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On Resonant Robin Problems with General Potential

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Abstract We consider a semilinear Robin problem with indefinite and unbounded potential and a reaction term which asymptotically at $\pm\infty$ is resonant with respect to any nonprincipal, nonnegative eigenvalue of the differential operator. Using critical point theory, Morse theory (critical groups) and the reduction method, we show that the problem has at least three nontrivial solutions.

Keywords Indefinite and undbounded potential · Robin boundary condition · Critical groups · Multiple smooth solutions · Resonance

Mathematics Subject Classification 35J20 · 35J60 · 58E05

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2- boundary $\partial\Omega$. In this paper, we study the following semilinear Robin problem

$$\begin{aligned} -\Delta u(z) + \xi(z)u(z) &= f(z, u(z)) \text{ in } \Omega \\ \frac{\partial u}{\partial n} + \beta(z)u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{1}$$

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In this problem, the potential function $\xi \in L^s(\Omega)$ $s > N$, is indefinite (that is, sign changing). So, the differential operator in problem (1) is not coercive. The reaction term $f(z, x)$ is a measurable function defined on $\Omega \times \mathbb{R}$ and for a.a. $z \in \Omega$ $f(z, \cdot) \in C^1(\mathbb{R})$. We assume that asymptotically as $x \to \pm\infty$ the function $f(z, \cdot)$ exhibits linear growth and can interact (resonance) with any nonprincipal, nonnegative eigenvalue of the differential operator $u \to -\Delta u + \xi(z)u$ with Robin boundary condition. In the boundary condition, $\frac{\partial u}{\partial n}$ denotes the usual normal derivative of u, defined by extension of the linear map

$$C^1(\overline{\Omega}) \ni u \to \frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The boundary coefficient $\beta(z)$ satisfies $\beta \in W^{1,\infty}(\partial\Omega)$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$. When $\beta \equiv 0$, we have the usual Neumann problem.

In this paper, using variational methods based on the critical point theory, together with Morse theory (critical groups) and the reduction technique of Amann [2] and Castro and Lazer [3], we prove a multiplicity theorem for problem (1), producing at least three nontrivial smooth solutions, two of which have constant sign (one positive and the other negative).

Recently, semilinear problems with an indefinite potential were studied by Kyristi and Papageorgiou [8], Li and Wang [9], Papageorgiou and Papalini [13], Qin et al. [19], Zhang and Liu [22] (Dirichlet problems), Papageorgiou and Radulescu [14], Papageorgiou and Smyrlis [17] (Neumann problems) and D'Agui et al. [5], Papageorgiou and Radulescu [16], Papageorgiou et al. [18], Shi and Li [20] (Robin problems). In D'Agui et al. [5], the reaction term is asymmetric, Papageorgiou and Radulescu [16] assume that the reaction $f(z, \cdot)$ has z-dependent zeros of constant sign, and arbitrary growth, Papageorgiou et al. [18] allow for double resonance to occur and prove only existence of nontrivial solutions, and finally Shi and Li [20] assume a superlinear reaction term satisfying the Ambrosetti–Rabinowitz condition.

2 Mathematical Background

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$, we denote the duality brackets for the pair (X^*, X). If $\varphi \in C^1(X, \mathbb{R})$, then we say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following property holds: "Every sequence $\{u_n\}_{n\geq 1} \subset X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subset \mathbb{R}$ is bounded and

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \text{ in } X^*,$$

admits a strongly convergent subsequence."

This compactness-type condition on φ leads to a deformation theorem from which one can derive the minimax theory of the critical values of φ. One of the main results in that theory is the so-called mountain pass theorem which we recall here.

Theorem 1 *If $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > \rho > 0$*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\left[\varphi(u) : \|u - u_0\| = \rho\right] = m_\rho$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} \varphi(\gamma(t))$ with $\Gamma := \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$, then $c \ge m_\rho$ and c is a critical value of φ

In the study of problem (1), we will use the following spaces:

- The Sobolev space $H^1(\Omega)$.
- The Banach space $C^1(\overline{\Omega})$.
- The boundary Lebesgue spaces $L^p(\partial\Omega)$ $(1 \le p \le \infty)$.

We know that $H^1(\Omega)$ is a Hilbert space with inner product

$$(u, h)_{H^1(\Omega)} = \int_\Omega uh\mathrm{d}z + \int_\Omega (Du, Dh)_{\mathbb{R}^N}\mathrm{d}z \quad \text{for all } u, h \in H^1(\Omega)$$

and corresponding norm

$$\|u\| = \left[\|u\|_2^2 + \|Du\|_2^2\right]^{\frac{1}{2}} \quad \text{for all } u \in H^1(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \ge 0 \quad \text{for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior which contains the set

$$D_+ = \{u \in C_+ : u(z) > 0 \quad \text{for all } z \in \overline{\Omega}\}.$$

On $\partial\Omega$, we consider the $(N-1)-$dimensional Hausdorff (surface) measure denoted by $\sigma(\cdot)$. Using this measure, we can define in the usual way the Lebesgue spaces $L^p(\partial\Omega)$ $(1 \le p \le \infty)$. The theory of Sobolev spaces gives us a unique continuous linear map $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$, known as the "trace map," such that

$$\gamma_0(u) = u|_{\partial\Omega} \quad \text{for all } u \in H^1(\Omega) \cap C(\overline{\Omega}).$$

Hence, the trace map defines "boundary values" for an arbitrary Sobolev function $u \in H^1(\Omega)$. We know that

$$ker\gamma_0 = H_0^1(\Omega) \text{ and } im\gamma_0 = H^{\frac{1}{2},2}(\partial\Omega).$$

The trace map is compact into $L^p(\partial\Omega)$ for all $p \in \left[1, \frac{2(N-1)}{N-2}\right)$ when $N \ge 3$ and into $L^p(\partial\Omega)$ for all $p \in [1, \infty)$ when $N = 1, 2$. In what follows, for the sake of notational simplicity we drop the use of the trace map γ_0. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

For every $x \in \mathbb{R}$, we set $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. Then, given $u \in H^1(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$ and we have

$$u = u^+ - u^-, \ |u| = u^+ + u^-, \ u^\pm \in H^1(\Omega).$$

Given $g : \Omega \times \mathbb{R} \to \mathbb{R}$ a measurable function, by $N_g(\cdot)$ we denote the Nemytski operator corresponding to $g(\cdot, \cdot)$, that is,

$$N_g(u)(\cdot) = g(\cdot, u(\cdot)) \quad \text{for all } u \in H^1(\Omega)$$

Evidently $z \to N_g(u)(z)$ is measurable. Also $A \in L(H^1(\Omega), H^1(\Omega)^*)$ is defined by

$$\langle A(u), h\rangle = \int_\Omega (Du, Dh)_{\mathbb{R}^N} \mathrm{d}z \quad \text{for all } u, h \in H^1(\Omega).$$

We introduce our hypotheses on the potential function $\xi(\cdot)$ and the boundary coefficient $\beta(\cdot)$

$\underline{H(\xi)}$: $\xi \in L^s(\Omega)$ $s > N$ and $\xi^+ \in L^\infty(\Omega)$.

$\underline{H(\beta)}$: $\beta \in W^{1,\infty}(\partial\Omega)$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$.

Remark We can have $\beta \equiv 0$, in which case we recover the usual Neumann problem.

Let $\gamma : H^1(\Omega) \to \mathbb{R}$ be the C^2–functional defined by

$$\gamma(u) := \|Du\|_2^2 + \int_\Omega \xi(z)u^2 \mathrm{d}z + \int_{\partial\Omega} \beta(z)u^2 \mathrm{d}\sigma \quad \text{for all } u \in H^1(\Omega).$$

From D'Agui et al. [5], we know that there exists $\mu > 0$ such that

$$\gamma(u) + \mu\|u\|_2^2 \geq c_0\|u\|^2 \text{ for some } c_0 > 0, \text{ all } u \in H^1(\Omega). \tag{2}$$

We consider the following linear eigenvalue problem

$$\begin{aligned} -\Delta u(z) + \xi(z)u(z) &= \hat{\lambda}u(z) \text{ in } \Omega, \\ \frac{\partial u}{\partial n} + \beta(z)u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{3}$$

Using (2) and the spectral theorem for compact self-adjoint operators, we show that problem (3) admits a sequence $\{\hat{\lambda}_k\}_{k\geq 1} \subset \mathbb{R}$ of distinct eigenvalues such that $\hat{\lambda}_k \to +\infty$. We know that the first eigenvalue $\hat{\lambda}_1$ is simple and the corresponding eigenfunctions do not change sign. Moreover, it admits the following variational characterization

$$\hat{\lambda}_1 = \inf\left[\frac{\gamma(u)}{\|u\|_2^2} : u \in H^1(\Omega), u \neq 0\right]. \tag{4}$$

The infimum in (4) is realized on the corresponding one dimensional eigenspace (recall that $\hat{\lambda}_1$ is simple). By $\hat{u}_1$, we denote the L^2-normalized (that is, $\|\hat{u}_1\|_2 = 1$) positive eigenfunction corresponding to $\hat{\lambda}_1$. We know that $\hat{u}_1 \in C_+\backslash\{0\}$ and in fact hypothesis $H(\xi)$ and the strong maximum principle imply that $\hat{u}_1 \in D_+$. Note that

- $\hat{\lambda}_1 = 0$ if $\xi \equiv 0,\ \beta \equiv 0$ (Neumann eigenvalue problem).
- $\hat{\lambda}_1 > 0$ if $\xi(z) \geq 0$ for a.a. $z \in \Omega,\ \xi \not\equiv 0$ or if $\xi \equiv 0, \beta \geq 0, \beta \not\equiv 0$.

By $E(\hat{\lambda}_k),\ k \in \mathbb{N}$, we denote the eigenspace corresponding to the eigenvalue $\hat{\lambda}_n$. We have the following variational characterizations for the other eigenvalues:

$$\begin{aligned}\hat{\lambda}_k &= \inf\left[\frac{\gamma(u)}{\|u\|_2^2} : u \in \overline{\bigoplus_{n\geq k} E(\hat{\lambda}_n)},\ u \neq 0\right]\\ &= \sup\left[\frac{\gamma(u)}{\|u\|_2^2} : u \in \bigoplus_{n=1}^{k} E(\hat{\lambda}_n),\ u \neq 0\right],\ k \geq 2.\end{aligned} \tag{5}$$

In (5), both the infimum and the supremum are realized on $E(\hat{\lambda}_k)$. We have $E(\hat{\lambda}_k) \subset C^1(\overline{\Omega})$ for all $k \in \mathbb{N}$, and that for every $k \geq 2$, the elements of $E(\hat{\lambda}_k)$ are nodal (that is, sign changing). Moreover, each eigenspace $E(\hat{\lambda}_k),\ k \in \mathbb{N}$, has the "unique continuation property" which says that if $u \in E(\hat{\lambda}_k)$ and vanishes on a set of positive measure, then $u \equiv 0$. For details, we refer to D'Agui et al. [5].

The properties outlined above lead to the following useful lemma (see Papageorgiou and Radulescu [16]).

Lemma 2 (a) *If* $\xi \in L^s(\Omega)$ $s > N$, *hypothesis* $H(\beta)$ *hold and* $\theta \in L^\infty(\Omega)$ *satisfies*

$$\theta(z) \leq \hat{\lambda}_m \text{ for a.a } z \in \Omega, \theta \not\equiv \hat{\lambda}_m$$

then there exists $c_1 > 0$ *such that*

$$\gamma(u) - \int_\Omega \theta(z)u^2\mathrm{d}z \geq c_1\|u\|^2 \quad \text{for all } u \in \overline{\bigoplus_{k\geq m} E(\hat{\lambda}_k)}$$

(b) *If* $\xi \in L^s(\Omega)$ $s > N$, *hypothesis* $H(\beta)$ *hold and* $\theta \in L^\infty(\Omega)$ *satisfies*

$$\theta(z) \geq \hat{\lambda}_m \text{ for a.a } z \in \Omega, \quad \theta \not\equiv \hat{\lambda}_m$$

then there exists $\hat{c}_1 > 0$ *such that*

$$\gamma(u) - \int_\Omega \theta(z)u^2 dz \leq -\hat{c}_1\|u\|^2 \quad \text{for all } u \in \bigoplus_{k=1}^{m} E(\hat{\lambda}_k).$$

In a similar way, we can analyze the following weighted version of the eigenvalue problem (3)

$$-\Delta u(z) + \xi(z)u(z) = \tilde{\lambda}\eta(z)u(z) \text{ in } \Omega, \quad \frac{\partial u}{\partial n} + \beta(z)u = 0 \text{ on } \partial\Omega. \tag{6}$$

In this problem, $\eta \in L^\infty(\Omega)$, $\eta \not\equiv 0$, $\eta(z) \geq 0$ for a.a. $z \in \Omega$. Again we have a whole sequence $\{\tilde{\lambda}_k(\eta)\}_{k\geq 1} \subset \mathbb{R}$ of distinct eigenvalues such that $\tilde{\lambda}_k(\eta) \to +\infty$ as $k \to +\infty$. These eigenvalues exhibit the same properties as those of problem (3), and in their variational characterization the Rayleigh quotient is $\frac{\gamma(u)}{\int_\Omega \eta u^2 dz}$ for all $u \in H^1(\Omega)$, $u \neq 0$. The unique continuation property leads to the following strict monotonicity property for the map $\eta \to \hat{\lambda}_k(\eta)$.

Lemma 3 *If $\xi \in L^s(\Omega)$ with $s > N$, hypothesis $H(\beta)$ holds and $\eta, \hat{\eta} \in L^\infty(\Omega)$ satisfy*

$$\eta(z) \leq \hat{\eta}(z) \textit{ for a.a. } z \in \Omega,\ \eta \neq \hat{\eta},$$

then for all $k \in \mathbb{N}$ we have

$$\tilde{\lambda}_k(\hat{\eta}) < \tilde{\lambda}_k(\eta).$$

Next let X be a Banach space and $\varphi \in C^1(X, \mathbb{R})$. We introduce the following sets.

$$\varphi^c = \{u \in X : \varphi(u) \leq c\} \quad \text{for all } c \in \mathbb{R},$$
$$K_\varphi = \{u \in X : \varphi'(u) = 0\} \quad \text{(the critical set of } \varphi),$$
$$K_\varphi^c = \{u \in K_\varphi : \phi(u) = c\} \quad \text{for all } c \in \mathbb{R}.$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subset Y_1 \subset X$. For every $k \in \mathbb{N}$, by $H_k(Y_1, Y_2)$ we denote the kth relative singular homology group for the pair (Y_1, Y_2) with integer coefficients (recall that if $k \in -\mathbb{N}$, then $H_k(Y_1, Y_2) = 0$). If $u \in K_\varphi^c$ is isolated, then the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \backslash \{u\}) \text{ for all } k \in \mathbb{N}_0.$$

Here U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology implies that the above definition is independent of the choice of the neighborhood U.

Finally, we will introduce our hypotheses on the reaction term $f(z, x)$. We set

$$m_0 = \min\{k \in \mathbb{N} : \hat{\lambda}_k \geq 0\}$$

Then, $\hat{\lambda}_{m_0}$ is the first nonnegative eigenvalue. Note that

- If $\xi \equiv 0$, $\beta \equiv 0$ (classical Neumann problem), then $m_0 = 1$ and $\hat{\lambda}_{m_0} = 0$
- If $\xi \geq 0$, then $m_0 = 1$

The hypotheses on the reaction term $f(z,x)$ are the following:

$\underline{H(f)}$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$ $f(z,0) = 0$, $f(z,\cdot) \in C^1(\mathbb{R})$ and

(i) There exists $m \geq m_0, m \neq 1$ such that

$$\hat{\lambda}_m \leq \liminf_{x \to \pm\infty} \frac{f(z,x)}{x} \text{ uniformly for a.a. } z \in \Omega;$$

(ii) There exists a function $\eta \in L^\infty(\Omega)$ such that

$$\begin{aligned} \eta(z) &\leq \hat{\lambda}_{m+1} \quad \text{for a.a. } z \in \Omega, \eta \not\equiv \hat{\lambda}_{m+1} \\ f'_x(z,x) &\leq \eta(z) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}; \end{aligned}$$

(iii) If $F(z,x) = \int_0^x f(z,s)\mathrm{d}s$,
then $f(z,x)x - 2F(z,x) \to -\infty$ uniformly for a.a. $z \in \Omega$ as $x \to \pm\infty$;

(iv) There exists $\theta \in L^\infty(\Omega)$ such that

$$\begin{aligned} \theta(z) \leq \hat{\lambda}_1 \quad &\text{for a.a. } z \in \Omega, \ \theta \not\equiv \hat{\lambda}_1, \\ \limsup_{x \to 0} \frac{f(z,x)}{x} &\leq \theta(z) \text{ uniformly for a.a. } z \in \Omega; \end{aligned}$$

(v) For every $\rho > 0$, there exist $\hat{\xi}_\rho > 0$ and $\alpha_\rho \in L^\infty(\Omega)$

$$f(z,x)x + \hat{\xi}_\rho x^2 \geq 0 \text{ and } |f(z,x)| \leq \alpha_\rho(z) \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \rho.$$

Remarks Hypothesis $H(f)(i)$ implies that at $\pm\infty$ we can have resonance with respect to any nonprincipal nonnegative eigenvalue of the differential operator. Hypothesis $H(f)(ii)$ and the mean value theorem imply that

$$(f(z,x) - f(z,y)) \leq \eta(z)(x-y)^2 \quad \text{for a.a. } z \in \Omega, \text{ all } x, y \in \mathbb{R} \tag{7}$$

Example 1 The following function satisfies hypotheses $H(f)$. For the sake of simplicity, we drop the z-dependence

$$f(x) = \begin{cases} \theta x + c|x|x & \text{if } |x| \leq 1 \\ \hat{\lambda}_m x + |x|^{q-2}x & \text{if } |x| > 1 \end{cases}$$

with $\theta < \hat{\lambda}_1$, $m \geq m_0$, $m \neq 1$, $1 < q < 2$ and $c = \hat{\lambda}_m + 1 - \theta$

3 Solutions of Constant Sign

In this section, we produce two nonsmooth solutions of constant sign. To this end, we introduce the following truncations–perturbations of $f(z,\cdot)$:

$$\hat{f}_+(z,x) = \begin{cases} 0 & \text{if } x \le 0 \\ f(z,x) + \mu x & \text{if } 0 < x \end{cases}$$

$$\hat{f}_-(z,x) = \begin{cases} f(z,x) + \mu x & \text{if } x < 0 \\ 0 & \text{if } 0 \le x \end{cases} \tag{8}$$

Here $\mu > 0$ is as in (2). Both $\hat{f}_\pm(z,x)$ are Caratheodory functions (that is, for all $x \in \mathbb{R}$ $z \to \hat{f}_\pm(z,x)$ are measurable and for a.a. $z \in \Omega$, $x \to \hat{f}_\pm(z,x)$ are continuous). We set $\hat{F}_\pm(z,x) = \int_0^x \hat{f}_\pm(z,s)ds$ and consider the C^1-functionals $\hat{\varphi}_\pm$: $H^1(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\pm(u) = \frac{1}{2}\gamma(u) + \frac{\mu}{2}\|u\|_2^2 - \int_\Omega \hat{F}_\pm(z,u)\mathrm{d}z \quad \text{for all } u \in H^1(\Omega)$$

Proposition 4 *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$ hold, then the functional $\hat{\varphi}_\pm$ satisfy the C-condition*

Proof We do the proof for the functional $\hat{\varphi}_+$, the proof for $\hat{\varphi}_-$ being similar.

Let $\{u_n\}_{n\ge 1} \subset H^1(\Omega)$ be a sequence such that

$$|\hat{\varphi}_+(u_n)| \le M_1 \text{ for some } M_1 > 0, \quad \text{all } n \in \mathbb{N}, \tag{9}$$

$$(1 + \|u_n\|)\hat{\varphi}_+'(u_n) \to 0 \text{ in } H^1(\Omega)^* \text{ as } n \to \infty \tag{10}$$

From (10), we have

$$\begin{aligned} &|\langle \hat{\varphi}_+'(u_n), h\rangle| \le \frac{\epsilon_n \|h\|}{1 + \|u_n\|} \quad \text{for all } h \in H^1(\Omega) \text{ with } \epsilon_n \to 0^+, \\ \implies &|\langle A(u_n), h\rangle + \int_\Omega (\xi(z) + \mu)u_n h\mathrm{d}z + \int_{\partial\Omega} \beta(z)u_n h\mathrm{d}\sigma \\ &- \int_\Omega \hat{f}_+(z,u_n)h\mathrm{d}z| \le \frac{\epsilon_n\|h\|}{1 + \|u_n\|} \text{ for all } h \in H^1(\Omega) \end{aligned} \tag{11}$$

In (11), we choose $h = -u_n^- \in H^1(\Omega)$ and we obtain

$$\begin{aligned} &\gamma(u_n^-) + \mu\|u_n^-\|_2^2 \le \epsilon_n \quad \text{for all } n \in \mathbb{N} \text{ (see (8))}, \\ \implies &c_0\|u_n^-\|^2 \le \epsilon_n \quad \text{for all } n \in \mathbb{N} \text{ (see (2))}, \\ \implies &u_n^- \to 0 \text{ in } H^1(\Omega) \quad \text{as } n \to \infty. \end{aligned} \tag{12}$$

From (9) and (12), it follows that

$$\gamma(u_n^+) - \int_\Omega 2F\left(z, u_n^+\right) \mathrm{d}z \geq -M_2 \quad \text{for some } M_2 > 0, \text{ all } n \in \mathbb{N} \tag{13}$$

In (11), we choose $h = u_n^+ \in H^1(\Omega)$ and have

$$-\gamma(u_n^+) + \int_\Omega f\left(z, u_n^+\right) u_n^+ \mathrm{d}z \geq -\epsilon_n \quad \text{for all } n \in \mathbb{N} \tag{14}$$

Adding (13) and (14), we obtain

$$\int_\Omega \left[f\left(z, u_n^+\right) u_n^+ - 2F\left(z, u_n^+\right)\right] \mathrm{d}z \geq -M_3 \text{ for some } M_3 > 0, \quad \text{all } n \in \mathbb{N}. \tag{15}$$

We will show that $\{u_n^+\}_{n\geq 1} \subset H^1(\Omega)$ is bounded. Arguing by contradiction, suppose that

$$\|u_n^+\| \to +\infty \tag{16}$$

We set $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. By passing to a suitable subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \text{ in } H^1(\Omega) \text{ and } y_n \to y \text{ in } L^{\frac{2s}{s-1}}(\Omega) \text{ and in } L^2(\partial\Omega) \tag{17}$$

From (11) and (12), we have

$$|\langle A(y_n), h\rangle + \int_\Omega \xi(z) y_n h \mathrm{d}z + \int_{\partial\Omega} \beta(z) y_n h \mathrm{d}\sigma - \int_\Omega \frac{N_f(u_n^+)}{\|u_n^+\|} h \mathrm{d}z| \leq \epsilon_n' \|h\| \\ \text{for all } h \in H^1(\Omega) \text{ with } \epsilon_n' \to 0^+ \tag{18}$$

Here $N_f(y)(\cdot) = f(\cdot, y(\cdot))$ for all $y \in H^1(\Omega)$ (the Nemitsky map corresponding to f). From (7) and hypothesis $H(f)(i)$, it follows that

$$\hat{\lambda}_m \leq \liminf_{x\to\pm\infty} \frac{f(z,x)}{x} \leq \limsup_{x\to\pm\infty} \frac{f(z,x)}{x} \leq \eta(z) \quad \text{uniformly for a.a. } z \in \Omega \tag{19}$$

From (19) and hypothesis H(f)(v), we see that

$$\left\{\frac{N_f(u_n^+)}{\|u_n^+\|}\right\}_{n\geq 1} \subset L^2(\Omega) \text{ is bounded}$$

Passing to a subsequence if necessaty and using (16) and (19), we have

$$\frac{N_f(u_n^+)}{\|u_n^+\|} \xrightarrow{w} k(z) y \text{ in } L^2(\Omega) \text{ with } \hat{\lambda}_m \leq k(z) \leq \eta(z) \quad \text{for a.a. } z \in \Omega \tag{20}$$

(see Aizicovici et al. [1], proof of Proposition 30). We return to (18), pass to the limit as $n \to \infty$ and use (17) and (20). Then,

$$\begin{aligned}
&\langle A(y), h\rangle + \int_\Omega \xi(z) y h \mathrm{d}z + \int_{\partial\Omega} \beta(z) y h \mathrm{d}\sigma = \int_\Omega k(z) y h \mathrm{d}z \quad \text{for all } h \in H^1(\Omega), \\
&\quad\implies -\Delta y(z) + \xi(z) y(z) = k(z) y(z) \quad \text{for a.a. } z \in \Omega \\
&\quad\frac{\partial y}{\partial n} + \beta(z) y = 0 \quad \text{on } \partial\Omega \\
&\quad\text{(see Papageorgiou and Radulescu [15]).}
\end{aligned} \tag{21}$$

If $k \not\equiv \hat{\lambda}_m$ (see (20)), then using Lemma 3 we have

$$1 = \tilde{\lambda}_m(\lambda_m) > \tilde{\lambda}(k) \text{ and } \tilde{\lambda}_{m+1}(k) \geq \tilde{\lambda}_{m+1}(\eta) > \tilde{\lambda}_{m+1}(\hat{\lambda}_{m+1}) = 1 \tag{22}$$

From (21) and (22), it follows that

$$y = 0 \tag{23}$$

On the other hand, if in (18) we choose $h = y_n - y \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (17) and (20), then

$$\begin{aligned}
&\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0, \\
&\quad\implies \|Dy_n\|_2 \to \|Dy\|_2, \\
&\quad\implies y_n \to y \text{ in } H^1(\Omega) \\
&\quad\text{(by the Kadec-Klee property of Hilbert spaces, see (17))}, \\
&\quad\implies \|y\| = 1.
\end{aligned} \tag{24}$$

Comparing (23) and (24), we have a contradiction.

Next suppose that $k(z) = \hat{\lambda}_m$ for a.a. $z \in \Omega$. From (21) and (24), we have that

$$\begin{aligned}
&y \in E(\hat{\lambda}_m)\backslash\{0\} \\
&\quad\implies y(z) \neq 0 \quad \text{for a.a. } z \in \Omega \\
&\quad\text{(by the unique continuation property).}
\end{aligned}$$

This implies that

$$u_n^+(z) \to +\infty \quad \text{for a.a. } z \in \Omega \text{ (see (16))},$$

$$\begin{aligned}
&\implies \int_\Omega \left[f\left(z, u_n^+\right) u_n^+ - 2F\left(z, u_n^+\right)\right] \mathrm{d}z \to -\infty \\
&\text{(see hypothesis H(f)(iii) and use Fatous lemma)}
\end{aligned} \tag{25}$$

Comparing (15) and (25), we have a contradiction.

This proves that $\{u_n^+\}_{n\geq 1} \subset H^1(\Omega)$ is bounded; hence, using (12), we conclude that

$$\{u_n\} \subset H^1(\Omega) \text{ is bounded}$$

Therefore, we may assume that

$$u_n \xrightarrow{w} u \text{ in } H^1(\Omega) \text{ and } u_n \to u \text{ in } L^{\frac{2s}{s-1}}(\Omega) \text{ and in } L^2(\partial\Omega). \tag{26}$$

In (11), we choose $h = u_n - u \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (26). Then,

$$\begin{aligned} &\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0, \\ &\quad \Longrightarrow u_n \to u \text{ in } H^1(\Omega) \text{ (as before via the Kadec-Klee property)}, \\ &\quad \Longrightarrow \hat{\varphi}_+ \text{ satisfies the C-condition.} \end{aligned}$$

Similarly, we show that $\hat{\varphi}_-$ satisfies the C-condition. □

Proposition 5 *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$ hold, then $u \equiv 0$ is a local minimizer of the functionals $\hat{\varphi}_\pm$.*

Proof Given $r > 2$ and $\epsilon > 0$ and using hypothesis H(f)(iv), we see that we can find $c_2 = c_2(r, \epsilon) > 0$ such that

$$F(z, x) \leq \frac{1}{2}(\theta(z) + \epsilon)x^2 + c_1|x|^r \quad \text{for a.a. } z \in \Omega, \quad \text{all } x \in \mathbb{R}. \tag{27}$$

For all $u \in H^1(\Omega)$, we have

$$\begin{aligned} \hat{\varphi}_+(u) &= \frac{1}{2}\gamma(u) + \frac{\mu}{2}\|u\|^2 - \int_\Omega \hat{F}_+(z, u)\mathrm{d}z \\ &\geq \frac{1}{2}\gamma(u) + \frac{\mu}{2}\|u\|^2 - \frac{1}{2}\int_\Omega \theta(z)(u^+)^2\mathrm{d}z - \frac{\epsilon}{2}\|u^+\|^2 \\ &- c_3\|u\|^r \\ &\text{for some } c_3 > 0 \text{ (see (27) and (8))} \\ &\geq \frac{1}{2}\gamma(u^-) + \frac{\mu}{2}\|u^-\|_2^2 + \frac{1}{2}\left[\gamma(u^+) - \int_\Omega \theta(z)(u^+)^2\mathrm{d}z\right] \\ &- \frac{\epsilon}{2}\|u^+\|^2 - c_3\|u\|^r \\ &\geq \frac{c_0}{2}\|u^-\|^2 + \frac{c_4 - \epsilon}{2}\|u^+\|^2 - c_3\|u\|^r \quad \textit{for some } c_4 > 0 \end{aligned}$$

(see (2) and Lemma 2)

Choosing $\epsilon \in (0, c_4)$, we infer that

$$\hat{\varphi}_+(u) \geq c_5\|u\|^2 - c_3\|u\|^r \quad \text{for some } c_5 > 0. \tag{28}$$

Since $r > 2$, from (28) we see that by choosing $\rho \in (0, 1)$ small we have

$$\hat{\varphi}_{+}(u) > 0 \quad \text{for all } u \in H^1(\Omega) \text{ with } 0 < \|u\| \le \rho,$$
$$\Longrightarrow \; u = 0 \text{ is a (strict) local minimizer of } \hat{\varphi}_{+}.$$

Similarly for the functional $\hat{\varphi}_{-}$. □

Recall that $\hat{u}_1 \in D_+$ (see Sect. 2) and that $m \neq 1$. So, using hypothesis H(f)(i) we have:

Proposition 6 *If hypotheses $H(f)$, $H(\beta)$, $H(g)$ hold, then $\hat{\varphi}_{\pm}(t\hat{u}_1) \to -\infty$ as $t \to \pm\infty$.*

Now we are ready to produce the two constant sign smooth solutions.

Proposition 7 *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$ hold, then problem (1) has two nontrivial solutions of constant sign*

$$u_0 \in D_+ \text{ and } v_0 \in -D_+$$

Proof Propositions 4, 5 and 6 permit the use of the mountain pass theorem (Theorem 1) for the functional $\hat{\varphi}_{+}$. So, we can find $u_0 \in H^1(\Omega)$ such that

$$u_0 \in K_{\hat{\varphi}_{+}}, u_0 \neq 0.$$

We have

$$\begin{aligned} &\hat{\varphi}'_{+}(u_0) = 0 \text{ in } H^1(\Omega)^*, \\ \Longrightarrow \; &\langle A(u_0), h\rangle + \int_\Omega (\xi(z) + \mu) u_0 h \mathrm{d}z + \int_{\partial\Omega} \beta(z) u_0 h \mathrm{d}\sigma = \int_\Omega \hat{f}_{+}(z, u_0) h \mathrm{d}z \\ &\text{for all } h \in H^1(\Omega). \end{aligned} \tag{29}$$

In (29), we choose $h = -u_0^- \in H^1(\Omega)$ and obtain

$$\begin{aligned} &\gamma(u_0^-) + \mu\|u_0^-\|_2^2 = 0 \text{ (see (8))}, \\ \Longrightarrow \; &c_0\|u_0^-\|^2 \le 0 \text{ (see (2))}, \\ \Longrightarrow \; &u_0 \ge 0, u_0 \neq 0. \end{aligned}$$

Then, from (8) and (29), we have

$$\begin{aligned} &\langle A(u_0), h\rangle + \int_\Omega \xi(z) u_0 h \mathrm{d}z + \int_{\partial\Omega} \beta(z) u_0 h \mathrm{d}\sigma = \int_\Omega f(z, u_0) h \mathrm{d}z \\ &\text{for all } h \in H^1(\Omega), \\ \Longrightarrow \; &-\Delta u_0(z) + \xi(z) u_0(z) = f(z, u_0(z)) \quad \text{for a.a. } z \in \Omega, \\ &\frac{\partial u_0}{\partial n} + \beta(z) u_0 = 0 \text{ on } \partial\Omega \\ &\text{(see Papageorgiou and Radulescu [15])} \end{aligned} \tag{30}$$

Hypotheses $H(f)$ imply that

$$|f(z,x)| \leq c_6|x| \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R} \text{ and some } c_6 > 0. \tag{31}$$

We set

$$g(z) := \begin{cases} \frac{f(z,u_0(z))}{u_0(z)} & \text{if } u_0(z) \neq 0 \\ 0 & \text{if } u_0(z) = 0. \end{cases}$$

Evidently $g \in L^{\infty}(\Omega)$ (see (31)). From (30), we have

$$\begin{cases} -\Delta u_0(z) = (g(z) - \xi(z))u_0(z) & \text{for a.a. } z \in \Omega, \\ \frac{\partial u_0}{\partial n} + \beta(z)u_0 = 0 \text{ on } \partial\Omega. \end{cases} \tag{32}$$

Note that $g - \xi \in L^s(\Omega)$ (see hypothesis $H(\xi)$). Then, from (32) and Lemma 5.1 of Wang [21], we have that

$$u_0 \in L^{\infty}(\Omega)$$

Then, from the Calderon–Zygmund estimates (see Lemma 5.2 of Wang [21]), we have

$$\begin{aligned} & u_0 \in W^{2,s}(\Omega), \\ & \implies u_0 \in C^{1,\alpha}(\overline{\Omega}) \text{ with } \alpha = 1 - \frac{N}{s} > 0 \\ & \text{(by the Sobolev embedding theorem)} \end{aligned}$$

Let $\rho = \|u_0\|_{\infty}$ and let $\hat{\xi}_{\rho} > 0$ as postulated by hypothesis H(f)(v). From (30), we have

$$\begin{aligned} & -\Delta u_0(z) + (\xi(z) + \hat{\xi}_{\rho})u_0(z) = f(z,u_0(z)) + \hat{\xi}_{\rho}u_0(z) \geq 0 \quad \text{for a.a. } z \in \Omega, \\ & \implies \Delta u_0(z) \leq (\|\xi^+\|_{\infty} + \hat{\xi}_{\rho})u_0(z) \quad \text{for a.a. } z \in \Omega, \\ & \implies u_0 \in D_+ \text{ (by the strong maximum principle).} \end{aligned}$$

Similarly working with the functional $\hat{\varphi}_-$, we obtain another constant sign solution $v_0 \in -D_+$ which is a critical point of $\hat{\varphi}_-$ of mountain pass type. □

Let $\varphi : H^1(\Omega) \to \mathbb{R}$ be the energy (Euler) functional for problem (1) defined by

$$\varphi(u) = \frac{1}{2}\gamma(u) - \int_{\Omega} F(z,u)\mathrm{d}z \quad \text{for all } u \in H^1(\Omega).$$

We have $\varphi \in C^2(H^1(\Omega))$. We will compute the critical groups of φ at the two constant sign solutions $u_0 \in D_+$, $v_0 \in -D_+$.

Proposition 8 *If hypothesis $H(\xi)$, $H(\beta)$, $H(f)$ hold, then $C_k(\varphi,u_0) = C_k(\varphi,v_0) = \delta_{k,1}\mathbb{Z}$ for all $k \in \mathbb{N}_0$.*

Proof From the proof of Proposition 7, we know that

$$u_0 \in D_+ \text{ is a critical point of } \hat{\varphi}_+ \text{ of mountain pass type,}$$
$$v_0 \in -D_+ \text{ is a critical point of } \hat{\varphi}_- \text{ of mountain pass type.}$$

Hence, we have

$$C_1(\hat{\varphi}_+, u_0) \neq 0 \, , C_1(\hat{\varphi}_-, v_0) \neq 0 \tag{33}$$

(see Motreanu et al. [11], Corollary 6.81, p. 168).

Note that

$$\hat{\varphi}_+\Big|_{C_+} = \varphi\Big|_{C_+} \text{ and } \hat{\varphi}_-\Big|_{-C_+} = \varphi\Big|_{-C_+} \text{ (see (8)).}$$

Since $u_0 \in D_+$ and $v_0 \in -D_+$, we have that

$$C_k(\varphi\Big|_{C^1(\overline{\Omega})}, u_0) = C_k(\hat{\varphi}_+\Big|_{C^1(\overline{\Omega})}, u_0) \quad \text{for all } k \in \mathbb{N}_0, \tag{34}$$

$$C_k(\varphi\Big|_{C^1(\overline{\Omega})}, v_0) = C_k(\hat{\varphi}_-\Big|_{C^1(\overline{\Omega})}, v_0) \quad \text{for all } k \in \mathbb{N}_0. \tag{35}$$

But from Palais [12] (see also Chang [4] (p. 14)), we know that

$$C_k(\varphi\Big|_{C^1(\overline{\Omega})}, u_0) = C_k(\varphi, u_0) \quad \text{for all } k \in \mathbb{N}_0,$$
$$C_k(\varphi\Big|_{C^1(\overline{\Omega})}, v_0) = C_k(\varphi, v_0) \quad \text{for all } k \in \mathbb{N}_0,$$
$$C_k(\hat{\varphi}_+\Big|_{C^1(\overline{\Omega})}, u_0) = C_k(\hat{\varphi}_+, u_0) \quad \text{for all } k \in \mathbb{N}_0,$$
$$C_k(\hat{\varphi}_-\Big|_{C^1(\overline{\Omega})}, v_0) = C_k(\hat{\varphi}_-, v_0) \quad \text{for all } k \in \mathbb{N}_0.$$

Combining these facts with (33), (34), (35), we obtain

$$C_1(\varphi, u_0) \neq 0 \text{ and } C_1(\varphi, v_0) \neq 0. \tag{36}$$

Since $\varphi \in C^2(H^1(\Omega))$, from (36) and Corollary 6.102, p. 177 of Motreanu et al. [11], we conclude that

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

□

4 Three Nontrivial Solutions

In this section, we produce a third nontrivial smooth solution, using the reduction method.

So, let

$$Y = \bigoplus_{i=1}^{m} E(\hat{\lambda}_i) \text{ and } V = Y^{\perp} = \overline{\bigoplus_{i \geq m+1} E(\hat{\lambda}_i)}$$

We have the following orthogonal direct sum decomposition

$$H^1(\Omega) = Y \oplus V$$

So, every $u \in H^1(\Omega)$ admits a unique sum decomposition of the form

$$u = y + v \text{ with } y \in Y, v \in V. \tag{37}$$

Proposition 9 *If hypothesis $H(\xi)$, $H(\beta)$, $H(f)$ hold, then there exists a continuous map $\hat{\tau} : Y \to V$ such that*

$$\varphi(y + \hat{\tau}(y)) = \inf[\varphi(y + v) : v \in V]$$

Proof We fix $y \in Y$ and consider the C^2-functional $\varphi_y : H^1(\Omega) \to \mathbb{R}$ defined by

$$\varphi_y(u) = \varphi(y + u) \quad \text{for all } u \in H^1(\Omega).$$

Let $i_V : V \to H^1(\Omega)$ be the embedding map and set

$$\tilde{\varphi}_y = \varphi_y \circ i_V : V \to \mathbb{R}.$$

From the chain rule, we have

$$\tilde{\varphi}_y' = \rho_{V^*} \varphi_y', \tag{38}$$

with ρ_{V^*} being the orthogonal projection of $H^1(\Omega)^*$ onto V^*. In what follows, by $\langle \cdot, \cdot \rangle_V$ we denote the duality brackets for the pair (V^*, V). For $v_1, v_2 \in V$, we have

$$\begin{aligned}
&\langle \tilde{\varphi}_y'(v_1) - \tilde{\varphi}_y'(v_2), v_1 - v_2 \rangle_V \\
&\quad = \langle \varphi_y'(v_1) - \varphi_y'(v_2), v_1 - v_2 \rangle \text{ (see (38))} \\
&\quad = \gamma(v_1 - v_2) - \int_\Omega (f(z, y + v_1) - f(z, y + v_2))(v_1 - v_2) \mathrm{d}z \\
&\quad \geq \gamma(v_1 - v_2) - \int_\Omega \eta(z)(v_1 - v_2)^2 \mathrm{d}z \text{ (see (7))} \\
&\quad \geq c_7 \|v_1 - v_2\|^2 \quad \text{for some } c_7 > 0 \text{ (see Lemma 2).}
\end{aligned} \tag{39}$$

Therefore,

$$\tilde{\varphi}_y' \text{ is strongly monotone hence } \tilde{\varphi}_y \text{ is strictly convex.} \tag{40}$$

For every $v \in V$, we have

$$\begin{aligned}\langle \tilde{\varphi}'_y(v), v\rangle_V &= \langle \varphi'_y(v), v\rangle \text{ (see 38))}\\ &= \langle \varphi'_y(v) - \varphi'_y(0), v\rangle + \langle \varphi'_y(0), v\rangle \\ &\geq c_7\|v\|^2 - c_8\|v\| \quad \text{for some } c_8 > 0 \text{ (see 39))}, \quad (41)\\ &\Longrightarrow \tilde{\varphi}'_y \text{ is coercive.} \quad (42)\end{aligned}$$

Since $\tilde{\varphi}'_y$ is continuous and strongly monotone (see (40)), it is maximal monotone. This fact and (42) imply that $\tilde{\varphi}'_y$ is surjective (see Gasinski and Papageorgiou [6] (p. 319)). So, we can find $v_0 \in V$ such that

$$\tilde{\varphi}'_y(v_0) = 0 \text{ in } V^*. \quad (43)$$

The strong monotonicity of $\tilde{\varphi}'_y$ implies that $v_0 \in V$ is unique. In fact, $v_0 \in V$ is the unique minimizer of the stricly convex function $\tilde{\varphi}_y$ (see (40)).

Let $\hat{\tau} : Y \to V$ be the map which to each $y \in Y$ assigns this unique minimizer. Then, from (38) and (43) we have

$$\rho_{V^*}\varphi'(y + \hat{\tau}(y)) = 0 \text{ and } \varphi(y + \hat{\tau}(y)) = \inf[\varphi(y + v) : v \in V]. \quad (44)$$

Next we show the continuity of the map $\hat{\tau} : Y \to V$. To this end, let $y_n \to y$ in Y. From (43), we have

$$\begin{aligned}&0 = \langle \tilde{\varphi}'_{y_n}(\hat{\tau}(y_n)), \hat{\tau}(y_n)\rangle_V \geq c_7\|\hat{\tau}(y_n)\|^2 - c_8\|\hat{\tau}(y_n)\| \quad \text{for all } n \in \mathbb{N} \text{ (see (41))},\\ &\Longrightarrow \{\hat{\tau}(y_n)\}_{n\geq 1} \subseteq V \text{ is bounded.}\end{aligned}$$

So, by passing to a suitable subsequence if necessary, we may assume that

$$\hat{\tau}(y_n) \xrightarrow{w} \hat{v} \in V \text{ in } H^1(\Omega) \quad (45)$$

Note that the Sobolev embedding theorem and the compactness of the trace map imply that the functional φ is sequentially weakly lower semicontinuous. So, using (45), we have

$$\begin{aligned}&\varphi(y + \hat{v}) \leq \liminf_{n\to\infty} \varphi(y_n + \hat{\tau}(y_n)) \leq \liminf_{n\to\infty} \varphi(y_n + v) \quad \text{for all } v \in V,\\ &\Longrightarrow \varphi(y + \hat{v}) \leq \varphi(y + v) \quad \text{for all } v \in V,\\ &\Longrightarrow \hat{v} = \hat{\tau}(y) \text{ (see ((44)))}.\end{aligned}$$

From the Uryshon criterion for convergence of sequences (see Gasinski and Papageorgiou [7] (p. 33)), for the original sequence we have

$$\hat{\tau}(y_n) \xrightarrow{w} \hat{\tau}(y) \text{ in } H^1(\Omega) \text{ and } \hat{\tau}(y_n) \to \hat{\tau}(y) \text{ in } L^{\frac{2s}{s-1}}(\Omega) \text{ and in } L^2(\partial\Omega). \quad (46)$$

We have

$$\begin{aligned} 0 &= \langle \tilde{\varphi}'_{y_n}(\hat{\tau}(y_n)), h\rangle_V \\ &= \langle \varphi'(y_n + \hat{\tau}(y_n)), h\rangle \quad \text{for all } h \in V, \text{ all } n \in \mathbb{N}. \end{aligned}$$

Choosing $h = \hat{\tau}(y_n) - \hat{\tau}(y) \in V$ and using (46) and (31), we obtain

$$\begin{aligned} &\lim_{n\to\infty} \langle A(y_n + \hat{\tau}(y_n)), \hat{\tau}(y_n) - \hat{\tau}(y)\rangle = 0, \\ &\implies \|D(y_n + \hat{\tau}(y_n))\|_2^2 \to \|D(y + \hat{\tau}(y))\|_2^2, \\ &\implies y_n + \hat{\tau}(y_n) \to y + \hat{\tau}(y) \text{ in } H^1(\Omega) \\ &\text{(by the Kadec–Klee property, see (46))}, \\ &\implies \hat{\tau}(y_n) \to \hat{\tau}(y) \text{ in } H^1(\Omega), \\ &\implies \hat{\tau} \text{ is continuous.} \end{aligned}$$

□

We define

$$\hat{\varphi}(y) = \varphi(y + \hat{\tau}(y)) \quad \text{for all } y \in Y. \tag{47}$$

Proposition 10 *If hypotheses $H(\xi), H(\beta), H(f)$ hold, then $\hat{\varphi} \in C^1(Y, \mathbb{R})$ and $\hat{\varphi}'(y) = \rho_{Y*}\varphi'(y + \hat{\tau}(y))$ for all $y \in Y$.*

Proof Let $y, w \in Y$ and $t > 0$. From the definition of $\hat{\varphi}$ (see (47))

$$\begin{aligned} &\frac{1}{t}\Big[\hat{\varphi}(y + tw) - \hat{\varphi}(y)\Big] \\ &\le \frac{1}{t}\Big[\varphi(y + tw + \hat{\tau}(y)) - \varphi(y + \hat{\tau}(y))\Big] \text{ (see Proposition 9)}, \\ &\implies \limsup_{t\to 0^+} \frac{1}{t}\Big[\hat{\varphi}(y + tw) - \hat{\varphi}(y)\Big] \le \langle \varphi'(y + \hat{\tau}(y)), w\rangle. \end{aligned} \tag{48}$$

Also, we have

$$\begin{aligned} &\frac{1}{t}\Big[\hat{\varphi}(y + tw) - \hat{\varphi}(y)\Big] \\ &\ge \frac{1}{t}\Big[\varphi(y + tw + \hat{\tau}(y + tw)) - \varphi(y + \hat{\tau}(y + tw))\Big] \text{ (see Proposition 9)}, \\ &\implies \liminf_{t\to 0^+} \frac{1}{t}\Big[\hat{\varphi}(y + tw) - \hat{\varphi}(y)\Big] \ge \langle \varphi'(y + \hat{\tau}(y)), w\rangle \\ &\text{(recall that } \varphi \in C^2(H^1(\Omega), \mathbb{R}) \text{ and } \hat{\tau}(\cdot) \text{ is continuous).} \end{aligned} \tag{49}$$

For (48),(49), we infer that

$$\lim_{t\to 0^+} \frac{1}{t}\Big[\hat{\varphi}(y + tw) - \hat{\varphi}(y)\Big] = \langle \varphi'(y + \hat{\tau}(y)), w\rangle \quad \text{for } w \in Y. \tag{50}$$

In a similar fashion, we show that

$$\lim_{t\to 0^-} \frac{1}{t}\Big[\hat{\varphi}(y+tw)-\hat{\varphi}(y)\Big] = \langle \varphi'(y+\hat{\tau}(y)), w\rangle \quad \text{for } w\in Y. \tag{51}$$

From (50) and (51), we conclude that

$$\hat{\varphi}\in C^1(Y,\mathbb{R}) \text{ and } \hat{\varphi}'(y) = \rho_{Y^*}\hat{\varphi}'(y+\hat{\tau}(y)) \quad \text{for all } y\in Y.$$

□

The next proposition is an easy observation about the critical points of $\hat{\varphi}$.

Proposition 11 *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$ hold, then $y\in K_{\hat{\varphi}}$ if and only if $y+\hat{\tau}(y)\in K_{\varphi}$.*

Proof $\Longrightarrow$ Let $y\in K_{\hat{\varphi}}$. We have

$$\begin{aligned} &\hat{\varphi}'(y)=0,\\ &\Longrightarrow \rho_{Y^*}\varphi'(y+\hat{\tau}(y))=0 \text{ (see Proposition 10)},\\ &\Longrightarrow \varphi'(y+\hat{\tau}(y))\in V^* \text{ (recall that } H^1(\Omega)^*=Y^*\oplus V^*). \end{aligned} \tag{52}$$

Also, from (44) we have

$$\begin{aligned} &\rho_{V^*}\varphi'(y+\hat{\tau}(y))=0,\\ &\Longrightarrow \varphi'(y+\hat{\tau}(y))\in Y^*. \end{aligned} \tag{53}$$

Recall that $Y^*\cap V^*=\{0\}$. So, from (52) and (53) it follows that

$$\begin{aligned} &\varphi'(y+\hat{\tau}(y))=0,\\ &\Longrightarrow y+\hat{\tau}(y)\in K_{\varphi}. \end{aligned}$$

$\Longleftarrow$ Suppose that $y+\hat{\tau}(y)\in K_{\varphi}$. Then

$$\begin{aligned} &\varphi'(y+\hat{\tau}(y))=0,\\ &\Rightarrow \rho_{Y^*}\varphi'(y+\hat{\tau}(y))=0,\\ &\Rightarrow \hat{\varphi}'(y)=0 \text{ (see Proposition 10)},\\ &\Rightarrow y\in K_{\hat{\varphi}}. \end{aligned}$$

□

Proposition 12 *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$, then the functional $\hat{\varphi}$ is anticoercive (that is, $\hat{\varphi}(y)\to-\infty$ if $\|y\|\to\infty$).*

Proof We argue by contradiction. So suppose that the proposition is not true. Then, we can find $\{y_n\}_{n\geq 1}\subseteq Y$ such that

$$\|y_n\|\to\infty \text{ and } \hat{\varphi}(y_n)\geq -c_9 \text{ for some } c_9>0,\ \text{all } n\in\mathbb{N}. \tag{54}$$

From (54) and Proposition 9, we have

$$-c_9 \leq \hat{\varphi}(y_n) \leq \varphi(y_n) = \frac{1}{2}\gamma(y_n) - \int_\Omega F(z, y_n)\mathrm{d}z \quad \text{for all } n \in \mathbb{N}. \tag{55}$$

Let $w_n = \frac{y_n}{\|y_n\|}$, $n \in \mathbb{N}$. Then, $\|w_n\| = 1$, $w_n \in Y$ for all $n \in \mathbb{N}$. Since Y is finite dimensional, we may assume that

$$w_n \to w \text{ in } H^1(\Omega), \ \|w\| = 1 \tag{56}$$

From (55), we have

$$-\frac{c_9}{\|y_n\|^2} \leq \frac{1}{2}\gamma(w_n) - \int_\Omega \frac{F(z, y_n)}{\|y_n\|^2}\mathrm{d}z \text{ for all } n \in \mathbb{N}. \tag{57}$$

From (31), we have

$$\begin{aligned}
&|F(z, y_n(z))| \leq c_{10} y_n(z)^2 \quad \text{for a.a. } z \in \Omega, \text{ all } n \in \mathbb{N}, \text{ some } c_{10} > 0, \\
&\Longrightarrow \frac{|F(z, y_n(z))|}{\|y_n\|^2} \leq c_{10} w_n(z)^2 \text{ for a.a. } z \in \Omega, \text{ all } n \in \mathbb{N}, \\
&\Longrightarrow \left\{\frac{F(\cdot, y_n(\cdot))}{\|y_n\|^2}\right\}_{n\geq 1} \subseteq L^1(\Omega) \text{ is uniformly integrable (see (56))}.
\end{aligned}$$

So, from the Dunford–Pettis theorem and (19), we have

$$\frac{F(\cdot, y_n(\cdot))}{\|y_n\|^2} \xrightarrow{w} \frac{1}{2}\eta_0(z)w^2 \text{ in } L^1(\Omega),\ \hat{\lambda}_m \leq \eta_0(z) \leq \eta(z) \quad \text{for a.a. } z \in \Omega \tag{58}$$

(see Aizicovici et al. [1], proof of Proposition 30). So, if in (57) we pass to the limit as $n \to \infty$ and we use (54), (56), (58), then

$$0 \leq \frac{1}{2}\gamma(w) - \frac{1}{2}\int_\Omega \eta_0(z)w^2\mathrm{d}z. \tag{59}$$

First suppose that $\eta_0 \neq \hat{\lambda}_m$ (see (58)). Then, from (59) and Lemma 2, we have

$$\begin{aligned}
&0 \leq -\hat{c}_1\|w\|^2, \\
&\Longrightarrow w = 0, \text{ a contradiction to (56)}.
\end{aligned}$$

Next assume that $\eta_0(z) = \hat{\lambda}_m$ for a.a. $z \in \Omega$. From (59) and since $w \in Y$, we have

$$\begin{aligned}
&\gamma(w) = \hat{\lambda}_m\|w\|_2^2 \text{ (see (5))}, \\
&\Longrightarrow w \in E(\hat{\lambda}_m)\backslash\{0\} \text{ (see (56))}, \\
&\Longrightarrow w(z) \neq 0 \quad \text{for a.a. } z \in \Omega \text{ (by the unique continuation property)}, \\
&\Longrightarrow |y_n(z)| \to +\infty \quad \text{for a.a. } z \in \Omega \text{ as } n \to \infty.
\end{aligned} \tag{60}$$

Hypothesis H(f)(iii) implies that given any $r > 0$, we can find $M_4 = M_4(r) > 0$ such that

$$f(z,x)x - 2F(z,x) \leq -r \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M_4.$$

Then, for a.a. $z \in \Omega$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{F(z,x)}{x^2}\right) &= \frac{f(z,x)x^2 - 2xF(z,x)}{x^4} \\
&= \frac{f(z,x)x - 2F(z,x)}{|x|^2 x} \\
&= \begin{cases} \leq -\frac{r}{x^3} & \text{if } x \geq M_4 \\ \geq \frac{r}{|x|^3} & \text{if } x \leq -M_4, \end{cases} \\
\Longrightarrow \frac{F(z,v)}{v^2} - \frac{F(z,u)}{u^2} &\leq \frac{r}{2}\left[\frac{1}{v^2} - \frac{1}{u^2}\right] \quad \text{for a.a. } z \in \Omega, \text{ all } |v| \geq |u| \geq M_4.
\end{aligned} \tag{61}$$

From (19), it follows that

$$\frac{\hat{\lambda}_m}{2} \leq \liminf_{x \to \pm\infty} \frac{F(z,x)}{x^2} \leq \limsup_{x \to \pm\infty} \frac{F(z,x)}{x^2} \leq \frac{1}{2}\eta(z) \text{ uniformly for a.a. } z \in \Omega \tag{62}$$

So, if in (61) we let $|v| \to \infty$ and use (62), then

$$\begin{aligned}
&\hat{\lambda}_m u^2 - 2F(z,u) \leq -r \quad \text{for a.a. } z \in \Omega, \text{ all } |v| \geq M_4, \\
&\Longrightarrow \hat{\lambda}_m u^2 - 2F(z,u) \to -\infty \text{ uniformly for a.a. } z \in \Omega, \text{ as } u \to \pm\infty.
\end{aligned} \tag{63}$$

From (55), we have

$$\begin{aligned}
&-2c_9 \leq \gamma(y_n) - \int_\Omega 2F(z,y_n)\mathrm{d}z \quad \text{for all } n \in \mathbb{N}, \\
&\Longrightarrow -2c_9 \leq \int_\Omega \left[\hat{\lambda}_m y_n^2 - 2F(z,y_n)\right]\mathrm{d}z \quad \text{for all } n \in \mathbb{N} \\
&(\text{recall } y_n \in Y \text{ and see } (5)).
\end{aligned} \tag{64}$$

From (60), (63) and Fatou's lemma, we have that

$$\int_\Omega \left[\hat{\lambda}_m y_n^2 - 2F(z,y_n)\right]\mathrm{d}z \to -\infty \text{ as } n \to \infty. \tag{65}$$

Comparing (64) and (65), we reach an contradiction.

So, (54) cannot occur and we conclude that $\hat{\varphi}$ is anticoercive. □

Now we can produce a third nontrivial smooth solution distinct from $u_0 \in D_+$ and from $v_0 \in -D_+$.

Proposition 13 *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$ hold, then problem* (1) *has a third solution $y_0 \in C^1(\overline{\Omega})$.*

Proof Let $u_0 \in D_+$ and $v_0 \in -D_+$ be the two constant sign solutions from Proposition 7. From Proposition 8, we know that

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{66}$$

Since Y is finite dimensional and $\hat{\varphi}$ is anticoercive (see Proposition 12), we can find $\hat{y} \in Y$ such that

$$\hat{\varphi}(\hat{y}) = \max\left[\hat{\varphi}(y) : y \in Y\right].$$

Then from Motreanu-Motreanu-Papageorgiou [11] (Example 6.45(b), p. 153), we have

$$C_k(\hat{\varphi}, \hat{y}) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0, \text{ with } d_m = dimY \geq 2. \tag{67}$$

From Lemma 2.3 of Liu [10], we have

$$\begin{aligned} &C_k(\hat{\varphi}, \hat{y}) = C_k(\varphi, \hat{y} + \hat{\tau}(\hat{y})) \quad \text{for all } k \in \mathbb{N}_0, \\ &\implies C_k(\varphi, y_0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0, \text{ with } y_0 = \hat{y} + \hat{\tau}(\hat{y}) \text{ (see (67))}. \end{aligned} \tag{68}$$

Reasoning as in the proof of Proposition 5, we show that

$$\begin{aligned} &u = 0 \text{ is a local minimizer of } \varphi, \\ &\implies C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \end{aligned} \tag{69}$$

Form (66), (68), (69) and Proposition 11, we have

$$\begin{aligned} &y_0 \in K_\varphi,\ y_0 \notin \{0, u_0, v_0\}, \\ &\implies y_0 \text{ is a third nontrivial solution of (1)} \end{aligned}$$

As before (see the proof of Proposition 7), using the regularity theory of Wang [21] we have that $y_0 \in C^1(\overline{\Omega})$. □

So we can state the following multiplicity theorem for problem (1).

Theorem 14 *If hypotheses $H(\xi)$, $H(\beta)$, $H(f)$ hold, then problem* (1) *has at least three nontrivial smooth solutions*

$$u_0 \in D_+, v_0 \in -D_+ \text{ and } y_0 \in C^1(\overline{\Omega})$$

Remark It is an open question if we can have that y_0 is nodal (sign changing).

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