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The Embedding of Annihilating-Ideal Graphs Associated to Lattices in the Projective Plane

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Abstract Let $(L, \wedge, \vee)$ be a lattice with a least element 0. The annihilating-ideal graph of L, denoted by $\mathbb{AG}(L)$, is a graph whose vertex set is the set of all non-trivial ideals of L and, for every two distinct vertices I and J, I is adjacent to J if and only if $I \wedge J = \{0\}$. In this paper, we completely determine all finite lattices L with projective annihilating-ideal graphs $\mathbb{AG}(L)$.

Keywords Annihilating-ideal graph · Lattice · Projective graph

Mathematics Subject Classification 05C10 · 06A07 · 06B10

1 Introduction

Recently, there has been considerable researches done on associating graphs with algebraic structures. For example, see [2,4,11–13]. The concept of a annihilating-ideal graph of a commutative ring R, denoted by $\mathbb{AG}(R)$, was introduced by Behboodi and Rakeei in [5] and [6]. Let $\mathbb{A}(R)$ be the set of annihilating-ideals of R, where a nonzero ideal I of R is called an annihilating-ideal, if there exists a nonzero ideal J of R such that $IJ = 0$. The annihilating-ideal graph of R is a simple graph with vertex set $\mathbb{A}(R)$, and two distinct vertices I and J are adjacent if and only if $IJ = 0$. The annihilating-

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ideal graph of a lattice L, denoted by $\mathbb{AG}(L)$, is defined by Khashyarmanesh et al in [1]. $\mathbb{AG}(L)$ is a graph whose vertex set is the set of all non-trivial ideals of L and, for every two distinct vertices I and J, I is adjacent to J, if and only if $I \wedge J = 0$. In this work, we assume that L is a finite lattice and $A(L) = \{a_1, a_2, \ldots, a_n\}$ is the set of all atoms of L. In the second section of this paper, we completely characterize all finite lattices L with projective annihilating-ideal graphs $\mathbb{AG}(L)$.

First, we recall some definitions and notations on lattices. For basis facts concerning lattice, we refer to [9]. Recall that a *lattice* is an algebra $L = (L, \wedge, \vee)$ with two binary operations $\wedge$ and $\vee$, satisfying the following conditions: for all $a, b, c \in L$,

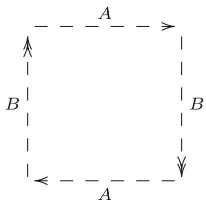
1. $a \wedge a = a,\ a \vee a = a$,
2. $a \wedge b = b \wedge a,\ a \vee b = b \vee a$,
3. $(a \wedge b) \wedge c = a \wedge (b \wedge c),\ a \vee (b \vee c) = (a \vee b) \vee c$, and
4. $a \vee (a \wedge b) = a \wedge (a \vee b) = a$.

By [15, Theorem 2.1], one can define an order $\leq$ on L as follows: for any $a, b \in L$, we set $a \leq b$ if and only if $a \wedge b = a$. Then $(L, \leq)$ is an ordered set in which every pair of elements has a greatest lower bound (g.l.b.) and a least upper bound (l.u.b.). Conversely, let P be an ordered set such that, for every pair $a, b \in P$, g.l.b.(a, b) and l.u.b.(a, b) belong to P. For each a and b in P, we define $a \wedge b :=$ g.l.b.(a, b) and $a \vee b :=$ l.u.b.(a, b). Then $(P, \wedge, \vee)$ is a lattice. A lattice L is said to be *bounded* if there are elements 0 and 1 in L such that $0 \wedge a = 0$ and $a \vee 1 = 1$, for all $a \in L$. Clearly, every finite lattice is bounded. Let $(L, \wedge, \vee)$ be a lattice with a least element 0 and I be a non-empty subset of L. We say that I is an *ideal* of L, denoted by $I \trianglelefteq L$, if

(i) For all $a, b \in I$, $a \vee b \in I$.
(ii) If $0 \leq a \leq b$ and $b \in I$, then $a \in I$.

For two distinct ideals I and J of a lattice L, we put $I \wedge J := \{x \wedge y\ ;\ \ x \in I, y \in J\}$. In a lattice $(L, \wedge, \vee)$ with a least element 0, an element a is called an *atom* if $a \neq 0$ and, for an element x in L, the relation $0 \leq x \leq a$ implies that either $x = 0$ or $x = a$. We denote the set of all atoms of L by $A(L)$. Also, for an ideal I of L, $A(I)$ denotes the set of all atoms contained in I.

Now we recall some definitions and notations on graphs. We use the standard terminology of graphs following [7]. In a graph G, for two distinct vertices a and b in G, the notation $a - b$ means that a and b are adjacent. For a positive integer r, an *r-partite graph* is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A *complete r-partite graph* is one in which each vertex is joined to every vertex that is not in the same subset. The *complete bipartite graph* (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A graph G is said to be *contracted* to a graph H if there exists a sequence of elementary contractions which transforms G into H, where an *elementary contraction* consists of deletion of a vertex or an edge or the identification of two adjacent vertices. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable simple characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (cf. [7, p.153]). By a surface, we mean a connected compact two-dimensional manifold without boundary. If a disc is cut from a sphere and it is closed by a Möbius band, then the obtained surface is called *projective plane*. The projective plane can also be obtained by identifying every point of an open disc with its antipodal points. A graph G is *embeddable* in a surface S if the vertices of G are assigned to distinct points in S such that every edge of G is a simple arc in S connecting the two vertices which are joined in G. If G can not be embedded in S, then G has at least two edges intersecting at a point which is not a vertex of G. We say a graph G is *irreducible* for a surface S if G does not embed in S, but any proper subgraph of G embeds in S. The set of 103 irreducible graphs for the projective plane has been found by Glover et al. in [10], and Archdeacon in [3] proved that this list is complete. This list also has been checked by Myrvold and Roth in [14]. Hence a graph embeds in the projective plane, which is called a *projective graph*, if and only if it contains no subdivision of 103 graphs in [3]. Note that a complete graph K_n is projective if $n = 5$ or 6, and the only projective complete bipartite graphs are $K_{3,3}$ and $K_{3,4}$ (see [8]). Note that a planar graph is not considered as a projective graph.



The canonical representation of a projective plane

2 Projective Annihilating-Ideal Graphs of Lattices

In this section, we study the projectivity of the annihilating-ideal graph $\mathbb{AG}(L)$. We begin this section with the following notation, which is needed in the rest of the paper.

Notation 2.1 *Let* $i_1, i_2, \ldots, i_n$ *be integers with* $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. *The notation* $U_{i_1 i_2 \ldots i_k}$ *stands for the following set:*

$$\left\{I \trianglelefteq L;\ \ \left\{a_{i_1}, a_{i_2}, \ldots, a_{i_k}\right\} \subseteq I \ \ and \ \ a_j \notin I, \text{for} \ \ j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}\right\}.$$

Note that no two distinct elements in $U_{i_1 i_2 \ldots i_k}$ are adjacent in $\mathbb{AG}(L)$. Also if the index sets $\{i_1, i_2, \ldots, i_k\}$ and $\{j_1, j_2, \ldots, j_{k'}\}$ of $U_{i_1 i_2 \ldots i_k}$ and $U_{j_1 j_2 \ldots j_{k'}}$, respectively, are distinct, then one can easily check that $U_{i_1 i_2 \ldots i_k} \cap U_{j_1 j_2 \ldots j_{k'}} = \emptyset$. Moreover, $V(\mathbb{AG}(L)) = \bigcup U_{i_1 i_2 \ldots i_k}$, for all $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Suppose that L has n atoms. Note that $U_{12\ldots n}$ consist of isolated vertices. Clearly, the isolated points do not affect projectivity. Hence, we ignore the set $U_{12\ldots n}$ from the vertex set of $\mathbb{AG}(L)$, and so we do not show these points in our figures.

In the following lemma, we determine an upper bound for the number of atoms of lattice L such that the graph $\mathbb{AG}(L)$ is projective.

Lemma 2.2 *If* $\mathbb{AG}(L)$ *is a projective graph, then* $2 \leq |A(L)| \leq 6$.

Proof Suppose on the contrary that $|A(L)| = 1$ or $|A(L)| > 6$. In the first situation, $\mathbb{AG}(L)$ is a totally disconnected graph, and so it is planar. Hence it is not a projective graph. In the second situation, since the induced subgraph of $\mathbb{AG}(L)$ on vertex set $\{\{0, a_i\}\}$, for $1 \leq i \leq 7$, is a complete graph, one can find a subgraph isomorphic to K_7. Therefore, the graph $\mathbb{AG}(L)$ is not projective. Hence we have $2 \leq |A(L)| \leq 6$. □

By Theorem 2.6 in [1], the graph $\mathbb{AG}(L)$ is complete bipartite if and only if $|A(L)| = 2$. In the following theorem, we state a necessary and sufficient condition for the projectivity of $\mathbb{AG}(L)$, when $|A(L)| = 2$.

Theorem 2.3 *Suppose that* $|A(L)| = 2$. *Then* $\mathbb{AG}(L)$ *is a projective graph if and only if* $|U_2| = 3$, *or* 4 *whenever* $|U_1| = 3$.

Proof Let the graph $\mathbb{AG}(L)$ be projective. Assume to the contrary that $|U_1| \leq 2$ or $|U_2| \leq 2$. By [16, Proposition 2.3], the graph $\mathbb{AG}(L)$ is planar, which is a contradiction. Also if $|U_1| > 3$ and $|U_2| > 3$, then the graph $\mathbb{AG}(L)$ contains a copy of $K_{4,4}$. And if $|U_1| = 3$ and $|U_2| > 4$, then the graph $\mathbb{AG}(L)$ contains a subgraph isomorphic to $K_{3,5}$. Hence $\mathbb{AG}(L)$ is not a projective graph. Therefore, we have $|U_2| = 3$, or 4 whenever $|U_1| = 3$.

The converse statement is clear. □

Now, we investigate the projectivity of $\mathbb{AG}(L)$, when $|A(L)| = 3$. In the following four cases, we probe the projectivity of $\mathbb{AG}(L)$ in the case that $|\bigcup_{i=1}^{3} U_i| \geq 5$. Additionally, in the rest of work, we do not consider the cases that $\mathbb{AG}(L)$ is planar. For planar cases see [16].

Case 1 $|\bigcup_{i=1}^{3} U_i| = 5$.

Without loss of generality, we may assume that $|U_1| = 1$ whenever U_{12} and U_{13} are non-empty. It is clear that $\mathbb{AG}(L)$ is projective. Also if $|U_1| = 3$ and $0 < |U_{23}| \leq 2$, then one can easily check that $\mathbb{AG}(L)$ is projective. In addition, if $|U_1| = 3$ and $|U_{23}| \geq 3$, then the contraction of $\mathbb{AG}(L)$ contains a copy of $K_{3,5}$. So the graph $\mathbb{AG}(L)$ is not projective.

Case 2 $|\bigcup_{i=1}^{3} U_i| = 6$.

Without loss of generality, we may assume that $|U_1| = |U_2| = |U_3| = 2$ whenever U_{12}, U_{13} and U_{23} are non-empty. Then it is not hard to see the graph $\mathbb{AG}(L)$ is projective. Also if $|U_1| = 3$ and $|U_{23}| \leq 1$, then we observe that $\mathbb{AG}(L)$ is projective. And we may assume that $|U_1| = 4$ and $|U_{23}| = 1$. Clearly, the graph $\mathbb{AG}(L)$ is projective. Finally, if $|U_1| = 3$ or 4 whenever $|U_{23}| \geq 2$, then we can find a copy of $K_{3,5}$ or $K_{4,4}$ in the structure of the contraction of $\mathbb{AG}(L)$, respectively. Hence $\mathbb{AG}(L)$ is not projective.

Case 3 $|\bigcup_{i=1}^{3} U_i| = 7$.

Without loss of generality, we may assume that $|U_1| \in \{3, 4\}$ and $U_{23} = \varnothing$. Then one can easily see that the graph $\mathbb{AG}(L)$ is projective. Otherwise, if U_{23} is non-empty, then the contraction of $\mathbb{AG}(L)$ contains a copy of $K_{3,5}$ or $K_{4,4}$. So the graph $\mathbb{AG}(L)$ is not projective. Also if $|U_1| = |U_2| = 3$ and $U_{13} = U_{23} = \varnothing$, then it is not hard to see that the graph $\mathbb{AG}(L)$ is projective. Otherwise, if U_{13} or U_{23} is non-empty, then we have a subgraph isomorphic to $K_{3,5}$ in the contraction of $\mathbb{AG}(L)$. Hence the graph $\mathbb{AG}(L)$ is not projective. Additionally, if $|U_1| = 5$ and $U_{23} = \varnothing$, then $\mathbb{AG}(L)$ is planar, which is not projective. And if $U_{23} \neq \varnothing$, then we can find a copy of $K_{3,5}$ in the structure of $\mathbb{AG}(L)$. So the graph $\mathbb{AG}(L)$ is not projective.

Case 4 $|\bigcup_{i=1}^{3} U_i| \geq 8$.

Without loss of generality, we may assume that $|U_1| = |\bigcup_{i=1}^{3} U_i|\backslash 2$ and $U_{23} = \varnothing$. Then $\mathbb{AG}(L)$ is planar, which is not projective. And if $U_{23} \neq \varnothing$, then we can find a copy of $K_{3,5}$ in the structure of the contraction of $\mathbb{AG}(L)$. So the graph $\mathbb{AG}(L)$ is not projective. Also if none of the U_i's has $|\bigcup_{i=1}^{3} U_i|\backslash 2$ elements, then the contraction of $\mathbb{AG}(L)$ contains a subgraph isomorphic to $K_{3,5}$ or $K_{4,4}$. Therefore, the graph $\mathbb{AG}(L)$ is not projective.

Now, by the above discussion, one can easily see that the following theorem holds.

Theorem 2.4 *Let $|A(L)| = 3$. Then $\mathbb{AG}(L)$ is a projective graph if and only if one of the following conditions holds:*

(i) $|\bigcup_{i=1}^{3} U_i| = 5$ *and one of the following cases is satisfied:*
  (a) There is U_i with $|U_i| = 3$ and $0 < |U_{jk}| \leq 2$, for $1 \leq i \neq j \neq k \leq 3$.
  (b) There is a unique U_i with $|U_i| = 1$ whenever U_{ij} and U_{ik} are non-empty sets, for $1 \leq i \neq j \neq k \leq 3$.

(ii) $|\bigcup_{i=1}^{3} U_i| = 6$ *and one of the following cases is satisfied:*
  (a) There exists i with $1 \leq i \leq 3$, such that $|U_i| = 4$ and $|U_{jk}| = 1$, for $1 \leq i \neq j \neq k \leq 3$.
  (b) There exists i with $1 \leq i \leq 3$, such that $|U_i| = 3$ and $|U_{jk}| \leq 1$, for $1 \leq i \neq j \neq k \leq 3$.
  (c) $|U_i| = 2$, for all i with $1 \leq i \leq 3$, and $U_{jk} \neq \varnothing$, for all $1 \leq j \neq k \leq 3$.

(iii) $|\bigcup_{i=1}^{3} U_i| = 7$ *and one of the following cases is satisfied:*
  (a) $|U_i| \in \{3, 4\}$, for some unique integer i with $1 \leq i \leq 3$, such that and U_{jk} is empty, for $1 \leq i \neq j \neq k \leq 3$.
  (b) $|U_i| = |U_j| = 3$, for some integers i and j, with $1 \leq i \neq j \leq 3$ whenever U_{ik} and U_{jk} are empty, for $1 \leq i \neq j \neq k \leq 3$.

In the sequel, we investigate the projectivity of $\mathbb{AG}(L)$, when $|A(L)| = 4$. Suppose that $|\bigcup_{i=1}^{4} U_i| \geq 8$. Then it is easy to see that $\mathbb{AG}(L)$ is not projective, because one can see that the contraction of $\mathbb{AG}(L)$ contains a copy of $K_{3,5}$ or $K_{4,4}$. And so it is not projective.

As a result of the above note, we have the following lemma.

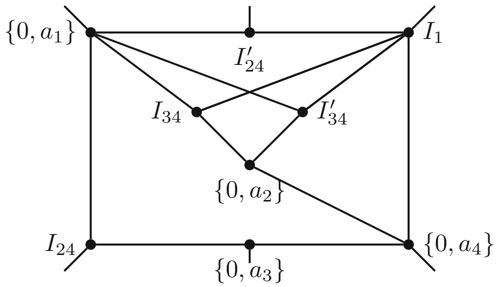
Lemma 2.5 *If $\mathbb{AG}(L)$ is projective, then $|\bigcup_{i=1}^{4} U_i| \leq 7$.*

Theorem 2.6 *Suppose that $|A(L)| = 4$. Then $\mathbb{AG}(L)$ is projective if and only if one of the following statements holds:*

(i) $|\bigcup_{i=1}^{4} U_i| = 5$ *and $|U_i| = 2$, for some unique integer i with $1 \leq i \leq 4$. If $U_{jk} = \varnothing$, then $|U_{jkl}| \neq \varnothing$, with $i, j, k \notin \{i\}$. And if the size of U_{jk} is 1 or 2 whenever at most one of the U_{jk}'s has exactly two elements, where $1 \leq i \neq j \neq k \leq 4$.*

(ii) $|\bigcup_{i=1}^{4} U_i| = 6$ *and one of the following cases holds:*
  (a) $|U_i| = 3$, for some integer i with $1 \leq i \leq 4$. If $|U_{jkl}| = 1$, with $1 \leq i \neq j \neq k \neq l \leq 4$, then $U_{jk} = \varnothing$, for all $j, k \notin \{i\}$. Also if $U_{jkl} = \varnothing$, with

Fig. 1 $\{0, a_1\}$, $I_1 \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, I_{24}, $I'_{24} \in U_{242}$ and I_{34}, $I'_{34} \in U_{34}$



$1 \le i \ne j \ne k \ne l \le 4$, *then* $|U_{jk}| \le 1$ *and at most one of the* U_{jk}*'s has exactly one element, where* $j, k \notin \{i\}$.

(b) $|U_i| = |U_j| = 2$, *for some integers* i *and* j *with* $1 \le i \ne j \le 4$ *whenever* $|U_{kl}| \le 1$, *where* $1 \le k < l \le 4$ *and* $k, l \notin \{i, j\}$. *Also* $U_{i'_1 i'_2} = \varnothing$ *whenever* $|U_{i_1 i_2}| = 1$, *for all* $1 \le i_1 \ne i'_1 \ne i_2 \ne i'_2 \le 4$, *with* $\{i'_1, i'_2\} = \{1, 2, 3, 4\} \backslash \{i_1, i_2\}$. *Moreover, if* $|U_{ik}|, |U_{il}| \le 1$ *or* $|U_{jk}|, |U_{jl}| \le 1$, *then* $|U_{kl}| \le 1$. *Also if* $|U_{ik}| = |U_{jk}| = 1$ *or* $|U_{il}| = |U_{jl}| = 1$, *then* $U_{kl} = \varnothing$.

(iii) $|\bigcup_{i=1}^4 U_i| = 7$ *and one of the following cases holds:*

(a) $|U_i| = 4$, *for some integer* i *with* $1 \le i \le 4$ *and* $U_{jkl} = U_{jk} = \varnothing$, *where* $1 \le i \ne j \ne k \ne l \le 4$.

(b) $|U_i| = 3$ *and* $|U_j| = 2$, *for some integers* i *and* j *with* $1 \le i \ne j \le 4$ *and* $U_{jkl} = \varnothing$, *where* $k, l \notin \{i, j\}$ *whenever* $U_{ii_1} = U_{ji_1} = \varnothing$, *where* $i_1 \notin \{i, j\}$, *with* $1 \le i_1 \le 4$, *and* $U_{kl} = \varnothing$, *where* $k, l \notin \{i, j\}$.

Proof First, assume that $\mathbb{AG}(L)$ is projective. Suppose on the contrary that none of the conditions (i), (ii) or (iii) holds. If $|\bigcup_{i=1}^4 U_i| = 5$ and the statement (i) does not hold, then one of the U_i's, $1 \le i \le 4$, say U_1, has two elements whenever U_{234}, U_{23}, U_{24} and U_{34} are empty. So $\mathbb{AG}(L)$ is planar, which is not projective. Additionally, if $|U_{23}|$, $|U_{24}|$ or $|U_{34}|$ is at least three, then the contraction of $\mathbb{AG}(L)$ contains a copy of the subdivision of $K_{3,5}$. Now, we may assume that at least two of the sets U_{23}, U_{24} or U_{34} have two elements, say U_{24} and U_{34}. Then a subgraph of $\mathbb{AG}(L)$ is isomorphic to E_5, one of the listed graphs in [10], as shown in Fig. 1. In this figure, we have $\{0, a_1\}$, $I_1 \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, I_{24}, $I'_{24} \in U_{24}$ and I_{34}, $I'_{34} \in U_{34}$.

If $|\bigcup_{i=1}^4 U_i| = 6$ and the statement (ii) does not hold, then there is only one of the U_i's, say U_1, such that $|U_1| = 3$ whenever $|U_{234}| \ge 2$. Hence the contraction of $\mathbb{AG}(L)$ contains a subgraph isomorphic to $K_{3,5}$. If $|U_{234}| = 1$ and at least one of the sets U_{23}, U_{24} or U_{34}, say U_{23}, has one element, then $\mathbb{AG}(L)$ contains a copy of E_{18}, one of the listed graphs in [10] (see Fig. 2). In this figure, we have $\{0, a_1\}$, I_1, $J_1 \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, $I_{23} \in U_{23}$ and $I_{234} \in U_{234}$.

If at least one of the sets U_{23}, U_{24} or U_{34} have two elements, then we can see that the contraction of $\mathbb{AG}(L)$ contains a copy of $K_{4,4}$. Additionally, we may assume that at least two of the sets U_{23}, U_{24} or U_{34}, say U_{23} and U_{24}, have one element. Then it is easy to find a copy of E_{18}, one of the listed graphs in [10], in the graph $\mathbb{AG}(L)$. Now, suppose that $|\bigcup_{i=1}^4 U_i| = 6$ and there exist distinct i and j such that $|U_i| = |U_j| = 2$.

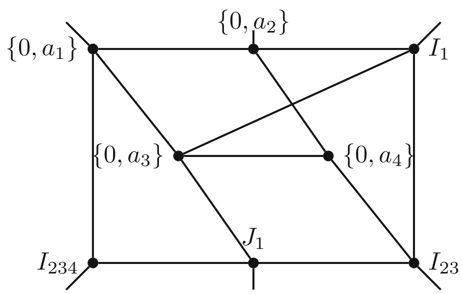


Fig. 2 $\{0, a_1\}$, I_1, $J_1 \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, $I_{23} \in U_{23}$ and $I_{234} \in U_{234}$

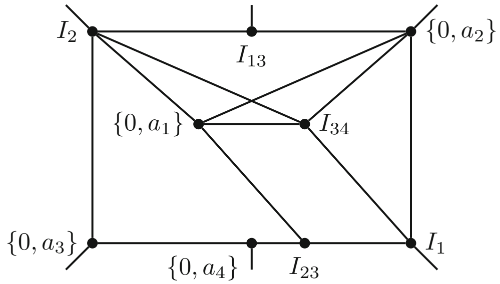


Fig. 3 $\{0, a_1\}$, $I_1 \in U_1$, $\{0, a_2\}$, $I_2 \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, $I_{13} \in U_{13}$, $I_{23} \in U_{23}$ and $I_{34} \in U_{34}$

Without loss of generality, we may assume that $|U_1| = |U_2| = 2$. When $|U_{34}| \geq 2$, we can find a subdivision of $K_{4,4}$ in the structure of the contraction of $\mathbb{AG}(L)$. If $|U_{13}| = 2$, then the contraction of the sets $U_2 \cup U_4$ and $U_1 \cup U_3 \cup U_{13}$ induces a copy of $K_{3,5}$. Moreover, we may assume that $|U_{12}| = |U_{34}| = 1$ or $|U_{14}| = |U_{23}| = 1$. In this case, the contraction of the graph $\mathbb{AG}(L)$ contains a copy of E_3 or E_{18}, two of the listed graphs in [10], respectively. Finally, suppose that U_{13}, U_{23} and U_{34} have one element. Consider the graph D_8, one of the listed graphs in [10], as shown in Fig. 3. In this figure, we have $\{0, a_1\}$, $I_1 \in U_1$, $\{0, a_2\}$, $I_2 \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, $I_{13} \in U_{13}$, $I_{23} \in U_{23}$ and $I_{34} \in U_{34}$.

Clearly, for the case that $|U_{14}| = |U_{24}| = |U_{34}| = 1$, we have the similar result.

Suppose that $|\bigcup_{i=1}^{4} U_i| = 7$ and the statement (iii) does not hold. First, assume that there exists only one U_i, say U_1, such that $|U_1| = 4$. If $U_{234} \neq \varnothing$, then the contraction of $\mathbb{AG}(L)$ contains a copy of $K_{4,4}$. If one of the sets U_{23}, U_{24} or U_{34} is not empty, then $K_{4,4}$ is isomorphic to a subgraph of $\mathbb{AG}(L)$. Now, assume that there is a unique U_i, say U_1, with $|U_1| = 3$. If $U_{234} \neq \varnothing$, then one can obtain a subdivision of $K_{3,5}$ in the structure of the contraction of $\mathbb{AG}(L)$. Also, we may assume that U_{13} or U_{14} is not empty. Then it is easy to see a copy of $K_{4,4}$ as a subgraph of $\mathbb{AG}(L)$. Moreover, if at least one of the sets U_{23}, U_{24} or U_{34} is not empty, then the contraction of $\mathbb{AG}(L)$ contains a copy of $K_{4,4}$ or $K_{3,5}$. Finally, assume that only one of the U_i's, where $1 \leq i \leq 4$, has one element, exactly. Then A_2, one of the listed graphs in [10], is isomorphic to a subgraph of $\mathbb{AG}(L)$, as shown in Fig. 4. In this figure, we have $\{0, a_1\}$, $I_1 \in U_1$, $\{0, a_2\}$, $I_2 \in U_2$, $\{0, a_3\}$, $I_3 \in U_3$ and $\{0, a_4\} \in U_4$.

Therefore, in all of the above situations, we have that $\mathbb{AG}(L)$ is not projective, which is a contradiction.

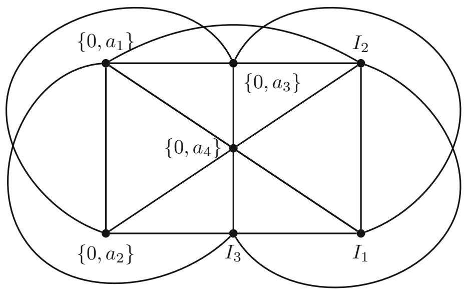


Fig. 4 $\{0, a_1\}$, $I_1 \in U_1$, $\{0, a_2\}$, $I_2 \in U_2$, $\{0, a_3\}$, $I_3 \in U_3$ and $\{0, a_4\} \in U_4$

Conversely, if one of the conditions (i), (ii) or (iii) holds, then one can easily check that the graph $\mathbb{AG}(L)$ embeds in a projective plane. So it is a projective graph and the proof is complete . □

In the rest of this paper, we need to consider the cases that $|A(L)| = 5$ and 6.

First, assume that $|A(L)| = 5$. If $|\bigcup_{i=1}^{5} U_i| \geq 8$, then one can easily realize that the contraction of $\mathbb{AG}(L)$ contains a copy of $K_{4,4}$ or $K_{3,5}$, which is not projective. Therefore, we investigate the cases that $|\bigcup_{i=1}^{5} U_i| = 5, 6$, or 7.

We continue this discussion with the following theorem.

Theorem 2.7 *Let* $|\bigcup_{i=1}^{5} U_i| = 5$. *Then* $\mathbb{AG}(L)$ *is a projective graph if and only if* $|U_{ij}| \leq 2$, *for all* $1 \leq i, j \leq 5$ *and one of the following conditions holds:*

(i) There is only one of the U_{ij}*'s, such that* $|U_{ij}| = 2$ *whenever* $U_{i'j'} = \varnothing$, *for* $\{i', j'\} = \{1, 2, \ldots, 5\} \backslash \{i, j\}$, *and also, for some* $k \in \{1, 2, \ldots, 5\} \backslash \{i, j\}$, *the number of such sets* U_{ki} *and* U_{kj} *with exactly one element is at most two. Moreover, for the sets* $U_{k_1 i}$ *and* $U_{k_2 j}$ *with* $\{k_1, i\} \cap \{k_2, j\} = \varnothing$, *we have* $U_{k_1 i} = \varnothing$ *or* $U_{k_2 j} = \varnothing$.

(ii) There is no U_{ij}, *such that* $|U_{ij}| = 2$, *and there exist at most three distinct sets* U_{ij}, $U_{ij'}$ *and* $U_{ij''}$ *with exactly one element, where* $1 \leq i, j, j', j'' \leq 5$ *whenever at most there is one* U_{kl} *with* $|U_{kl}| \leq 1$, *where* $1 \leq k, l \leq 5$ *and if* U_{kl} *has a vertex, then it is adjacent to at most one of the vertices in the sets* U_{ij}, $U_{ij'}$ *or* $U_{ij''}$.

Proof First, suppose that the graph $\mathbb{AG}(L)$ is projective and to the contrary that none of the conditions of theorem holds. Without loss of generality, assume that $|U_{12}| \geq 3$. Then the contraction of $\mathbb{AG}(L)$ contains a subgraph isomorphic to $K_{3,5}$. If $|U_{12}| = |U_{23}| = 2$, then the graph $\mathbb{AG}(L)$ contains a subgraph isomorphic to subdivision of E_5, one of the listed graphs in [10]. If $|U_{12}| = 2$ and $|U_{34}| = 1$, then the contraction of $\mathbb{AG}(L)$ contains a copy of $K_{4,4}$. If $|U_{12}| = 2$ and $|U_{13}| = |U_{24}| = 1$, then $\mathbb{AG}(L)$ contains a copy of F_1, one of the listed graphs in [10] (see Fig. 5). In this figure, we have $\{0, a_1\} \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, $\{0, a_5\} \in U_5$, $I_{12}, I'_{12} \in U_{12}$, $I_{13} \in U_{13}$ and $I_{24} \in U_{24}$.

If $|U_{13}| = |U_{24}| = 1$ and $|U_{14}| = |U_{23}| = 1$, then the graph $\mathbb{AG}(L)$ contains a copy of F_1, one of the listed graphs in [10]. If $|U_{12}| = |U_{13}| = |U_{14}| = 1$ and $|U_{25}| = 1$, then F_1, one of the listed graphs in [10], is isomorphic to subgraph of

Fig. 5 $\{0, a_1\} \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, $\{0, a_5\} \in U_5$, $I_{12}, I'_{12} \in U_{12}$, $I_{13} \in U_{13}$ and $I_{24} \in U_{24}$

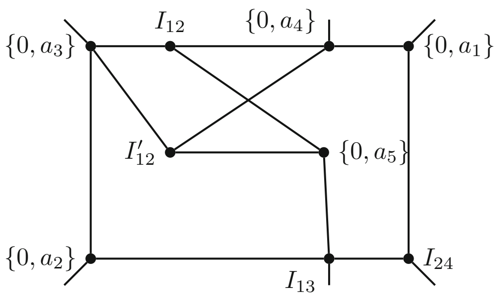


Fig. 6 $\{0, a_1\} \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, $\{0, a_5\} \in U_5$, $I_{12} \in U_{12}$, $I_{13} \in U_{13}$, $I_{14} \in U_{14}$, $I_{23} \in U_{23}$ and $I_{34} \in U_{34}$

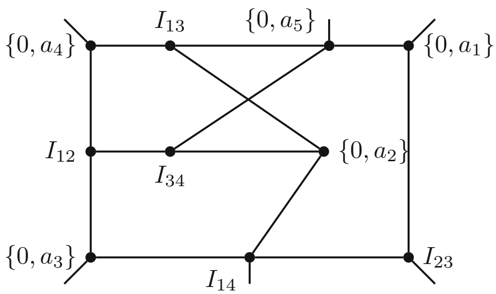
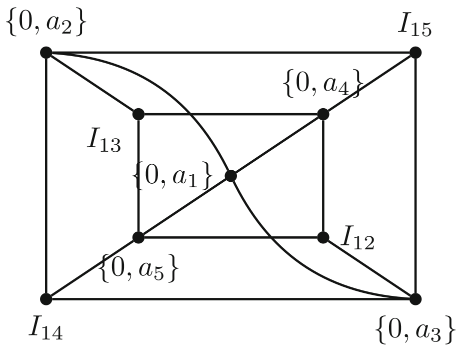


Fig. 7 $\{0, a_1\} \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, $\{0, a_5\} \in U_5$, $I_{12} \in U_{12}$, $I_{13} \in U_{13}$, $I_{14} \in U_{14}$ and $I_{15} \in U_{15}$



$\mathbb{AG}(L)$. If $|U_{12}| = |U_{13}| = |U_{14}| = 1$ and $|U_{23}| = |U_{34}| = 1$, then the graph $\mathbb{AG}(L)$ contains a copy of F_5, one of the listed graphs in [10] (see Fig. 6). In this figure, we have $\{0, a_1\} \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, $\{0, a_5\} \in U_5$, $I_{12} \in U_{12}$, $I_{13} \in U_{13}$, $I_{14} \in U_{14}$, $I_{23} \in U_{23}$ and $I_{34} \in U_{34}$.

If $|U_{12}| = |U_{13}| = |U_{14}| = |U_{15}| = 1$, then $\mathbb{AG}(L)$ contains a copy of E_{22}, one of the listed graphs in [10] (see Fig. 7). In this figure, we have $\{0, a_1\} \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, $\{0, a_5\} \in U_5$, $I_{12} \in U_{12}$, $I_{13} \in U_{13}$, $I_{14} \in U_{14}$ and $I_{15} \in U_{15}$.

So, by the above situations, the graph $\mathbb{AG}(L)$ is not projective, which is a contradiction.

Conversely, one can easily check that if one of the conditions (i) and (ii) holds, then $\mathbb{AG}(L)$ is a projective graph. □

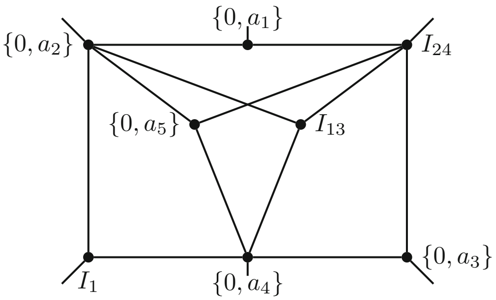


Fig. 8 $\{0, a\}, I_1 \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, $\{0, a_5\} \in U_5$, $I_{13} \in U_{13}$ and $I_{24} \in U_{24}$

Now, suppose that $|\bigcup_{i=1}^{5} U_i| = 6$. Then there is only one of the U_i's, say U_1, such that $|U_1| = 2$. It is clear that U_2, U_3, U_4 and U_5 have one element, exactly.

Theorem 2.8 *Suppose that* $|\bigcup_{i=1}^{5} U_i| = 6$ *and* $|U_1| = 2$ *whenever* $|U_{ijk}| \leq 1$*, for* $2 \leq i, j, k \leq 5$ *and* $|U_{ij}| \leq 1$*, for* $1 \leq i, j \leq 5$*. Then* $\mathbb{AG}(L)$ *is a projective graph if and only if one of the following conditions holds:*

(i) There exist at most two distinct sets U_{1i} *and* U_{1j}*, where* $2 \leq i, j \leq 5$*, such that* $|U_{1i}| = |U_{1j}| = |U_{ij}| = 1$.
(ii) There exist at most two distinct sets with exactly one element U_{ijk}, $U_{i'j'k'}$*, where* $2 \leq i, i', j, j', k, k' \leq 5$*. Moreover, we have at most two sets* U_{1j_1}, U_{1j_2} *with exactly one element such that* $j_1, j_2 \in \{i, j, k\} \cap \{i', j', k'\}$.

Proof Suppose that the graph $\mathbb{AG}(L)$ is projective and on the contrary that none of the conditions of theorem holds. Without loss of generality, assume that $|U_{12}| = 2$, then the contraction of $\mathbb{AG}(L)$ contains a copy of $K_{3,5}$. If $|U_{23}| = 2$, then the contraction of $\mathbb{AG}(L)$ contains a copy of $K_{4,4}$. If $|U_{13}| = |U_{24}| = 1$, then E_3, one of the listed graphs in [10], is isomorphic to a subgraph of $\mathbb{AG}(L)$ (see Fig. 8). In this figure, we have $\{0, a\}, I_1 \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, $\{0, a_5\} \in U_5$, $I_{13} \in U_{13}$ and $I_{24} \in U_{24}$.

If $|U_{23}| = |U_{24}| = 1$, then the graph $\mathbb{AG}(L)$ contains a subgraph isomorphic to E_3, one of the listed graphs in [10]. If $|U_{12}| = |U_{13}| = |U_{14}| = 1$, then the graph $\mathbb{AG}(L)$ contains a copy of E_{22}, one of the listed graphs in [10] (see Fig. 9). In this figure, we have $\{0, a_1\}, I_1 \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, $\{0, a_5\} \in U_5$, $I_{12} \in U_{12}$, $I_{13} \in U_{13}$ and $I_{14} \in U_{14}$.

If $|U_{234}| = 2$, then the contraction of $\mathbb{AG}(L)$ contains a copy of $K_{3,5}$.

If $|U_{234}| = |U_{235}| = |U_{245}| = 1$, then the graph $\mathbb{AG}(L)$ contains a copy of E_{20}, one of the listed graphs in [10] (see Fig. 10). In this figure, we have $\{0, a\}, I_1 \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, $\{0, a_5\} \in U_5$, $I_{234} \in U_{234}$, $I_{235} \in U_{235}$ and $I_{245} \in U_{245}$.

If $|U_{235}| = |U_{35}| = 1$ or $|U_{235}| = |U_{24}| = 1$, then the graph $\mathbb{AG}(L)$ contains a copy of E_3 or D_3, two of the listed graphs in [10]. The second case is pictured in Fig. 11. In this figure, we have $\{0, a\}, I_1 \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, $\{0, a_5\} \in U_5$, $I_{24} \in U_{24}$ and $I_{235} \in U_{235}$.

Obviously, we conclude that in each of the above statements, the graph $\mathbb{AG}(L)$ is not projective, which is a contradiction.

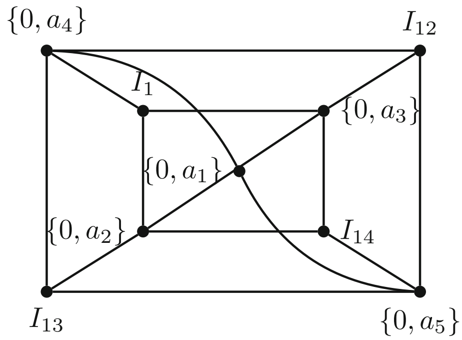


Fig. 9 $\{0, a_1\}$, $I_1 \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, $\{0, a_5\} \in U_5$, $I_{12} \in U_{12}$, $I_{13} \in U_{13}$ and $I_{14} \in U_{14}$

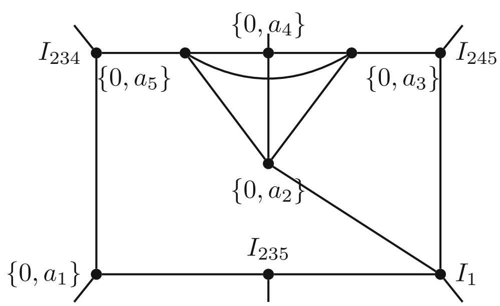


Fig. 10 $\{0, a\}$, $I_1 \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, $\{0, a_5\} \in U_5$, $I_{234} \in U_{234}$, $I_{235} \in U_{235}$ and $I_{245} \in U_{245}$

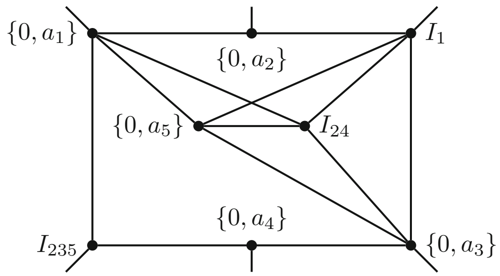


Fig. 11 $\{0, a\}$, $I_1 \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, $\{0, a_5\} \in U_5$, $I_{24} \in U_{24}$ and $I_{235} \in U_{235}$

Conversely, it is not hard to see that if one of the conditions (i) and (ii) holds, then $\mathbb{AG}(L)$ is a projective graph. □

Now, assume that $|\bigcup_{i=1}^{5} U_i| = 7$. If $|U_i| = |U_j| = 2$, for some $1 \leq i \neq j \leq 5$, then the contraction of $\mathbb{AG}(L)$ is isomorphic to B_1, one of the listed graphs in [10] (see Fig. 12). In this figure, we have $\{0, a_1\}$, $I_1 \in U_1$, $\{0, a_2\}$, $I_2 \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$ and $\{0, a_5\} \in U_5$. In this situation, $\mathbb{AG}(L)$ is not projective.

Therefore, it is enough to consider the case that $|U_i| = 3$, for some $1 \leq i \leq 5$. Without loss of generality, we may assume that $|U_1| = 3$. Hence, we have the following theorem.

Theorem 2.9 *Suppose that* $|\bigcup_{i=1}^{5} U_i| = 7$ *and* $|U_1| = 3$. *Then* $\mathbb{AG}(L)$ *is a projective graph if and only if, for all* $2 \leq i, j, k, l \leq 5$, $U_{ijkl} = U_{ijk} = \varnothing$ *whenever* $U_{ij} = \varnothing$, *for* $1 \leq i, j \leq 5$.

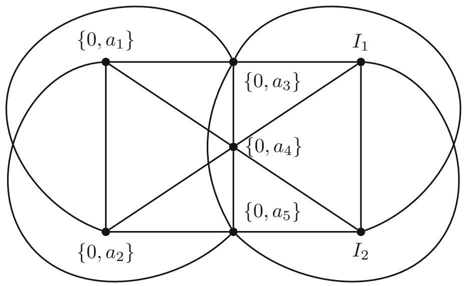


Fig. 12 $\{0, a_1\}, I_1 \in U_1$, $\{0, a_2\}, I_2 \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$ and $\{0, a_5\} \in U_5$

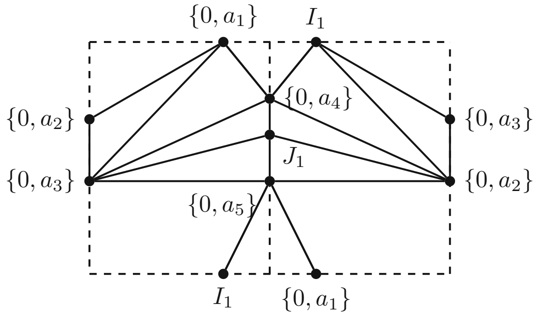


Fig. 13 $\{0, a_1\}, I_1, J_1 \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$ and $\{0, a_5\} \in U_5$

Proof First, assume that $\mathbb{AG}(L)$ is a projective graph and to the contrary that $U_{ijkl} \neq \varnothing$, $U_{ijk} \neq \varnothing$, for $2 \leq i, j, k, l \leq 5$ or $U_{ij} \neq \varnothing$, for $1 \leq i, j \leq 5$. If $U_{ijkl} \neq \varnothing$, for $2 \leq i, j, k, l \leq 5$, then the contraction of $\mathbb{AG}(L)$ contains a copy of $K_{3,5}$. Also if $U_{ijk} \neq \varnothing$, for $2 \leq i, j, k \leq 5$, then $K_{4,4}$ is isomorphic to a subgraph of the contraction of $\mathbb{AG}(L)$. Finally, if $U_{ij} \neq \varnothing$, for $1 \leq i, j \leq 5$, then the contraction of $\mathbb{AG}(L)$ contains a copy of $K_{3,5}$. Therefore, it is not projective, which is a contradiction.

Conversely, assume that $U_{ijkl} = U_{ijk} = \varnothing$, for all $2 \leq i, j, k, l \leq 5$, and $U_{ij} = \varnothing$, for $1 \leq i, j \leq 5$. Then one can easily check that the graph $\mathbb{AG}(L)$ is isomorphic to Fig. 13, which is projective. In this figure, we have $\{0, a_1\}, I_1, J_1 \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$ and $\{0, a_5\} \in U_5$. The proof is complete. □

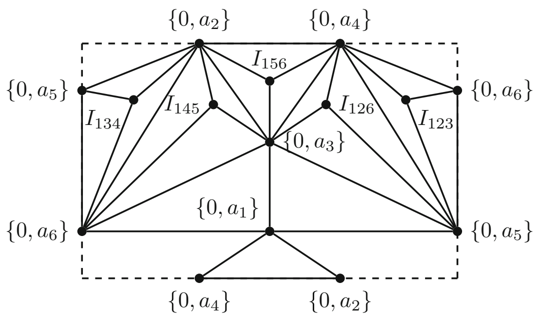
Now, by Theorems 2.7, 2.8 and 2.9, we completely characterized the projectivity of $\mathbb{AG}(L)$ in the case that $|A(L)| = 5$.

In order to complete the study of the projectivity of $\mathbb{AG}(L)$, we assume that $|A(L)| = 6$. First, suppose that $|\bigcup_{i=1}^{6} U_i| \geq 7$. Then $\mathbb{AG}(L)$ contains a copy of B_1, one of the listed graphs in [10], and so it is not projective. Hence, we may assume that $|\bigcup_{i=1}^{6} U_i| = 6$. Clearly, for all $1 \leq i \leq 6$, we have $|U_i| = 1$.

Theorem 2.10 *Suppose that* $|\bigcup_{i=1}^{6} U_i| = 6$ *and* $|U_{ijk}| \leq 1$ *whenever* $U_{ij} = \varnothing$, *for all* $1 \leq i \neq j \neq k \leq 6$. *Then* $\mathbb{AG}(L)$ *is a projective graph if and only if one of the following conditions holds:*

- *(i) If* $|U_{ijk}| = 1$, *then* $U_{i'j'k'} = \varnothing$, *where* $\{i', j', k'\} = \{1, 2, \ldots, 6\} \backslash \{i, j, k\}$.
- *(ii) There exist at most two distinct sets* U_{ijk} *and* $U_{ijk'}$ *with one element, exactly, where* $1 \leq i \neq j \neq k \neq k' \leq 6$.

Fig. 14 $\{0, a_1\} \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, $\{0, a_5\} \in U_5$, $\{0, a_6\} \in U_6$, $I_{123} \in U_{123}$, $I_{156} \in U_{156}$, $I_{134} \in U_{134}$, $I_{126} \in U_{126}$ and $I_{145} \in U_{145}$



(iii) There exist at most five distinct sets U_{ijk} with one element, exactly, where $1 \leq i \neq j \neq k \leq 6$, such that the intersection of all the sets at their indices has one element, exactly.

Proof Suppose that the graph $\mathbb{AG}(L)$ is projective and on the contrary that none of the conditions of theorem holds. Without loss of generality, assume that $|U_{123}| \geq 2$. Then the contraction of $\mathbb{AG}(L)$ contains a copy of $K_{3,5}$. Also if U_{12} is non-empty, then $\mathbb{AG}(L)$ contains a copy of B_1, one of the listed graphs in [10]. In addition, if $|U_{156}| = |U_{234}| = 1$, then E_{18}, one of the listed graphs in [10], is isomorphic to a subgraph of $\mathbb{AG}(L)$. Moreover, if $|U_{123}| = |U_{124}| = |U_{125}| = 1$, then $\mathbb{AG}(L)$ contains a copy of E_{22}, one of the listed graphs in [10]. In each of the above situations, the graph $\mathbb{AG}(L)$ is not projective, which is a contradiction.

Conversely, by considering the embedding of the graph $\mathbb{AG}(L)$ in a projective plane in Fig. 14, we conclude that it is projective graph. In this figure, we have $\{0, a_1\} \in U_1$, $\{0, a_2\} \in U_2$, $\{0, a_3\} \in U_3$, $\{0, a_4\} \in U_4$, $\{0, a_5\} \in U_5$, $\{0, a_6\} \in U_6$, $I_{123} \in U_{123}$, $I_{156} \in U_{156}$, $I_{134} \in U_{134}$, $I_{126} \in U_{126}$ and $I_{145} \in U_{145}$. □

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