Bull. Malays. Math. Sci. Soc. (2019) 42:1657–1673
https://doi.org/10.1007/s40840-017-0571-z

BULLETIN of the MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

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Multiplicity of Solutions for Kirchhoff-Type Problem with Two-Superlinear Potentials

Guanggang Liu[1] · **Shaoyun Shi**[2] · **Yucheng Wei**[3]

Received: 15 June 2017 / Revised: 13 September 2017 / Published online: 13 November 2017

Abstract In this paper we consider a class of Kirchhoff-type problem with 2-superlinear potentials. The existence of one positive solution and one negative solution will be established by using iterative technique and the Mountain Pass theorem, and a sign changing solution will be obtained by combining iterative technique and the Nehari method.

Keywords Kirchhoff-type problem · Mountain Pass theorem · Nehari method · Iterative technique

Mathematics Subject Classification 35J20 · 35J25 · 35J60

Communicated by Syakila Ahmad.

This work was supported by Doctoral foundation of Liaocheng University, NSFC Grant (No. 11771177), China Automobile Industry Innovation and Development Joint Fund (No. U1664257), Program for Changbaishan Scholars of Jilin Province and Program for JLU Science and Technology Innovative Research Team (No. 2017TD-20).

✉ Shaoyun Shi
shishaoyun2013@163.com

Guanggang Liu
lgg112@163.com

Yucheng Wei
ychengwei@126.com

1 School of Mathematical Sciences, Liaocheng University, Liaocheng 252000, People's Republic of China

2 School of Mathematics and State Key Laboratory of Automotive Simulation and Control, Jilin University, Changchun 130012, People's Republic of China

3 Department of Mathematics, Hechi University, Yizhou 546300, People's Republic of China

1 Introduction

In this paper we consider the following Kirchhoff-type problem

$$\begin{cases} -(a+b\int_\Omega |\nabla u|^2 \mathrm{d}x)\Delta u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded smooth domain. The problem (1.1) is related to the stationary analogue of the equation

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = f(x,u) \tag{1.2}$$

proposed by Kirchhoff [8] as an existence of the classical D'Alembert's wave equations for free vibration of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Equation (1.2) received much attention only after Lions [12] introduced an abstract framework to the problem. In recent years, the Kirchhoff-type problem on a bounded domain $\Omega \subset \mathbb{R}^N$ or on $\mathbb{R}^N$ has been studied by many authors, see [1–3,5–7,9–11,14,15,17,19,21–28] and references therein. To obtain the existence of solution by applying the Mountain Pass theorem or Morse theory, the authors have to impose a 4-superlinear or 4-asymptotically linear growth condition on the nonlinearity. For example, Perera and Zhang [17] considered the case where $f(x,\cdot)$ is asymptotically linear near zero and asymptotically 4-linear at infinity; they obtained a nontrivial solution of the problem by using the Yang index and critical group. For the cases when $f(x,\cdot)$ is 4-sublinear, 4-superlinear and asymptotically 4-linear at infinity, Zhang and Perera [26] obtained the existence of multiple and sign changing solutions by using variational methods and invariant sets of descent flow. He and Zou [5,6] obtained infinitely many solutions by using the local minimax methods and the fountain theorems under the 4-superlinear condition. Sun and Liu [20] obtained nontrivial solutions via Morse theory when the nonlinearity is superlinear near zero but asymptotically 4-linear at infinity, and the nonlinearity is asymptotically linear near zero but 4-superlinear at infinity. Recently, Li et al. [9] discussed the existence of positive solutions to the 2-superlinear Kirchhoff problem in $\mathbb{R}^N$, $N \geq 3$; they obtained at least one positive radial solution by using truncation technique, Pohozaev-type identity and variational methods. Later, Zhang et al. [28] considered the 2-superlinear Kirchhoff problem in a smooth bounded convex domain; they established the existence of one positive solution using iterative technique, Pohozaev-type identity and variational methods.

Motivated by Li et al. [9] and Zhang et al. [28], in this paper we consider the 2-superlinear Kirchhoff problem in a bounded smooth domain but not necessarily convex. By using the iterative technique proposed in Figuereido et al. [4] and the Mountain Pass theorem, one positive solution and one negative solution for (1.1) will be obtained. Moreover, we will get a sign changing solution by combining the iterative technique and the Nehari method.

We make the following assumptions:

(H0) $f \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and there exist $a_1 > 0$, $p \in (1, \frac{N+2}{N-2})$ such that

$$f'(x,t) \leq a_1(1+|t|^{p-1}), \forall x \in \overline{\Omega}, t \in \mathbb{R},$$

where $f' = \frac{\partial f}{\partial t}$;

(H1) $\lim_{t\to 0} \frac{f(x,t)}{t} = 0$ uniformly for $x \in \overline{\Omega}$;

(H2) There exist constant $\theta > 2$ and $t_0 > 0$ such that

$$0 < \theta F(x,t) < tf(x,t), \forall x \in \overline{\Omega}, \quad |t| \geq t_0$$

where $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$;

(H3) $f'(x,t) > \frac{f(x,t)}{t}$ for all $t \neq 0$, $x \in \overline{\Omega}$.

Our main result is the following theorem.

Theorem 1.1 *Assume that (H0)–(H3) hold. Then, there exists a constant $b_0 > 0$ such that for any $b \in [0, b_0]$, the problem (1.1) has at least three nontrivial solutions; among them, one is positive, one is negative, and one is sign changing.*

The paper is organized as follows. In Sect. 2 we will prove the existence of positive and negative solutions for (1.1) by using the Mountain Pass theorem and the iterative technique. In Sect. 3, the existence of sign changing solution for (1.1) will be proved via the Nehari method and the iterative technique, and we will prove Theorem 1.1.

2 Positive and Negative Solutions

The aim of this section is to prove the positive and negative solutions for (1.1). We consider only the existence of positive solutions for (1.1), and the existence of negative solutions can be done by a similar argument.

Let $E = H_0^1(\Omega)$ be the usual Sobolev space equipped with the following inner product and norm,

$$\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x, \|u\| = \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

Let

$$f_+(x,t) = \begin{cases} f(x,t), & \text{if } t \geq 0, \\ 0, & \text{if } t < 0, \end{cases}$$

and

$$F_+(x,t) = \int_0^t f_+(x,s)\mathrm{d}s,$$

then we can conclude easily that $f_+(x,t)$ also satisfies (H0), (H1), (H2) for $t \geq t_0$ and (H3) for $t > 0$.

For any given $w \in E$, let us consider the following problem

$$\begin{cases} -(a + b\int_\Omega |\nabla w|^2 dx)\Delta u = f_+(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Define $I_{w+}(u)$ by

$$I_{w+}(u) = \frac{1}{2}\left(a + b\int_\Omega |\nabla w|^2 dx\right)\int_\Omega |\nabla u|^2 dx - \int_\Omega F_+(x, u)dx.$$

By (H0), $I_{w+} \in C^2(E, \mathbb{R})$ is weakly lower semi-continuous, and it is clearly that the weak solution of the problem (2.1) corresponds to the critical point of the functional I_{w+}, see [18].

First, we prove that $I_{w+}(u)$ satisfies the (PS) condition.

Lemma 2.1 *Assume that (H0), (H2) hold, then $I_{w+}(u)$ satisfies the (PS) condition.*

Proof Let $\{u_n\}$ be a sequence such that $|I_{w+}(u_n)| \le M$ for some constant $M > 0$ and $I'_{w+}(u_n) \to 0$ as $n \to \infty$. By a standard argument (see [18]), it suffices to prove that $\{u_n\}$ is bounded. From (H0) and (H2), there exists $C_0 > 0$ such that

$$\begin{aligned} M + o(\|u_n\|) &\ge I_{w+}(u_n) - \frac{1}{\theta}(I'_{w+}(u_n), u_n) \\ &\ge \left(\frac{1}{2} - \frac{1}{\theta}\right) a \int_\Omega |\nabla u_n|^2 dx + \int_\Omega \left(\frac{1}{\theta} f_+(x, u_n)u_n - F_+(x, u_n)\right) dx \\ &\ge \left(\frac{1}{2} - \frac{1}{\theta}\right) a\|u_n\|^2 + \int_{u_n > t_0} \left(\frac{1}{\theta} f_+(x, u_n)u_n - F_+(x, u_n)\right) dx - C_0 \\ &\ge \left(\frac{1}{2} - \frac{1}{\theta}\right) a\|u_n\|^2 - C_0. \end{aligned} \tag{2.2}$$

This implies that $\{u_n\}$ is bounded in E. □

Next, we prove that I_{w+} has the geometry of the Mountain Pass theorem.

Lemma 2.2 *Assume that (H0), (H1) hold, then there exist $\rho > 0$ and $\alpha > 0$, which are independent of w, such that for any $u \in E$ and $\|u\| = \rho$,*

$$I_{w+}(u) \ge \alpha.$$

Proof By (H0) and (H1), for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|F_+(x, u(x))| \le \frac{\varepsilon}{2}|u|^2 + C_\varepsilon |u|^{p+1}.$$

Choose ε small enough, by Sobolev inequality we have that

$$\begin{aligned} I_{w+}(u) &\geq \frac{a}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega F_+(x,u)\mathrm{d}x \\ &\geq \frac{a}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \frac{\varepsilon}{2}\int_\Omega |u|^2 \mathrm{d}x - C_\varepsilon \int_\Omega |u|^{p+1}\mathrm{d}x \\ &\geq \frac{1}{2}\left(a - \frac{\varepsilon}{\lambda_1}\right)\|u\|^2 - C_\varepsilon^{'}\|u\|^{p+1} \\ &\geq \left(\frac{a}{4} - C_1^{'}\|u\|^{p-1}\right)\|u\|^2 \end{aligned} \tag{2.3}$$

where $C_1^{'}$ is a constant independent of w and λ_1 is the first eigenvalue of $-\Delta$. Since $p > 1$, then we can choose $\rho > 0$ such that $C_1^{'}\rho^{p-1} \leq \frac{a}{8}$. Let $\alpha = \frac{a}{8}\rho^2$, then by (2.3), for all $u \in \partial B_\rho(0)$, we have that $I_{w+}(u) \geq \alpha$. □

Lemma 2.3 *Assume that (H0), (H2) hold, then for any given positive function* $v_0 \in E$ *with* $\|v_0\| = 1$, *there exists* $T > 0$ *such that for all* $s \geq T$,

$$I_{w+}(sv_0) \leq 0.$$

Proof By (H2), there exist positive constants C_1, C_2 such that

$$F_+(x,t) \geq C_1|t|^\theta - C_2.$$

Hence, for $s \geq 0$ we have

$$\begin{aligned} I_{w+}(sv_0) &= \frac{1}{2}\left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)\|sv_0\|^2 - \int_\Omega F_+(x,sv_0)\mathrm{d}x \\ &\leq \frac{1}{2}\left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)s^2 - C_1 s^\theta \int_\Omega |v_0|^\theta \mathrm{d}x + C_2|\Omega|, \end{aligned}$$

where $|\Omega|$ is the Lebesgue measure of Ω, combining with $\theta > 2$, we conclude that there exists $T > 0$ such that $I_{w+}(sv_0) < 0$ for $s \geq T$. □

Theorem 2.1 *Assume (H0)–(H2) hold, then there exists a constant* $b_1 > 0$ *such that for* $b \in [0, b_1]$, (1.1) *has at least one positive solution and one negative solution.*

Proof The proof will be divided into three steps.

Step 1 For any given $w \in E$, (2.1) has a positive solution u_w with $\|u_w\| \geq c_1$ for some constant $c_1 > 0$ independent of w and b.

Let

$$c_w = \inf_{g\in\Gamma} \max_{u\in g([0,1])} I_{w+}(u), \tag{2.4}$$

where

$$\Gamma = \{g \in C([0,1], E) \mid g(0) = 0, g(1) = Tv_0\}.$$

By Lemmas 2.1, 2.2, 2.3 and the well-known Mountain Pass theorem [18], c_w is a critical value of I_{w+}, so there is a $u_w \in E$ such that $I_{w+}(u_w) = c_w$ and $I'_{w+}(u_w) = 0$. Hence, u_w satisfies

$$\begin{cases} -(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x)\Delta u_w = f_+(x, u_w), & \text{in } \Omega, \\ u_w = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Multiplying Eq. (2.5) by u_w^- with $u_w^- = \min\{u_w, 0\}$ and integrating on Ω, we get

$$\left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)\int_\Omega |\nabla u_w^-|^2 \mathrm{d}x = \int_\Omega f_+(x, u_w)u_w^- = 0,$$

so $u_w^- \equiv 0$. This shows u_w is a positive solution of (2.1).

On the other hand, by (H0) and (H1), given $\varepsilon > 0$, there exists a positive constant C_ε, such that

$$|f_+(x, t)| \le \varepsilon|t| + C_\varepsilon|t|^p,$$

then using Eq. (2.5), by Sobolev inequality we obtain

$$\begin{aligned} \left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)\int_\Omega |\nabla u_w|^2 \mathrm{d}x &= \int_\Omega f_+(x, u_w)u_w \mathrm{d}x \\ &\le \varepsilon\int_\Omega |u_w|^2 \mathrm{d}x + C_\varepsilon\int_\Omega |u_w|^{p+1} \mathrm{d}x \\ &\le \frac{\varepsilon}{\lambda_1}\int_\Omega |\nabla u_w|^2 \mathrm{d}x + C'_\varepsilon\left(\int_\Omega |\nabla u_w|^2 \mathrm{d}x\right)^{\frac{p+1}{2}}; \end{aligned}$$

thus,

$$\left(a - \frac{\varepsilon}{\lambda_1}\right)\|u_w\|^2 \le C'_\varepsilon\|u_w\|^{p+1},$$

which implies that there exists a positive constant c_1, independent of w and b, such that

$$\|u_w\| \ge c_1. \tag{2.6}$$

Step 2 We construct a bounded positive functions sequence $\{u_n\}$ in E such that $I'_{u_{n-1}+}(u_n) = 0$ for any $n \ge 2$.

We fix a constant $L > 0$ throughout this paper. Let $b(R) = \frac{L}{R^2}$ for $R > 0$, then for any $w \in E$ with $\|w\| \le R$, any positive function $v_0 \in E$ with $\|v_0\| = 1$ and $b \in [0, b(R)]$, by (2.4) and (H2), we have

$$\begin{aligned} c_w &= I_{w+}(u_w) \\ &\le \max_{t\ge 0} I_{w+}(tv_0) \end{aligned}$$

$$
\begin{aligned}
&\leq \max_{t\geq 0}\left\{\frac{1}{2}(a+bR^2)\|tv_0\|^2-\frac{1}{2}\int_\Omega F_+(tv_0)\mathrm{d}x\right\}\\
&\leq \max_{t\geq 0}\left\{\frac{1}{2}(a+L)t^2-\frac{1}{2}\int_\Omega (C_1t^\theta|v_0|^\theta-C_2)\mathrm{d}x\right\}\\
&\leq C_1(L),
\end{aligned}
$$

where $C_1(L)$ is a constant independent of b, R and w. On the other hand, since $\langle I'_{w+}(u_w), u_w\rangle = 0$, by (H2),

$$
\begin{aligned}
c_w &= I_{w+}(u_w)-\frac{1}{\theta}\langle I'_{w+}(u_w),u_w\rangle\\
&=\left(\frac{1}{2}-\frac{1}{\theta}\right)\left(a+b\int_\Omega|\nabla w|^2\mathrm{d}x\right)\int_\Omega|\nabla u_w|^2\mathrm{d}x\\
&\quad+\int_\Omega\left(\frac{1}{\theta}f_+(x,u_w)u_w-F_+(x,u_w)\right)\mathrm{d}x\\
&\geq\left(\frac{1}{2}-\frac{1}{\theta}\right)\left(a+b\int_\Omega|\nabla w|^2\mathrm{d}x\right)\int_\Omega|\nabla u_w|^2\mathrm{d}x-C_0.
\end{aligned}
$$

Hence,

$$
\int_\Omega|\nabla u_w|^2\mathrm{d}x\leq\frac{c_w+C_0}{\left(\frac{1}{2}-\frac{1}{\theta}\right)\left(a+b\int_\Omega|\nabla w|^2\mathrm{d}x\right)}\leq\frac{2\theta(c_w+C_0)}{(\theta-2)a}\leq\frac{2\theta C_1'(L)}{(\theta-2)a},
$$

where $C_1'(L)=C_1(L)+C_0$.

Set $R_1=\sqrt{\frac{2\theta C_1'(L)}{(\theta-2)a}}$ and $b_1=b(R_1)$, then for any $w\in E$ with $\|w\|\leq R_1$ and $b\in[0,b_1]$, I_{w+} has a critical point u_w with $u_w>0$ and $c_1\leq\|u_w\|\leq R_1$. Let $w=u_1$ for some $u_1\in E$ with $u_1>0$ and $\|u_1\|\leq R_1$, then I_{u_1+} has a critical point u_2 with $u_2>0$ and $c_1\leq\|u_2\|\leq R_1$. Again, let $w=u_2$, then I_{u_2+} has a critical point u_3 with $u_3>0$ and $c_1\leq\|u_3\|\leq R_1$. By induction, we get a sequence $\{u_n\}$ with $I'_{u_{n-1}+}(u_n)=0$, $u_n>0$ and $c_1\leq\|u_n\|\leq R_1$.

Step 3 We prove that $u_n\to\bar u$ in E for some $\bar u\in E$ up to a subsequence and $\bar u$ is a positive solution of (1.1).

Since $\|u_n\|\leq R_1$, then there exists $\bar u\in E$ such that $u_n\rightharpoonup\bar u$ in E and $u_n\to\bar u$ in $L^{p+1}(\Omega)$ up to a subsequence. By (H0), we have

$$
\begin{aligned}
I'_{u_{n-1}+}(\bar u)(u_n-\bar u)&=\left(a+b\int_\Omega|\nabla u_{n-1}|^2\mathrm{d}x\right)\int_\Omega\nabla\bar u\cdot\nabla(u_n-\bar u)\mathrm{d}x\\
&\quad-\int_\Omega f_+(x,\bar u)(u_n-\bar u)\mathrm{d}x\\
&\to 0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
0 &= \lim_{n\to\infty}[I'_{u_{n-1}+}(u_n)(u_n-\bar{u}) - I'_{u_{n-1}+}(\bar{u})(u_n-\bar{u})] \\
&= \lim_{n\to\infty}\left[\left(a+b\int_\Omega |\nabla u_{n-1}|^2 \mathrm{d}x\right)\int_\Omega |\nabla(u_n-\bar{u})|^2 \mathrm{d}x\right. \\
&\quad \left. -\int_\Omega (f_+(x,u_n)-f_+(x,\bar{u}))(u_n-\bar{u})\mathrm{d}x\right] \\
&= \lim_{n\to\infty}\left(a+b\int_\Omega |\nabla u_{n-1}|^2 \mathrm{d}x\right)\|u_n-\bar{u}\|^2,
\end{aligned}
$$

which means that $u_n \to \bar{u}$ in E as $n \to \infty$. Then, for any $\varphi \in E$, we have

$$
\begin{aligned}
0 &= \lim_{n\to\infty} I'_{u_{n-1}+}(u_n)\varphi \\
&= \lim_{n\to\infty}\left(a+b\int_\Omega |\nabla u_{n-1}|^2 \mathrm{d}x\right)\int_\Omega \nabla u_n\cdot\nabla\varphi \mathrm{d}x - \int_\Omega f_+(x,u_n)\varphi \mathrm{d}x \\
&= \left(a+b\int_\Omega |\nabla \bar{u}|^2 \mathrm{d}x\right)\int_\Omega \nabla \bar{u}\cdot\nabla\varphi - \int_\Omega f_+(x,\bar{u})\varphi \mathrm{d}x \\
&= I'_{\bar{u}+}(\bar{u})\varphi.
\end{aligned}
$$

Hence, $\bar{u}$ is a critical point of $I_{\bar{u}+}$, and $\bar{u}$ satisfies

$$
\begin{cases}
-(a+b\int_\Omega |\nabla\bar{u}|^2\mathrm{d}x)\Delta\bar{u} = f_+(x,\bar{u}), & \text{in } \Omega, \\
\bar{u}=0, & \text{on } \partial\Omega.
\end{cases} \tag{2.7}
$$

Again by multiplying equation (2.7) by $\bar{u}^-$ with $\bar{u}^- = \min\{\bar{u}, 0\}$ and integrating on Ω, we get

$$
\left(a+b\int_\Omega |\nabla\bar{u}|^2\mathrm{d}x\right)\int_\Omega |\nabla\bar{u}^-|^2\mathrm{d}x = \int_\Omega f_+(x,\bar{u})\bar{u}^-\mathrm{d}x = 0,
$$

which means that

$$
\bar{u}^- \equiv 0. \tag{2.8}
$$

On the other hand, since $u_n \to \bar{u}$ in E and $\|u_n\| \geq c_1$, we have $\|\bar{u}\| \geq c_1$. Combined with (2.8) it proves that $\bar{u}$ is a positive solution of (2.7). Since $f_+(x,\bar{u}) = f(x,\bar{u})$, $\bar{u}$ is also a positive solution of (1.1).

By a similar argument, we can prove that for $b \in [0, b_1]$, (1.1) also has at least one negative solution. □

3 Sign Changing Solution

In this section, we first study the sign changing solution for (1.1) using the Nehari method proposed by Nehari[16] and then give the proof of Theorem 1.1.

For any $w \in E$, we consider the following problem

$$\begin{cases} -(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x)\Delta u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

The associated functional corresponding to (3.1) is $I_w : E \to \mathbb{R}$,

$$I_w(u) = \frac{1}{2}\left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)\int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega F(x, u)\mathrm{d}x.$$

By (H0), $I_w \in C^2(E, \mathbb{R})$ is weakly lower semi-continuous and the weak solution of the problem (3.1) corresponds to the critical point of the functional I_w, see [18].

Define

$$\begin{aligned} G_w(u) &= \langle I_w'(u), u\rangle = \left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)\int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega f(x, u)u\mathrm{d}x, \\ \mathcal{N}_w &= \{u \in H_0^1(\Omega)\backslash\{0\} \mid G_w(u) = 0\}, \\ \mathcal{S}_w &= \{u \in \mathcal{N}_w \mid u^+ \in \mathcal{N}_w, u^- \in \mathcal{N}_w\}, \end{aligned}$$

where $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$. The set $\mathcal{N}_w$ is called Nehari manifold. Obviously, any sign changing solutions of (3.1) must be on $\mathcal{S}_w$.

Lemma 3.1 *Assume that (H0)–(H3) hold, then for each $u \in E\backslash\{0\}$ there exists unique $t = t(u) > 0$ such that $t(u)u \in \mathcal{N}_w$.*

Proof Similar as Lemma 2.2, there exist $\alpha > 0$ and $\delta > 0$ such that $I_w(u) > 0$ for all $u \in B_\delta(0)\backslash\{0\}$ and $I_w(u) \geq \alpha$ for all $u \in \partial B_\delta(0)$.

Next we prove that for any $u \in E\backslash\{0\}$, $I_w(tu) \to -\infty$, as $t \to \infty$. By (H2),

$$\begin{aligned} I_w(tu) &= \frac{t^2}{2}\left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)\int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega F(x, tu)\mathrm{d}x \\ &\leq \frac{t^2}{2}\left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)\int_\Omega |\nabla u|^2 \mathrm{d}x - C_1 t^\theta \int_\Omega |u|^\theta \mathrm{d}x + C_2|\Omega|. \end{aligned}$$

Since $\theta > 2$, we have $I_w(tu) \to -\infty$, as $t \to \infty$.

For each fixed $u \in E\backslash\{0\}$, let $g_w(t) = I_w(tu)$ for $t > 0$, then from the above argument, $g_w(t)$ has at least one maximum point with maximum value greater than α. We will prove that $g_w(t)$ has a unique critical point for $t > 0$. Noticed that

$$\begin{aligned} g_w'(t) &= (I_w'(tu), u) = \left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)\int_\Omega t|\nabla u|^2 \mathrm{d}x - \int_\Omega f(x, tu)u\mathrm{d}x, \\ g_w''(t) &= \left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)\int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega f'(x, tu)u^2 \mathrm{d}x, \end{aligned}$$

by (H3), for every critical point $\bar{t}$ of $g_w(t)$, we have

$$\begin{aligned} g_w^{''}(\bar{t}) &= \left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)\int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega f^{'}(x, \bar{t}u)u^2 \mathrm{d}x \\ &= \int_\Omega \frac{f(x, \bar{t}u)u}{\bar{t}} \mathrm{d}x - \int_\Omega f^{'}(x, \bar{t}u)u^2 \mathrm{d}x \\ &= \frac{1}{\bar{t}^2}\int_\Omega f(x, \bar{t}u)\bar{t}u - f^{'}(x, \bar{t}u)(\bar{t}u)^2 \mathrm{d}x < 0. \end{aligned}$$

This means that every critical point of $g_w(t)$ must be a strict local maximum; hence, $g_w(t)$ has a unique critical point, which is denoted by $t(u)$.

Finally, by

$$(I_w^{'}(t(u)u), t(u)u) = t(u)(I_w^{'}(t(u)u), u) = t(u)g_w^{'}(t(u)) = 0,$$

we obtain that $t(u)u \in \mathcal{N}_w$. □

Lemma 3.2 *There exists a constant* $c_2 > 0$ *independent of* w *such that* $\|u\| \geq c_2$ *for all* $u \in \mathcal{N}_w$.

Proof It follows from $u \in \mathcal{N}_w$ that

$$\left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)\int_\Omega |\nabla u|^2 \mathrm{d}x = \int_\Omega f(x, u)u \mathrm{d}x.$$

By (H0) and (H1), for given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x, u)| \leq \varepsilon|u| + C_\varepsilon|u|^p,$$

then we have

$$\left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)\int_\Omega |\nabla u|^2 \mathrm{d}x \leq \varepsilon\int_\Omega |u|^2 \mathrm{d}x + C_\varepsilon\int_\Omega |u|^{p+1} \mathrm{d}x.$$

Using Sobolev inequality, we obtain

$$\left(a - \frac{\varepsilon}{\lambda_1}\right)\|u\|^2 \leq C_\varepsilon^{'}\|u\|^{p+1},$$

which implies that there exists a constant $c_2 > 0$ independent of w such that $\|u\| \geq c_2$ for all $u \in \mathcal{N}_w$. □

Define $m_1 = \inf_{\mathcal{S}_w} I_w$, then it is clear that

$$m_1 \geq \inf_{\partial B_\delta(0)} I_w \geq \alpha > 0.$$

Lemma 3.3 *m_1 is achieved at some $u_w \in \mathcal{S}_w$.*

Proof Let $\{u_n\}$ be a minimizing sequence on $\mathcal{S}_w$ such that $I_w(u_n) \to m_1$, then

$$\frac{1}{2}\left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)\int_\Omega |\nabla u_n|^2 \mathrm{d}x - \int_\Omega F(x, u_n)\mathrm{d}x = m_1 + o(1) \le C_3, \tag{3.2}$$

$$\left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)\int_\Omega |\nabla u_n|^2 \mathrm{d}x - \int_\Omega f(x, u_n)u_n \mathrm{d}x = 0, \tag{3.3}$$

where $C_3 > 0$ is a constant. From (3.2), (3.3) and (H2) we have that

$$\begin{aligned}\frac{\theta - 2}{2}\left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)\int_\Omega |\nabla u_n|^2 \mathrm{d}x &\le C_3 \\ + \int_\Omega (\theta F(x, u_n) - f(x, u_n)u_n)\mathrm{d}x &\le C_4;\end{aligned}$$

thus, $\{u_n\}$ is bounded in E. Then, up to a subsequence, $u_n \rightharpoonup u$ and $u_n^\pm \rightharpoonup u^\pm$ in E.

Now we claim that $u^+ \not\equiv 0$ and $u^- \not\equiv 0$. In fact, if $u^+ \equiv 0$, then $u_n^+ \to 0$ in $L^{p+1}(\Omega)$, so by $u_n^+ \in \mathcal{N}_w$ and (H0), we get

$$\left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)\int_\Omega |\nabla u_n^+|^2 \mathrm{d}x = \int_\Omega f(x, u_n^+)u_n^+ \mathrm{d}x \to 0,$$

this is a contradiction with Lemma 3.2. Similarly, we can prove that $u^- \not\equiv 0$.

By Lemma 3.1, there exist $t, s > 0$, such that $tu^+ \in \mathcal{N}_w$ and $su^- \in \mathcal{N}_w$, and hence $tu^+ + su^- \in \mathcal{S}_w$. Furthermore, since $I_w(u)$ is weakly lower semi-continuous, we get

$$\begin{aligned} m_1 \le I_w(tu^+ + su^-) &= I_w(tu^+) + I_w(su^-) \\ &\le \liminf_{n\to\infty} I_w(tu_n^+) + \liminf_{n\to\infty} I_w(su^-) \\ &\le \liminf_{n\to\infty} I_w(u_n^+) + \liminf_{n\to\infty} I_w(u_n^-) \\ &\le \liminf_{n\to\infty} (I_w(u_n^+) + I_w(u_n^-)) \\ &= \liminf_{n\to\infty} (I_w(u_n)) = m_1. \end{aligned}$$

Let $u_w = tu^+ + su^-$, then $I_w(u_w) = m_1$. □

In what follows we prove that the minimizer u_w of I_w on $\mathcal{S}_w$ is a critical point of I_w, here we use an argument similar as [13].

Lemma 3.4 *If $I_w(u_w) = m_1$ for some $u_w \in \mathcal{S}_w$, then u_w is a critical point of I_w.*

Proof If u_w is not a critical point of I_w, then there exists $\varphi \in C_0^\infty(\Omega)$ such that

$$\langle I_w'(u_w), \varphi \rangle \le -1;$$

thus, there exists $\varepsilon_0 > 0$ such that for $|t-1| \le \varepsilon_0$, $|s-1| \le \varepsilon_0$, and $|\sigma| \le \varepsilon_0$,

$$\langle I_w'(tu_w^+ + su_w^- + \sigma\varphi), \varphi\rangle \le -\frac{1}{2}.$$

Let $\eta = \eta(t,s) \ge 0$, $(t,s) \in T = [\frac{1}{2}, \frac{3}{2}] \times [\frac{1}{2}, \frac{3}{2}]$ be a cut-off function such that $\eta(t,s) = 1$, if $|t-1| \le \frac{1}{2}\varepsilon_0$ and $|s-1| \le \frac{1}{2}\varepsilon_0$, $\eta(t,s) = 0$, if $|t-1| \ge \varepsilon_0$ or $|s-1| \ge \varepsilon_0$. If $|t-1| \le \varepsilon_0$ and $|s-1| \le \varepsilon_0$, then

$$\begin{aligned} & I_w(tu_w^+ + su_w^- + \varepsilon_0\eta(t,s)\varphi) \\ & \quad = I_w(tu_w^+ + su_w^-) + \int_0^1 \langle I_w'(tu_w^+ + su_w^- + \mu\varepsilon_0\eta(t,s)\varphi), \varepsilon_0\eta(t,s)\varphi\rangle \mathrm{d}\mu \\ & \quad \le I_w(tu_w^+ + su_w^-) - \frac{1}{2}\varepsilon_0\eta(t,s). \end{aligned} \tag{3.4}$$

For $|t-1| \ge \varepsilon_0$ or $|s-1| \ge \varepsilon_0$, since $\eta(t,s) = 0$, the above estimate is trivial. Since $u_w \in \mathcal{S}_w$, for $(t,s) \ne (1,1)$, we have $I_w(tu_w^+ + su_w^-) < I_w(u_w)$. Then, by (3.4), for $(t,s) \ne (1,1)$,

$$I_w(tu_w^+ + su_w^- + \varepsilon_0\eta(t,s)\varphi) \le I_w(tu_w^+ + su_w^-) < I_w(u_w),$$

for $(t,s) = (1,1)$,

$$I_w(u_w^+ + u_w^- + \varepsilon_0\eta(t,s)\varphi) \le I_w(u_w) - \frac{1}{2}\varepsilon_0\eta(1,1) = I_w(u_w) - \frac{1}{2}\varepsilon_0.$$

Hence,

$$\sup_{(t,s)\in T} I_w(tu_w^+ + su_w^- + \varepsilon_0\eta(t,s)\varphi) < I_w(u_w) = m_1,$$

which implies that for all $(t,s) \in T$, $tu_w^+ + su_w^- + \varepsilon_0\eta(t,s)\varphi \notin \mathcal{S}_w$. We will show that there must be $(t_0, s_0) \in T$ such that $t_0u_w^+ + s_0u_w^- + \varepsilon_0\eta(t_0,s_0)\varphi \in \mathcal{S}_w$, then this gives a contradiction.

For $0 \le \varepsilon \le \varepsilon_0$, define $h_\varepsilon : T \to H_0^1(\Omega)$ by

$$h_\varepsilon(t,s) = tu_w^+ + su_w^- + \varepsilon\eta(t,s)\varphi$$

and $H_\varepsilon : T \to \mathbb{R}^2$ by

$$H_\varepsilon(t,s) = (G_w(h_\varepsilon(t,s)^+), G_w(h_\varepsilon(t,s)^-)).$$

For all $(t,s) \in \partial T$, $\eta(t,s) = 0$, then $h_\varepsilon(t,s) = tu_w^+ + su_w^-$; thus, for all $(t,s) \in \partial T$ and $0 \le \varepsilon \le \varepsilon_0$,

$$\begin{aligned} H_\varepsilon(t,s) &= (G_w((tu_w^+ + su_w^-)^+), G_w((tu_w^+ + su_w^-)^-)) \\ &= (G_w(tu_w^+), G_w(su_w^-)) \\ &\neq (0,0). \end{aligned}$$

Hence, by the homotopy invariance of Brouwer degree, we have

$$\deg(H_{\varepsilon_0}(t,s), T, (0,0)) = \deg(H_0(t,s), T, (0,0)). \tag{3.5}$$

Next we shall prove that

$$\deg(H_0(t,s), T, (0,0)) = 1.$$

Notice that $H_0(t,s) = (G_w(tu_w^+), G_w(su_w^-))$, and we denote

$$a(t) = G_w(tu_w^+), b(s) = G_w(su_w^-).$$

By $u_w^+ \in \mathcal{N}_w$ and (H3) we have

$$\begin{aligned} a^{'}(1) &= \langle G_w^{'}(u_w^+), u_w^+ \rangle \\ &= \left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)\int_\Omega 2|\nabla u_w^+|^2 \mathrm{d}x - \int_\Omega \left[f^{'}(x,u_w^+)(u_w^+)^2 + f(x,u_w^+)u_w^+\right]\mathrm{d}x \\ &= \int_\Omega 2f(x,u_w^+)u_w^+ \mathrm{d}x - \int_\Omega \left[f^{'}(x,u_w^+)(u_w^+)^2 + f(x,u_w^+)u_w^+\right]\mathrm{d}x \\ &= \int_\Omega \left[f(x,u_w^+)u_w^+ - f^{'}(x,u_w^+)(u_w^+)^2\right]\mathrm{d}x < 0. \end{aligned}$$

Similarly, $b^{'}(1) < 0$. By Lemma 3.1, $(t,s) = (1,1)$ is the unique solution of $H_0(t,s) = (a(t), b(s)) = (0,0)$. Then, by the definition of Brouwer degree, clearly we have

$$\deg(H_0(t,s), T, (0,0)) = 1.$$

Thus, by (3.5) we have

$$\deg(H_{\varepsilon_0}(t,s), T, 0) = 1 \neq 0.$$

Therefore, there must exists $(t_0, s_0) \in T$ such that

$$\begin{aligned} H_{\varepsilon_0}(t_0,s_0) = (&G_w((t_0u_w^+ + s_0u_w^- + \varepsilon_0\eta(t_0,s_0)\varphi)^+), \\ &G_w((t_0u_w^+ + s_0u_w^- + \varepsilon_0\eta(t_0,s_0)\varphi)^-)) \\ = (&0,0), \end{aligned}$$

which means that $(t_0u_w^+ + s_0u_w^- + \varepsilon_0\eta(t_0,s_0)\varphi \in \mathcal{S}_w$. □

Now we can state and prove the existence of sign changing solution for (1.1).

Theorem 3.1 *Assume (H0)–(H3) hold, then there exists a constant $b_2 > 0$ such that for any $b \in [0, b_2]$, (1.1) has at least one sign changing solution.*

Proof First we fix a function $v \in E$ with $v^+ \neq 0$ and $v^- \neq 0$. Thanks to Lemma 3.3 and Lemma 3.4, we get a minimizer u_w of I_w on $\mathcal{S}_w$ and $I'_w(u_w) = 0$. For the fixed constant $L > 0$ as in the proof of Theorem 2.1, recall that $b(R) = \frac{L}{R^2}$ for $R > 0$. By Lemma 3.1, (H2), and notice that u_w is a minimizer of I_w on $\mathcal{S}_w$, for $w \in E$ with $\|w\| \leq R$ and $b \in [0, b(R)]$, clearly we have

$$\begin{aligned}
I_w(u_w) &\leq \sup_{t,s>0} I_w(tv^+ + sv^-) \\
&\leq \sup_{t>0}\left(\frac{t^2}{2}\left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)\int_\Omega |\nabla v^+|^2 \mathrm{d}x - C_1 t^\theta \int_\Omega |v^+|^\theta + C_2|\Omega|\right) \\
&\quad + \sup_{s>0}\left(\frac{s^2}{2}\left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)\int_\Omega |\nabla v^-|^2 \mathrm{d}x - C_1 s^\theta \int_\Omega |v^-|^\theta + C_2|\Omega|\right) \\
&\leq \sup_{t>0}\left(\frac{t^2}{2}(a + L)\int_\Omega |\nabla v^+|^2 \mathrm{d}x - C_1 t^\theta \int_\Omega |v^+|^\theta + C_2|\Omega|\right) \\
&\quad + \sup_{s>0}\left(\frac{s^2}{2}(a + L)\int_\Omega |\nabla v^-|^2 \mathrm{d}x - C_1 s^\theta \int_\Omega |v^-|^\theta + C_2|\Omega|\right) \\
&\leq C_2(L),
\end{aligned}$$

where $C_2(L) > 0$ is a constant independent of b, R and w, and the last inequality follows from $\theta > 2$. Since u_w is a critical point of I_w, by (H2) we have

$$\begin{aligned}
&\frac{1}{2}\left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)\int_\Omega |\nabla u_w|^2 \mathrm{d}x \\
&\quad = I_w(u_w) + \int_\Omega F(x, u_w)\mathrm{d}x \\
&\quad \leq C_2(L) + \int_\Omega F(x, u_w)\mathrm{d}x \\
&\quad \leq C_2(L) + \frac{1}{\theta}\int_\Omega f(x, u_w)u_w \mathrm{d}x + C_0 \\
&\quad = C'_2(L) + \frac{1}{\theta}\left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)\int_\Omega |\nabla u_w|^2 \mathrm{d}x,
\end{aligned}$$

where $C'_2(L) = C_2(L) + C_0$, then

$$\|u_w\|^2 \leq \frac{C'_2(L)}{\left(\frac{1}{2} - \frac{1}{\theta}\right)\left(a + b\int_\Omega |\nabla w|^2 \mathrm{d}x\right)} \leq \frac{2\theta C'_2(L)}{(\theta - 2)a}. \tag{3.6}$$

Set $R_2 = \sqrt{\frac{2\theta C'_2(L)}{(\theta-2)a}}$ and $b_2 = b(R_2)$. For any $w \in E$ with $\|w\| \leq R_2$ and $0 \leq b \leq b_2$, by Lemma 3.4 and (3.6), I_w has a critical point $u_w \in \mathcal{S}_w$ with $\|u_w\| \leq R_2$.

Let $w = u_1$ for some $u_1 \in E$ with $\|u_1\| \leq R_2$, then I_{u_1} has a critical point u_2 with $u_2 \in \mathcal{S}_{u_1}$ and $\|u_2\| \leq R_2$. Again, let $w = u_2$, then I_{u_2} has a critical point $u_3 \in \mathcal{S}_{u_2}$ with $\|u_3\| \leq R_2$. By induction, we get a sequence $\{u_n\}$ with $I'_{u_{n-1}}(u_n) = 0, u_n \in \mathcal{S}_{u_{n-1}}$ and $\|u_n\| \leq R_2$.

Since $\|u_n\| \leq R_2$, we have $u_n \rightharpoonup \tilde{u}$ in E and $u_n \to \tilde{u}$ in $L^{p+1}(\Omega)$ up to a subsequence. Then, by (H0), we have

$$\begin{aligned} I'_{u_{n-1}}(\tilde{u})(u_n - \tilde{u}) &= \left(a + b\int_\Omega |\nabla u_{n-1}|^2 \mathrm{d}x\right)\int_\Omega \nabla\tilde{u}\cdot\nabla(u_n - \tilde{u})\mathrm{d}x \\ &\quad - \int_\Omega f(x,\tilde{u})(u_n - \tilde{u})\mathrm{d}x \\ &\to 0. \end{aligned}$$

Thus,

$$\begin{aligned} 0 &= \lim_{n\to\infty}[I'_{u_{n-1}}(u_n)(u_n - \tilde{u}) - I'_{u_{n-1}}(\tilde{u})(u_n - \tilde{u})] \\ &= \lim_{n\to\infty}\left[(a + b\int_\Omega |\nabla u_{n-1}|^2 \mathrm{d}x)\int_\Omega |\nabla(u_n - \tilde{u})|^2 \mathrm{d}x \right. \\ &\quad \left. - \int_\Omega (f(x,u_n) - f(x,\tilde{u}))(u_n - \tilde{u})\mathrm{d}x\right] \\ &= \lim_{n\to\infty}\left(a + b\int_\Omega |\nabla u_{n-1}|^2 \mathrm{d}x\right)\|u_n - \tilde{u}\|^2, \end{aligned}$$

which means that $u_n \to \tilde{u}$ in E as $n \to \infty$. Then, for any $\varphi \in E$, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} I'_{u_{n-1}}(u_n)\varphi \\ &= \lim_{n\to\infty}\left(a + b\int_\Omega |\nabla u_{n-1}|^2 \mathrm{d}x\right)\int_\Omega \nabla u_n\cdot\nabla\varphi - \int_\Omega (f(x,u_n)\varphi \\ &= \left(a + b\int_\Omega |\nabla\tilde{u}|^2 \mathrm{d}x\right)\int_\Omega \nabla\tilde{u}\cdot\nabla\varphi - \int_\Omega (f(x,\tilde{u})\varphi \\ &= I'_{\tilde{u}}(\tilde{u})\varphi, \end{aligned}$$

so $\tilde{u}$ is a critical point of $I_{\tilde{u}}$, and $\tilde{u}$ satisfies (1.1). Since $u_n \in \mathcal{S}_{u_{n-1}}$, we have $u_n^+ \in \mathcal{N}_{u_{n-1}}$ and $u_n^- \in \mathcal{N}_{u_{n-1}}$. By Lemma 3.2, $\|u_n^+\| \geq c_2$ and $\|u_n^-\| \geq c_2$; hence, $\|\tilde{u}^+\| \geq c_2$ and $\|\tilde{u}^-\| \geq c_2$. Therefore, $\tilde{u}$ is a sign changing solution of (1.1). □

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1 Let $b_0 = \min\{b_1, b_2\}$, then by Theorems 2.1 and 3.1, for any $b \in [0, b_0]$, the problem (1.1) has at least three nontrivial solutions; among them, one is positive, one is negative, and one is sign changing. □

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