. Hence,

$$\Delta^* < \Delta(3A, 4A, 7A) - \Sigma_{U_3(3)}(3A, 4A, 7A) = 14 - 14 < |C_{J_2}(7A)| = 7.$$

Then $3A - 4A \notin E(\Omega(J_2))$. Therefore, $\Omega(J_2) \cong K_{20} - H_2$.

3. The Q-generating graph of M_{12} has 14 vertices with two conjugacy classes of involutions. By Atlas [14],

$$V(\Omega(M_{12})) = \{2A, 2B, 3A, 3B, 4A, 4B, 5A, 6A, 6B, 8A, 8B, 10A, 11A, 11B\}.$$

For $n > 4$, M_{12} is nX-complementary generated and so all vertices nX, $(n > 4)$ are connected in the graph. Obviously, $2A - 2B \notin E(\Omega(M_{12}))$. Also the vertex $2A$ is not adjacent with $3B$. Note that $\Delta(2A, 3B, 5A) = 20 > |C_{M_{12}}(5A)| = 10$ and there exists a maximal subgroup $L_2(11)$ which intersects conjugacy classes $2A$, $3B$ and $5A$. Then,

$$\Delta(2A, 3B, 5A) - \Sigma_{L_2(11)}((2a, 3b, 5a) + (2a, 3b, 5b)) = 20 - (10 + 10) = 0$$

and so $2A - 3B \notin E(\Omega(M_{12}))$. Also the vertex $2B$ is not adjacent to the vertices $3A$, $4A$ and $4B$. Since M_{11} is the maximal subgroup of M_{12} which has intersection with $2B$, $3A$ and $5A$, and

$$\Delta(2B,3A,5A) - \Sigma_{M_{11}}(2B,3A,5A) = 20 - 15 < |C_{M_{12}}(5A)| = 10,$$

$\Delta^*(2B,3A,5A) = 0$ and $2B - 3A \notin E(\Omega(M_{12}))$. Also, $\Delta(2B,4A,5A) = 20 > |C_{M_{12}}(5A)| = 10$ and the maximal subgroup $M2$ of M_{12} (Gap notation) has non-empty intersection with $2B$, $4A$ and $5A$. Hence

$$\Delta(2B,4A,5A) - \Sigma_{M2}(2B,4A,5A) = 20 - 15 < |C_{M_{12}}(5A)| = 10.$$

This shows that $\Delta^*(2B,4A,5A) = 0$ and $2B - 4A \notin E(\Omega(M_{12}))$. Again by GAP, one can see that

$$\Delta(2B,4B,5A) - \Sigma_{M_{11}}(2B,4B,5A) = 20 - 15 < |C_{M_{12}}(5A)| = 10.$$

Then $\Delta^*(2B,4B,5A) = 0$ and $2B - 4B$ is not an edge in $\Omega(M_{12})$ and $\Omega(M_{12}) \cong K_{14} - H_3$.

4. For the Mathieu group M_{24}, the graph $\Omega(M_{24})$ has 25 vertices as follows: For $n > 2$, all vertices nX are adjacent except $3A$ and $4B$, since M_{24} is nX-

$2A$	$2B$	$3A$	$3B$	$4A$	$4B$	$4C$	$5A$	$6A$	$6B$	$7A$
$7B$	$8A$	$10A$	$11A$	$12A$	$12B$	$14A$	$14B$	$15A$	$15B$	$21A$
$21B$	$23A$	$23B$								

complementary generated for $n > 2$ and $nX \neq 3A, 4B$. Clearly, $2A - 2B \notin E(\Omega(M_{24}))$, and for each conjugacy class tZ we have that $\Delta(2A,3A,tZ) < |C_{M_{24}}(tZ)|$. Then, $\Delta^*(2A,\, 3A, tZ) = 0$. Hence, M_{24} is not $(2A, 3A)$-generated. Moreover, $2A - 4B \notin E(\Omega(M_{24}))$, since $\Delta(2A,4B,3A) = 1215$, $|C_{M_{24}}(3A)| = 1080$ and the maximal subgroup M_{23} in triple $(2a, 4a, 3a)$ has this property that $\Sigma_{M_{23}}(2a,4a,3a) = 540$. On the other hand, $1215 - 540 = 675 < 1080$ which implies that $\Delta^*(2A,4B,3A) = 0$ and $\Omega(M_{24}) \cong K_{25} - H_4$.

5. The sporadic group McL has 23 non-identity conjugacy classes as vertices; According to Lemma 4.2(1), McL is nX-complementary generated for $n \geq 4$, and

$2A$	$3A$	$3B$	$4A$	$5A$	$5B$	$6A$	$6B$	$7A$	$7B$
$8A$	$9A$	$9B$	$10A$	$11A$	$11B$	$12A$	$14A$	$14B$	$15A$
$15B$	$30A$	$30B$							

it has only one conjugacy class of involutions. Then, we should check the adjacency between the vertices $2A$ and $3Y$, $Y \in \{A, B\}$. Our computations show that for any conjugacy class tZ, $\Delta(2A,3Y,tZ) < |C_{McL}(tZ)|$, which implies that $2A - 3Y \notin E(\Omega(McL))$, $Y \in \{A, B\}$. Hence, $\Omega(McL) \cong K_{23} - \{2A - 3A, 2A - 3B\}$.

6. The Harada-Norton group HN has 53 non-identity conjugacy classes as follows: Based on Lemma 4.2(11), HN is nX-complementary generated when $nX \neq 2A$,

$2A$	$2B$	$3A$	$3B$	$4A$	$4B$	$4C$	$5A$	$\cdots$	$5E$	$6A$
$6B$	$6C$	$7A$	$8A$	$8B$	$9A$	$10A$	$\cdots$	$10H$	$11A$	$12A$
$12B$	$12C$	$14A$	$15A$	$15B$	$15C$	$19A$	$19B$	$20A$	$\cdots$	$20E$
$21A$	$22A$	$25A$	$25B$	$30A$	$30B$	$30C$	$35A$	$35B$	$40A$	$40B$

$2B$, $3A$, $5A$ and $5B$, which means that the degree of the other vertices equals 52. Also based on Lemma 4.1(3) HN is (p, q, r)-generated when $(p, q, r) \neq (2, 3, 5)$. Thus, each vertex pX has degree 52 where p is a prime divisor of $|HN|$ except 2, 3 or 5. For each conjugacy class tZ, we have $\Delta(2A, nY, tZ) < |C_{HN}(tZ)|$, where $nY \in \{3A, 5A, 5B\}$. Then, $\{2A - 3A, 2A - 5A, 2A - 5B\} \nsubseteq E(\Omega(HN))$ and also it is obvious that $2A - 2B \notin E(\Omega(HN))$. Our computations show that these are the only pairs of conjugacy classes that cannot generate HN and do not belong to the set of edges of Q-generating graph $\Omega(HN)$. Hence, $\Omega(HN) \cong K_{53} - H_6$.

7. Fi_{22} has 64 non-identity conjugacy classes as the vertices of $\Omega(Fi_{22})$ with three conjugacy classes of involutions.

$2A$	$2B$	$2C$	$3A$	$3B$	$3C$	$3D$	$4A$	$\cdots$	$4E$	$5A$
$6A$	$\cdots$	$6K$	$7A$	$8A$	$8B$	$8C$	$8D$	$9A$	$9B$	$9C$
$10A$	$10B$	$11A$	$11B$	$12A$	$\cdots$	$12K$	$13A$	$13B$	$14A$	$15A$
$16A$	$16B$	$18A$	$18B$	$18C$	$18D$	$20A$	$21A$	$22A$	$22B$	$24A$
$24B$	$30A$									

By Lemma 4.1(3) and 4.2(8), we conclude that in this graph the vertices pX and qY are adjacent to all other vertices except when $p = 2$ and $q = 3$. Since for every non-identity conjugacy class tZ,

$$\Delta(2A, 3Y, tZ) < |C_{Fi_{22}}(tZ)|, \quad Y \neq A,$$
$$\Delta_{Fi_{22}}(2A, 4Y, tZ) < |C_{Fi_{22}}(tZ)|, \quad Y \in \{A, \ldots, E\},$$

we conclude that for these triples $\Delta^* = 0$ and $\{2A - 3B, 2A - 3C, 2A - 3D, 2A - 4Y\} \nsubseteq E(\Omega(Fi_{22}))$, where $Y \in \{A, \ldots, E\}$. Also our computations show that

$$\Delta(2A, 5A, 30A) - \Sigma_{2 \cdot U_6(2)}(2A, 5A, 30A) = 36 - 30 < |C_{Fi_{22}}(30A)| = 30,$$

which means $\Delta^* = 0$ and $2A - 5A \notin E(\Omega(Fi_{22}))$. Also $2A - 6Y \notin E(\Omega(Fi_{22}))$, $Y \in \{A, \ldots, I\}$ because for each vertex tZ, $\Delta(2A, 6Y, tZ) < |C_{Fi_{22}}(tZ)|$. Fi_{22} is not $6J$-complementary generated and

$$\Delta(2A, 6J, 14A) = 14 = \Sigma_{2 \cdot U_6(2)}(2A, 6J, 14A)$$

$$\Delta(2A, 6J, 21A) = 21 = \Sigma_{O_8^+(2):S_3}(2A, 6J, 21A)$$
$$\Delta(2A, 6J, 24A) = 24 = \Sigma_{O_8^+(2):S_3}(2A, 6J, 24A)$$

then $2A - 6J \notin E(\Omega(Fi_{22}))$. Besides

$$\Delta(2A, 8A, 10B) - \Sigma(2 \cdot U_6(2)) = 45 - 25 < |C_{Fi_{22}}(10B)| = 40$$
$$\Delta(2A, 8A, 12K) - \Sigma(O_8^+(2) : S_3) = 36 - 12 < |C_{Fi_{22}}(12K)| = 36,$$
$$\Delta(2A, 8A, 18D) - \Sigma(2 \cdot U_6(2)) = 36 - 18 < |C_{Fi_{22}}(18D)| = 36$$
$$\Delta(2A, 8A, 22A) - \Sigma(2 \cdot U_6(2)) = 22 - 22 < |C_{Fi_{22}}(22A)|,$$

which means that in each case $\Delta^* = 0$, so $2A$ and $8A$ are not adjacent. Since

$$\Delta(2A, 8B, 11Z) - \Sigma(2 \cdot U_6(2)) = 22 - 22 < |C_{Fi_{22}}(11Z)|,$$
$$\Delta(2A, 8B, 14A) - \Sigma(2 \cdot U_6(2)) = 14 - 14 < |C_{Fi_{22}}(14A)|,$$
$$\Delta(2A, 8B, 21A) - \Sigma(O_8^+(2) : S_3) = 21 - 21 < |C_{Fi_{22}}(21A)|,$$

where $Z \in \{A, B\}$, $\Delta^* = 0$ and $2A - 8B \notin E(\Omega(Fi_{22}))$. Our computations show that $\{2A - 9A, 2A - 12A, 2A - 12B, 2A - 12C, 2A - 12D\} \nsubseteq E(\Omega(Fi_{22}))$.

8. The group Fi_{23} has 97 non-identity conjugacy classes.
   By Lemma 4.2(9), for $n > 12$ or $n \in \{7, 8, 10, 11\}$, Fi_{23} is nX-complementary

$2A$	$2B$	$2C$	$3A$	$3B$	$3C$	$3D$	$4A$	$4B$	$4C$	$4D$
$5A$	$6A$	$6B$	$\cdots$	$6N$	$6O$	$7A$	$8A$	$8B$	$8C$	$9A$
$9B$	$9C$	$9D$	$9E$	$10A$	$10B$	$10C$	$11A$	$12A$	$12B$	$\cdots$
$12O$	$13A$	$13B$	$14A$	$14B$	$15A$	$15B$	$16A$	$16B$	$17A$	$18A$
$18B$	$\cdots$	$18H$	$20A$	$20B$	$21A$	$22A$	$22B$	$22C$	$23A$	$23B$
$24A$	$24B$	$24C$	$26A$	$26B$	$27A$	$28A$	$30A$	$30B$	$30C$	$35A$
$36A$	$36B$	$39A$	$39B$	$42A$	$60A$					

generated which means $\deg(nX) = 96$ for such n. If $nX \in \{6N, 6O, 9D, 9E, 12C, \ldots, 12O\}$, $\deg(nX) = 96$. Clearly, Fi_{23} is not $2X$-complementary generated, and then $\{2A - 2B, 2A - 2C, 2B - 2C\} \nsubseteq E(\Omega(Fi_{23}))$. For each conjugacy class tZ, we have that

$$\Delta(2A, 3Y, tZ) < |C_{Fi_{23}}(tZ)|, \quad Y \in \{A, B, C, D\},$$
$$\Delta(2A, 4Y, tZ) < |C_{Fi_{23}}(tZ)|, \quad Y \in \{A, B, C, D\},$$
$$\Delta(2A, 5Y, tZ) < |C_{Fi_{23}}(tZ)|, \quad Y = A,$$
$$\Delta(2A, 6Y, tZ) < |C_{Fi_{23}}(tZ)|, \quad Y \neq N, O,$$
$$\Delta(2A, 12Y, tZ) < |C_{Fi_{23}}(tZ)|, \quad Y \in \{A, B\},$$
$$\Delta(2A, 9Y, tZ) < |C_{Fi_{23}}(tZ)|, \quad Y \in \{A, B\}.$$

For the pair $(2A, 9C)$, our computations show that $\Delta(2A, 9C, 30C) = 30 = |C_{Fi_{23}}(30C)|$. Besides, the maximal subgroup $M = O_8^+(3) \cdot 3 \cdot 2$ has intersection with the conjugacy classes $2A$, $9C$ and $30C$, such that

$$\Delta(2A, 9C, 30C) - \Sigma_M(2A, 9C, 30C) = 30 - 30 < |C_{Fi_{23}}(30C)|,$$

then $\Delta^*(2A, 9C, 30C) = 0$ and $2A - 9C \notin E(\Omega(Fi_{23}))$. For vertex $2B$ we obtained that for each conjugacy class tZ

$$\Delta(2B, 3Y, tZ) < |C_{Fi_{23}}(tZ)|, \quad Y \in \{A, B, C\},$$
$$\Delta(2B, 6A, tZ) < |C_{Fi_{23}}(tZ)|,$$

and also for $tZ = 20A$, since

$$\Delta(2B, 4A, 20A) - \Sigma_{2.Fi_{22}}(2B, 4A, 20A) = 165 - 55 < |C_{Fi_{23}}(20A)| = 120,$$

$2B - 4A \notin E(\Omega(Fi_{23}))$. Moreover,

$$\Delta(2B, 6B, 18G) - \Sigma_{O_8^+(3)\cdot 3\cdot 3}(2B, 6B, 18G) = 81 - 81 < |C_{Fi_{23}}(18G)| = 54,$$

and $2B - 6B \notin E(\Omega(Fi_{23}))$. The conjugacy class $2C$ is not adjacent to $4Y$, $Y \in \{A, B\}$, because for each vertex tZ, $\Delta(2C, 4Y, tZ) < |C_{Fi_{23}}(tZ)|$. Consequently, $\Omega(Fi_{23}) \cong K_{97} - H_8$.

9. The Conway group Co_1 has 100 non-identity conjugacy classes as follows,

$2A$	$2B$	$2C$	$3A$	$3B$	$3C$	$3D$	$4A$	$\cdots$	$4F$	$5A$
$5B$	$5C$	$6A$	$\cdots$	$6I$	$7A$	$7B$	$8A$	$\cdots$	$8F$	$9A$
$9B$	$9C$	$10A$	$\cdots$	$10F$	$11A$	$12A$	$\cdots$	$12M$	$14A$	$14B$
$15A$	$\cdots$	$15E$	$16A$	$16B$	$18A$	$18B$	$18C$	$20A$	$20B$	$20C$
$21A$	$21B$	$21C$	$22A$	$23A$	$23B$	$24A$	$\cdots$	$24F$	$26A$	$28A$
$28B$	$30A$	$\cdots$	$30E$	$33A$	$35A$	$36A$	$39A$	$39B$	$40A$	$42A$
$60A$										

which are the vertices of $\Omega(Co_1)$. Based on Lemma 4.2(5), $\deg(nX) = 99$, for $n \geq 4$ and $nX \notin \{4A, 4B, 4C, 4D, 5A, 6A\}$. Since for each vertex tZ, we have

$$\Delta(2A, 3A, tZ) < |C_{Co_1}(tZ)|,\ \Delta(2B, 3A, tZ) < |C_{Co_1}(tZ)|,$$
$$\Delta(3A, 5A, tZ) < |C_{Co_1}(tZ)|,\ \Delta(3A, 4D, tZ) < |C_{Co_1}(tZ)|,$$
$$\Delta(2A, 4C, tZ) < |C_{Co_1}(tZ)|,\ \Delta(2A, 6A, tZ) < |C_{Co_1}(tZ)|,$$

then $\Delta^* = 0$ and $\{2A - 3A, 2B - 3A, 3A - 5A, 3A - 4D, 2A - 4C, 2A - 6A\} \not\subseteq E(\Omega(Co_1))$. For two vertices $2B$ and $3B$, $\Delta(2B, 3B, tZ) = 84 > |C_{Co_1}(tZ)| = 72$ and there is a maximal subgroup $M = 3.Suz.2$, such that $\Sigma_M(2B, 3B, 12L) = 24$ and $\Delta(2B, 3B, 12L) - \Sigma_M(2B, 3B, 12L) = 84 - 24 <$

72 which implies that $\Delta^*(2B, 3B, 12L) = 0$ and $2B - 3B \notin E(\Omega(Co_1))$. Also we have $\Delta(2A, 4B, 12L) = 96 > |C_{Co_1}(12L)| = 72$, but

$$\Delta(2A, 4B, 12L) - \Sigma_M(2A, 4B, 12L) = 96 - 24 < |C_{Co_1}(12L)|,$$

which means that $(2A, 4B)$ is not a generating pair of Co_1. Similarly, we have

$$\Delta(2B, 4A, 12E) - \Sigma_M(2B, 4A, 12E) = 243 - 81 < |C_{Co_1}(12E)| = 216,$$

so the pair of $(2B, 4A)$ does not generate Co_1. For the conjugacy classes $2C$ and $4A$, our calculations show that $\Delta(2C, 4A, tZ)$ is greater than $|C_{Co_1}(tZ)|$, for every vertex tZ. On the other hand, there is no maximal subgroup containing the subgroup $\langle 2C, 4A\rangle$ and so $(2C, 4A)$ is a generating pair for Co_1. This concludes that $\Omega(Co_1) \cong K_{100} - H_9$.

10. The Conway group Co_2 has 59 non-identity conjugacy classes as the vertices of $\Omega(Co_2)$ which are

2A	2B	2C	3A	3B	4A	···	4G	5A	5B	6A
···	6F	7A	8A	···	8F	9A	10A	10B	10C	11A
12A	···	12H	14A	14B	14C	15A	15B	15C	16A	16B
18A	20A	20B	23A	23B	24A	24B	28A	30A	30B	30C

According to our calculation with GAP, we can see that for each vertex tZ, $\Delta(2A, 3Y, tZ)$, $Y \in \{A, B\}$, $\Delta(2A, 4Y, tZ)$, $Y \in \{A, B, C, D\}$, $\Delta(2B, 3A, tZ)$ and $\Delta(2B, 4A, tZ)$ are less than $|C_{Co_2}(tZ)|$. Since the group generated by two involutions is isomorphic to dihedral group, the conjugacy classes of the involutions are not adjacent in $\Omega(Co_2)$. Up to isomorphism, the Conway group Co_2 has eleven maximal subgroups $m_1, m_2, \ldots, m_{11}$ as follows:

$$m_1 = U_6(2) \cdot 2 \qquad m_2 = 2^{10} : M_{22} : 2 \qquad m_3 = McL$$
$$m_4 = 2^1 + 8 : s6f2 \qquad m_5 = HS \cdot 2 \qquad m_6 = 2^1 + 4 + 6 \cdot a8$$
$$m_7 = U_4(3) \cdot D_8 \qquad m_8 = 2^{(4+10)}(S_5 \times S_3) \qquad m_9 = M_{23}$$
$$m_{10} = 3^1 + 4 : 2^1 + 4 \cdot s5 \quad m_{11} = 5^{(1+2)} : 4S_4$$

For two vertices $2B, 3B$ we have $\Delta(2B, 3B, 7A) = 91 > |C_{Co_2}(7A)| = 56$. The maximal subgroup m_1 has non-empty intersection with these conjugacy classes and $\Sigma_{m_1}(2B, 3B, 7A) = 63$, so $91 - 63 < 56$, which implies that $\Delta^*(2B, 3B, 7A) = 0$. For vertices $2A$, $4E$ and $4F$ we have,

$$\Delta(2A, 4E, 10C) - \Sigma_{m_4}(2A, 4E, 10C) = 50 - 17 < |C_{Co_2}(10C)| = 40,$$
$$\Delta(2A, 4F, 11A) - \Sigma_{m_2}(2A, 4F, 11A) = 11 - 11 < |C_{Co_2}(11A)| = 11.$$

Then $\Delta^*(2A, 4E, 10C) = 0$ and $\Delta^*(2A, 4F, 11A) = 0$. Since

$$\Delta(2B, 4B, 10C) - \Sigma_{m_2}(2B, 4B, 10C) = 40 - 30 < |C_{Co_2}(10C)| = 30,$$

$$\Delta(2B, 4C, 7A) - \Sigma_{m_1}(2B, 4C, 7A) = 91 - 63 < |C_{Co_2}(7A)| = 56,$$
$$\Delta(2B, 4D, 15A) - \Sigma_{m_1}(2B, 4D, 15A) = 45 - 30 < |C_{Co_2}(15A)| = 30,$$
$$\Delta(2B, 4E, 11A) - \Sigma_{m_1}(2B, 4E, 11A) = 11 - 11 < |C_{Co_2}(11A)| = 11,$$

$2B - 4Y \notin E(\Omega(Co_2))$, where $Y \in \{B, C, D, E\}$. For the vertices $2C$ and $4A$, we have that $\Delta(2C, 4A, 12G) = 84 > |C_{Co_2}(12G)| = 48$ and the maximal subgroups which have non-empty intersection with three conjugacy classes $2C$, $4A$ and $12G$ are conjugate to m_1 or m_4. Then

$$[\Delta - (\Sigma_{m_1} + \Sigma_{m_4})](2C, 4A, 12G) = 84 - (24 + 36) < |C_{Co_2}(12G)| = 48.$$

By Lemma 4.2(6), Co_2 is $4G$-, $5A$-, $5B$-generated, then they are adjacent to all other vertices in $\Omega(Co_2)$. On the other hand, the group Co_2 is $6A$-, $6B$-, $6E$- and $6F$-complementary generated, but for the pair $(2A, 6C)$, one can see that for each vertex tZ,

$$\Delta^*(2A, 6C, tZ) \leq (\Delta_{Co_2} - \Sigma_{m_1})(2A, 6C, tZ) < |C_{Co_2}(tZ)|,$$

where $tZ \in \{14A, 16B, 18A, 24A\}$ and m_1 is the only maximal subgroup of Co_2 with non-empty intersection by $2A$ and $6C$. Moreover, for the pair $(2A, 6D)$, again we have

$$\Delta^*(2A, 6D, tZ) \leq (\Delta_{Co_2} - \Sigma_{m_1})(2A, 6D, tZ) < |C_{Co_2}(tZ)|,$$

where $tZ \in \{7A, 9A, 10B, 11A, 16A, 18A\}$. Furthermore $\{2A-6C, 2A-6D\} \nsubseteq E(\Omega(Co_2))$. Since Co_2 is nX-complementary generated for $n \geq 7$, for these vertices, we have $\deg(nX) = 58$ and $\Omega(Co_2) \cong K_{59} - H_{10}$.

11. The Conway group Co_3 has 41 non-identity conjugacy classes as the vertices of $\Omega(Co_3)$ which are

$2A$	$2B$	$3A$	$3B$	$3C$	$4A$	$4B$	$5A$	$5B$	$6A$
$6B$	$6C$	$6D$	$6E$	$7A$	$8A$	$8B$	$8C$	$9A$	$9B$
$10A$	$10B$	$11A$	$12A$	$12B$	$12C$	$14A$	$15A$	$15B$	$18A$
$20A$	$20B$	$21A$	$22A$	$22B$	$23A$	$23B$	$24A$	$24B$	$30A$

By Lemma 4.1(3), Co_3 is $(pX, qY, 23Z)$-generated for the primes $p \leq q$ and $pX \neq qY$, if and only if $(pX, qY) \notin \{(2A, 3A), (2A, 3B), (2B, 3A)\}$, then we should obtain the adjacency of these pairs. Since for each conjugacy class tZ, $\Delta(2A, 3A, tZ) < |C_{Co_3}(tZ)|$, then $2A - 3A \notin E(\Omega(Co_3))$. For two classes $2A$ and $4A$ and each class tZ, we have $\Delta(2A, 4A, tZ) \geq |C_{Co_3}(tZ)|$, but there is a maximal subgroup $McL:2$ which is of order divisible by 8 such that

$$\Delta(2A, 4A, 8B) - \Sigma_{McL:2}(2A, 4A, 8B) = 260 - 164 < |C_{Co_3}(8B)|,$$
$$\Delta(2A, 4A, 10B) - \Sigma_{McL:2}(2A, 4A, 10B) = 30 - 25 < |C_{Co_3}(10B)|,$$

$$\Delta(2A, 4A, 24A) - \Sigma_{McL:2}(2A, 4A, 24A) = 24 - 24 < |C_{Co_3}(24A)|.$$

Then, $2A - 4A \notin E(\Omega(Co_3))$. For the vertices $2A$ and $3B$, $\Delta(2A, 3B, 7A) = 63 > |C_{Co_3}(7A)| = 42$, but for the maximal subgroup $McL : 2$ we have $\Sigma_{McL:2}(2a, 3a, 7a) = 49$, then

$$\Delta^* < \Delta(2A, 3B, 7A) - \Sigma_{McL:2}(2a, 3a, 7a) = 63 - 49 < |C_{Co_3}(7A)| = 42$$

and $2A - 3B \notin E(\Omega(Co_3))$. Also for the conjugacy classes $2B$ and $3A$, since $\Delta(2B, 3A, 10B) = |C_{Co_3}(10B)| = 20$ and $\Sigma_{McL:2}(2b, 3a, 10b) = 10$,

$$\Delta^* < \Delta(2B, 3A, 10B) - \Sigma_{McL:2}(2b, 3a, 10b) = 20 - 10 < |C_{Co_3}(10B)| = 20,$$

which means that $\Delta^* = 0$ and $2B - 3A \notin E(\Omega(Co_3))$. Our calculations with GAP show that for each $x \in 2A$ and $y \in 4B$, $\langle x, y\rangle$ is a proper subgroup in Co_3 and so it is not a generating pair. Hence, $2A - 4B \notin E(\Omega(Co_3))$. Also, the Conway group Co_3 is $3C$-complementary generated, $\deg(3C) = 40$ and similarly for all nX, $n > 4$, $\deg(nX) = 40$. As a result, we can see that $\Omega(Co_3) \cong K_{41} - H_{11}$.

12. The Suzuki group *Suz* has 42 non-identity conjugacy classes in which there are two classes of involutions that are not adjacent in Q-generating graph $\Omega(Suz)$. The vertices are listed as follows

$2A$	$2B$	$3A$	$3B$	$3C$	$4A$	$4B$	$4C$	$4D$	$5A$	$5B$
$6A$	$\cdots$	$6E$	$7A$	$8A$	$8B$	$8C$	$9A$	$9B$	$10A$	$10B$
$11A$	$12A$	$\cdots$	$12E$	$13A$	$13B$	$14A$	$15A$	$15B$	$15C$	$18A$
$18B$	$18B$	$20A$	$21A$	$21B$	$24A$					

By Lemma 4.2(10), since *Suz* is $3C$-complementary generated, $\deg(3C) = 41$. Also for $n \geq 4$, $\deg(nX) = 41$ except when $nX = 4A$ or $6A$. For each conjugacy class tZ, we have that

$$\Delta(2A, 3A, tZ) < |C_{Suz}(tZ)|, \qquad \Delta(2A, 4A, tZ) < |C_{Suz}(tZ)|,$$
$$\Delta(3A, 3B, tZ) < |C_{Suz}(tZ)|, \qquad \Delta(3A, 4A, tZ) < |C_{Suz}(tZ)|.$$

Then, $\{2A - 3A, 2A - 4A, 3A - 3B, 3A - 4A\} \nsubseteq E(\Omega(Suz))$ and for the pair $(3A, 6A)$, our calculations show that

$$\Delta(3A, 6A, 7A) - \Sigma_{G_2(4)}(3A, 6A, 7A) = 112 - 63 < |C_{Suz}(7A)| = 84,$$

where $G_2(4)$ is the maximal subgroup of *Suz* which contains this triple. Hence, $\Delta^* = 0$ and $3A - 6A \notin E(\Omega(Suz))$. Thus $\Omega(Suz) \cong K_{42} - H_{12}$.

13. The Higman–Sims group *HS* has 23 non-identity conjugacy classes as the vertices of $\Omega(HS)$, where $2A - 2B \notin E(\Omega(HS))$ and based on Lemma 4.2(12), *HS* is nX-complementary generated for $nX = 4C$ or $n \geq 5$, so for these vertices $\deg(nX) =$

22. But according to the computations by GAP and [26], we see that for every conjugacy class tZ, $\Delta(2A, 3A, tZ) < |C_{HS}(tZ)|$, $\Delta(2A, 4A, tZ) < |C_{HS}(tZ)|$ and $\Delta(2A, 4B, tZ) < |C_{HS}(tZ)|$. For example

$2A$	$2B$	$3A$	$4A$	$4B$	$4C$	$5A$	$5B$	$5C$	$6A$
$6B$	$7A$	$8A$	$8B$	$8C$	$10A$	$10B$	$11A$	$11B$	$12A$
$15A$	$20A$	$20B$							

$$\Delta(2A, 4B, 6B) - \Sigma_{M_{22}}(2A, 4B, 6B) = 48 - 36 < |C_{HS}(6B)|,$$
$$\Delta(2A, 4B, 8A) - \Sigma_{M_{22}}(2A, 4B, 8A) = 46 - 44 < |C_{HS}(8A)|,$$
$$\Delta(2A, 4B, 11X) - \Sigma_{M_{22}}(2A, 4B, 11X) = 22 - 22 < |C_{HS}(11X)|,$$
$$\Delta(2A, 4B, 12A) - \Sigma_{S_8}(2A, 4B, 12A) = 18 - 18 < |C_{HS}(12A)|,$$
$$\Delta(2A, 4B, 15A) - \Sigma_{S_8}(2A, 4B, 15A) = 15 - 15 < |C_{HS}(15A)|,$$

and for the triple $(2A, 4B, 7A)$, the maximal subgroup M_{22} has two conjugacy classes with non-empty intersection with these three classes, say m_1 and m_2. Then, we have

$$\Delta^* < \Delta - (\Sigma_{m_1} + \Sigma_{m_2} - \Sigma_{m_1 \cap m_2})(2A, 4B, 7A) = 7 - (28 - 21) = 0.$$

Then, $\{2A - 3A, 2A - 4A, 2A - 4B\} \nsubseteq E(\Omega(HS))$. The only conjugacy class which has a non-empty intersection with $2B$ and $4A$ is $7A$ and $\Delta(2B, 4A, 7A) = 7 = |C_{HS}(7A)|$. The maximal subgroup $U_3(5) \cdot 2$ has non-empty intersection with the classes of $2B$, $4A$ and $7A$ such that

$$\Delta(2B, 4A, 7A) - \Sigma_{U_3(5)\cdot 2}(2B, 4A, 7A) = 7 - 7 < |C_{HS}(7A)|,$$

then $2B - 4A \notin E(\Omega(HS))$. Hence $\Omega(HS) \cong K_{23} - H_{13}$.

This completes our argument. □

We end this paper with the following theorem that its proof is similar to those cases given in Theorem 4.3 and so omitted.

Theorem 4.4 *If G is one of the following groups, then $\Omega(G)$ is obtained by removing an edge from a complete graph and we have that*

1. $\Omega(M_{11}) \cong K_9 - \{2A - 3A\}$,
2. $\Omega(M_{22}) \cong K_{11} - \{2A - 3A\}$,
3. $\Omega(M_{23}) \cong K_{16} - \{2A - 3A\}$,
4. $\Omega(Ru) \cong K_{35} - \{2A - 2B\}$,
5. $\Omega(J_4) \cong K_{61} - \{2A - 2B\}$,
6. $\Omega(Ly) \cong K_{52} - \{2A - 3A\}$.

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