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Unique Solutions for Fractional q-Difference Boundary Value Problems Via a Fixed Point Method

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Abstract In this paper, by applying the cone theory in ordered Banach spaces associated with the characters of increasing $\varphi-(h,e)$-concave operators, we investigate the existence and uniqueness of nontrivial solutions for a nonlinear fractional q-difference equation boundary value problem. The main results show that we can construct an iterative scheme approximating the unique nontrivial solution. Relying on an example, we show the efficiency and applicability of the main result.

Keywords Fractional q-difference equation · Existence and uniqueness · Nontrivial solution · $\varphi-(h,e)$-Concave operator

Mathematics Subject Classification 34B18 · 33D05 · 39A13

1 Introduction

Fractional differential equation has been of great interest, and fruits from research into it emerge continuously. For example, see [1–25,28] and references therein. Recently, there have already appeared many extensive studies on fractional q-difference equation boundary value problems because of its popularity and importance applications in many different professional situations, see, for instance, [1,7,8,13,16,19] and the reference therein. The fractional q-difference equation boundary value problem has

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been proved to be a very rich and promising field. Since Al-Salam [7] and Agarwal [1] proposed the fractional q-calculus, related research interests including the extension of the theory for fractional q-difference calculus, the existence and multiplicity of solutions for fractional q-difference equation boundary value problems and the coupled system of boundary value problems for fractional q-difference equations have been considered by using various methods, see [2–6,8,10–18] and the reference therein. For example, Ahmad et al. [3] obtained the existence and uniqueness of the solutions for impulsive fractional q-integro-difference equations with separated boundary conditions via fixed point theorems due to Krasnoselskii and O'Regan. In [24], Yang considered the coupled integral boundary value problem for systems of fractional q-difference equations by using the nonlinear alternative of Leray–Schauder-type and Krasnoselskii's fixed point theorems. Furthermore, Miao and Liang [18] obtained the uniqueness of positive solutions for boundary value problem of fractional q-difference equation with p-Laplacian operator by using a fixed point theorem in partially ordered set. In [11], by applying a fixed point theorem in cones, Ferreira investigated the existence of positive solutions to the nonlinear fractional q-difference equation boundary value problem

$$\begin{cases} D_q^\alpha y(x) = -f(x, y(x)), & x \in (0, 1), \\ y(0) = D_q y(0), & D_q y(1) = \beta \geq 0. \end{cases}$$

From the literature, there are still few papers that reported the uniqueness of solutions for nonlinear fractional q-difference equations. Different from the works [11,12,18], we study the existence and uniqueness of nontrivial solutions for a nonlinear fractional q-difference equation boundary value problem:

$$\begin{cases} D_q^\alpha u(t) + f(t, u(t)) = b, & t \in (0, 1), \\ u(0) = D_q u(0), & D_q u(1) = \beta \geq 0, \end{cases} \tag{1.1}$$

where $0 < q < 1, 2 < \alpha \leq 3, b > 0$ is a constant, $f : [0, 1] \times (-\infty, +\infty) \to (-\infty, +\infty)$ is continuous, and D_q^α denotes the Riemann–Liouville-type fractional q-derivative of order α. Our method is a new fixed point theorem of $\varphi - (h, e)$-concave operator. The related operator is a new concept defined on a new set $P_{h,e}$ (see [27]). Based on this new method, we will prove the uniqueness and existence of nontrivial solutions for problem (1.1). Moreover, we can construct an iterative scheme approximating the unique nontrivial solution. In the last section, we give an example to show the efficiency and applicability of the main result.

2 Preliminaries and Previous Results

In this section, we present some necessary definitions and lemmas of fractional q-calculus. These details can be found in the recent literature [1,7,8,13,19].

Let q be a real number with $0 < q < 1$, the definition of q-analog for $\alpha \in \mathbf{R}$ is

$$[\alpha]_q = \frac{1-q^\alpha}{1-q}.$$

The q-gamma function Γ_q is defined by

$$\Gamma_q(\alpha) = \frac{(1-q)^{(\alpha-1)}}{(1-q)^{\alpha-1}}, \quad \alpha \in \mathbf{R}\backslash\{0, -1, -2, \ldots\},$$

and we have

$$\Gamma_q(\alpha+1) = [\alpha]_q \Gamma_q(\alpha), \quad \Gamma_q(1) = 1, \quad q \in (0, 1).$$

The q-analog of the power function $(a-b)^{(\alpha)}$ is defined by

$$(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{1-(b/a)q^n}{1-(b/a)q^{n+\alpha}}, \quad a, b, \alpha \in \mathbf{R}.$$

Then $(a-b)^{(0)} = 1,\ a^{(\alpha)} = a^\alpha$, when $b = 0$ and $(a(t-s))^{(\alpha)} = a^\alpha (t-s)^{(\alpha)}$.
The q-integral of a function f in the interval $[0, b]$ is defined by

$$(I_q f)(t) = \int_0^t f(s) d_q s = (1-q) \sum_{n=0}^{\infty} f(tq^n) tq^n, \quad t \in [0, b].$$

For any $t, s > 0$, the q-beta function is defined by

$$B_q(t, s) = \int_0^1 u^{(t-1)} (1-qu)^{(s-1)} d_q u, \quad q \in (0, 1),$$

where the expression of q-beta function in terms of the q-gamma function is

$$B_q(t, s) = \frac{\Gamma_q(t) \Gamma_q(s)}{\Gamma_q(t+s)}.$$

Definition 2.1 (See [8]) Let $\alpha \geq 0$ and f be a function defined on $[0, 1]$. The fractional q-integral of Riemann–Liouville type is $(I_q^0 f)(t) = f(t)$ and

$$(I_q^\alpha f)(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f(s) d_q s, \quad \alpha > 0.$$

Note that $(I_q^\alpha f)(t) = (I_q f)(t)$ when $\alpha = 1$.
It is clear that f is q-integral on the closed interval $[a, b]$ and one has

$$\int_a^b f(t) d_q t = \int_a^c f(t) d_q t + \int_c^b f(t) d_q t, \quad c \in [a, b].$$

Definition 2.2 (See [8]) The fractional q-derivative of Riemann–Liouville type of order $\alpha \geq 0$ is defined by

$$\left(D_q^\alpha f\right)(t) = \left(D_q^{\lceil\alpha\rceil} I_q^{\lceil\alpha\rceil-\alpha} f\right)(t), \quad \alpha > 0, \quad t \in [0, 1],$$

where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α.
Evidently, $(D_q^\alpha f)(t) = D_q f(t)$, when $\alpha = 1$. Further analysis showed that

$$\left(I_q^\alpha D_q^p f\right)(t) = \left(D_q^p I_q^\alpha f\right)(t) - \sum_{n=0}^{p-1} \frac{t^{\alpha-p+n}}{\Gamma_q(\alpha - p + n + 1)} \left(D_q^n f\right)(0), \quad p \in \mathbf{N}.$$

Lemma 2.1 (See [25]) *If f, g are continuous on the interval $[0, s]$ and $f(t) \leq g(t)$, for all $t \in [0, s]$, the following properties are valid*

(i) $\int_0^s f(t)d_q t \leq \int_0^s g(t)d_q t$. *Further, if $\alpha > 1$, we have $I_q^\alpha f(s) \leq I_q^\alpha g(s)$, for $t \in [0, s]$;*
(ii) $\left|\int_0^s f(t)d_q t\right| \leq \int_0^s |f(t)|d_q t$, *for $t \in [0, s]$.*

Remark 2.1 (See [16]) If $\alpha > 0$ and $a \leq b \leq t$, then $(t-a)^{(\alpha)} \geq (t-b)^{(\alpha)}$.

Lemma 2.2 (See [19]) *For $\lambda \in (-1, \infty)$ and $\alpha \geq 0$, we have the following equality*

$$I_q^\alpha (t-a)^{(\lambda)} = \frac{\Gamma_q(\lambda + 1)}{\Gamma_q(\alpha + \lambda + 1)} (t-a)^{(\alpha+\lambda)}, \quad 0 < a < t.$$

Particularly, for $\lambda = 0$, $a = 0$, we have $I_q^\alpha(1)(t) = \frac{t^\alpha}{\Gamma_q(\alpha+1)}$. In conclusion, we obtain

$$\int_0^t (t - qs)^{(\alpha-1)} d_q s = \Gamma_q(\alpha) I_q^\alpha(1)(t) = \frac{1}{[\alpha]_q} t^\alpha. \tag{2.1}$$

Lemma 2.3 *Suppose that $f(t)$ is a continuous function on $[0, 1]$ and there exists $t_0 \in (0, 1)$ such that $f(t_0) \neq 0$. If $f(t) \geq 0$, then we have*

$$\int_0^1 f(t)d_q t > 0, \quad t \in [0, 1],$$

where

$$\int_0^1 f(t)d_q t = (1 - q) \sum_{n=0}^{\infty} q^n f(q^n), \quad 0 < q < 1.$$

Proof Since $f(t) \geq 0$ and $f(t_0) \neq 0$, there exists $n_0 \in \mathbf{N}$ such that $t_0 = q^{n_0}$, then we have

$$f(q^{n_0}) q^{n_0} > 0, \quad 0 < q < 1.$$

This implies

$$(1-q)\sum_{n=0}^{\infty} q^n f(q^n) \geq (1-q) f(q^{n_0}) q^{n_0} = (1-q) f(t_0) t_0 > 0.$$

Hence, we have $\int_0^1 f(t) d_q t > 0$. The proof is completed. □

Next, we list some notations and properties that are already known in the literature [25–28] and reference therein.

Let $(E, \|\cdot\|)$ be a real Banach space which is partially ordered by a cone $P \subset E$. For any $x, y \in E$, the notation $x \sim y$ means that there exist $\mu > 0$ and $\nu > 0$ such that $\mu x \leq y \leq \nu x$. Given $h > \theta (i.e., h \geq \theta$ and $h \neq \theta)$, we have the set $P_h = \{x \in E \mid x \sim h\}$. Clearly, $P_h \subset P$. Take $e \in P$ with $\theta \leq e \leq h$, we define $P_{h,e} = \{x \in E | x + e \in P_h\}$, that is

$$P_{h,e} = \{x \in E | \text{ there exist } \mu = \mu(h, e, x) > 0, \nu = \nu(h, e, x) > 0 \text{ such that } \mu h \leq x + e \leq \nu h\}.$$

Further, the following definition of $\varphi - (h, e)$-concave operators and fixed point theorem in partially ordered sets is fundamental to the proof of our main results.

Definition 2.3 (See [27]) Let $A : P_{h,e} \to E$ be a given operator. For any $x \in P_{h,e}$ and $\lambda \in (0, 1)$, there exists $\varphi(\lambda) > \lambda$ such that

$$A(\lambda x + (\lambda - 1)e) \geq \varphi(\lambda) Ax + (\varphi(\lambda) - 1)e. \tag{2.2}$$

Then A is called a $\varphi - (h, e)$-concave operator.

Lemma 2.4 (See [27]) *Let P be normal and A be an increasing $\varphi - (h, e)$-concave operator with $Ah \in P_{h,e}$. Then A has a unique fixed point x^* in $P_{h,e}$. Moreover, for any $w_0 \in P_{h,e}$, making the sequence $w_n = Aw_{n-1}, n = 1, 2, \ldots,$ then $\|w_n - x^*\| \to 0$ as $n \to \infty$.*

Lemma 2.5 (See [26]) *Let P be normal and A be an increasing $\varphi - (h, \theta)$-concave operator with $Ah \in P_h$. Then A has a unique fixed point x^* in P_h. Moreover, for any $w_0 \in P_h$, making the sequence $w_n = Aw_{n-1}, n = 1, 2, \ldots,$ we get $\|w_n - x^*\| \to 0$ as $n \to \infty$.*

3 Main Results

Lemma 3.1 (See [11]) *Assume $2 < \alpha \leq 3$ and $g \in C[0, 1]$, and then the following boundary value problem*

$$\begin{cases} D_q^{\alpha} u(t) + g(t) = 0, & 0 < t < 1, \\ u(0) = D_q u(0) = 0, & D_q u(1) = \beta \geq 0 \end{cases}$$

has a unique solution

$$u(t) = \frac{\beta}{[\alpha-1]_q} t^{\alpha-1} + \int_0^1 G(t, qs) g(s) d_q s,$$

where

$$G(t,s) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} (1-s)^{(\alpha-2)} t^{\alpha-1} - (t-s)^{(\alpha-1)}, & 0 \le s \le t \le 1, \\ (1-s)^{(\alpha-2)} t^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases}$$

Lemma 3.2 (See [11]) *The function* $G(t, qs)$ *has the following properties:*

(i) $G(t, qs) \ge 0, G(t, qs) \le G(1, qs), 0 \le t, s \le 1$;
(ii) $G(t, qs) \ge t^{\alpha-1} G(1, qs), 0 \le t, s \le 1$;
(iii) $G(t, qs) \le \frac{1}{\Gamma_q(\alpha)} (1-qs)^{(\alpha-2)} t^{\alpha-1} \le \frac{1}{\Gamma_q(\alpha)}, 0 \le t, s \le 1.$

In our considerations, we work in the Banach space $X = C[0, 1]$ *endowed with the norm* $\|u\| = \sup\{|u(t)| : t \in [0, 1]\}$. *Define the cone* $P = \{x \in C[0, 1] | x(t) \ge 0,\ t \in [0, 1]\}$, *the standard cone. Set*

$$h(t) = H t^{\alpha-1} \text{ with } H \ge \frac{b}{(1-q^{\alpha-1})^2 \Gamma_q(\alpha-1)}, \tag{3.1}$$

$$e(t) = \frac{b(1-q)^2}{\Gamma_q(\alpha-1)} \left[\frac{t^{\alpha-1}}{(1-q^{\alpha-1})^2} - \frac{t^{\alpha}}{(1-q^{\alpha})(1-q^{\alpha-1})} \right], \quad t \in [0, 1]. \tag{3.2}$$

Theorem 3.1 *Boundary value problem* (1.1) *has a unique nontrivial solution* u^* *in* $P_{h,e}$, *if the following conditions are satisfied:*

(H_1) $f : [0, 1] \times [-\hat{e}, +\infty) \to (-\infty, +\infty)$ *is continuous and increasing with respect to the second variable, where* $\hat{e} = \max\{e(t) : t \in [0, 1]\}$;
(H_2) *for any* $\lambda \in (0, 1)$, *there exists* $1 \ge \varphi(\lambda) > \lambda$ *such that*

$$f(t, \lambda x + (\lambda - 1) y) \ge \varphi(\lambda) f(t, x), \forall\, t \in [0, 1], x \in (-\infty, +\infty), y \in [0, \hat{e}];$$

(H_3) $f(t, 0) \ge 0$ *with* $f(t, 0) \not\equiv 0$ *for* $t \in [0, 1]$.

Further, we can construct a sequence

$$\begin{aligned} v_n(t) = {} & \frac{\beta}{[\alpha-1]_q} t^{\alpha-1} + \int_0^1 G(t, qs) f(s, v_{n-1}(s)) d_q s \\ & - \frac{b(1-q)^2}{(1-q^{\alpha-1})^2 \Gamma_q(\alpha-1)} t^{\alpha-1} \\ & + \frac{b(1-q)^2}{(1-q^{\alpha})(1-q^{\alpha-1}) \Gamma_q(\alpha-1)} t^{\alpha}, n = 1, 2, \ldots, \end{aligned}$$

for any given $v_0 \in P_{h,e}$, *and one has* $v_n(t) \to u^*(t)$ *as* $n \to \infty$.

Proof For $t \in [0, 1]$, one can see that

$$
\begin{aligned}
e(t) &= \frac{b(1-q)^2}{\Gamma_q(\alpha-1)}\left[\frac{t^{\alpha-1}}{(1-q^{\alpha-1})^2} - \frac{t^{\alpha}}{(1-q^{\alpha})(1-q^{\alpha-1})}\right] \\
&= \frac{b(1-q)^2 t^{\alpha-1}}{\Gamma_q(\alpha-1)} \cdot \frac{1-q^{\alpha}-t(1-q^{\alpha-1})}{(1-q^{\alpha})(1-q^{\alpha-1})^2} \\
&\geq \frac{b(1-q)^2 t^{\alpha-1}}{\Gamma_q(\alpha-1)} \cdot \frac{q^{\alpha-1}-q^{\alpha}}{(1-q^{\alpha})(1-q^{\alpha-1})^2} \geq 0;
\end{aligned}
$$

thus, we have $e \in P$. Moreover, for $t \in [0, 1]$,

$$
\begin{aligned}
e(t) &= \frac{b(1-q)^2}{(1-q^{\alpha-1})^2\Gamma_q(\alpha-1)}t^{\alpha-1} - \frac{b(1-q)^2}{(1-q^{\alpha})(1-q^{\alpha-1})\Gamma_q(\alpha-1)}t^{\alpha} \\
&\leq \frac{b}{(1-q^{\alpha-1})^2\Gamma_q(\alpha-1)}t^{\alpha-1} \leq Ht^{\alpha-1} = h(t).
\end{aligned}
$$

Hence, $0 \leq e(t) \leq h(t)$. Further, $P_{h,e} = \{u \in C[0, 1] | u + e \in P_h\}$.

In view of Lemmas 2.2 and 3.1, the solution $u(t)$ of problem (1.1) can be expressed as

$$
\begin{aligned}
u(t) &= \frac{\beta}{[\alpha-1]_q}t^{\alpha-1} + \int_0^1 G(t, qs) f(s, u(s)) d_q s - b\int_0^1 G(t, qs) d_q s \\
&= \frac{\beta}{[\alpha-1]_q}t^{\alpha-1} + \int_0^1 G(t, qs) f(s, u(s)) d_q s \\
&\quad - \frac{b(1-q)^2}{\Gamma_q(\alpha-1)}\left[\frac{t^{\alpha-1}}{(1-q^{\alpha-1})^2} - \frac{t^{\alpha}}{(1-q^{\alpha})(1-q^{\alpha-1})}\right] \\
&= \frac{\beta}{[\alpha-1]_q}t^{\alpha-1} + \int_0^1 G(t, qs) f(s, u(s)) d_q s - e(t).
\end{aligned}
$$

After that, for any $u \in P_{h,e}$, we consider the following operator

$$
Au(t) = \frac{\beta}{[\alpha-1]_q}t^{\alpha-1} + \int_0^1 G(t, qs) f(s, u(s)) d_q s - e(t), \quad t \in [0, 1].
$$

So $u(t)$ is the solution of problem (1.1) if and only if u is the fixed point of A.

Now we first show that $A : P_{h,e} \to E$ is a $\varphi - (h, e)$-concave operator. By the condition (H_2), for any $\lambda \in (0, 1)$, $u \in P_{h,e}$, we have

$$\begin{aligned}
&A\left(\lambda u + (\lambda - 1)e\right)(t) \\
&= \frac{\beta}{[\alpha-1]_q} t^{\alpha-1} + \int_0^1 G(t, qs) f\left(s, \lambda u(s) + (\lambda - 1)e(s)\right) d_q s - e(t) \\
&\geq \varphi(\lambda) \frac{\beta}{[\alpha-1]_q} t^{\alpha-1} + \varphi(\lambda) \int_0^1 G(t, qs) f(s, u(s)) d_q s - e(t) \\
&= \varphi(\lambda) \left[\frac{\beta}{[\alpha-1]_q} t^{\alpha-1} + \int_0^1 G(t, qs) f(s, u(s)) d_q s - e(t) \right] + [\varphi(\lambda) - 1] e(t) \\
&= \varphi(\lambda) Au(t) + [\varphi(\lambda) - 1] e(t).
\end{aligned}$$

Hence, we obtain

$$A(\lambda u + (\lambda - 1)e) \geq \varphi(\lambda) Au + [\varphi(\lambda) - 1]e, \lambda \in (0, 1), u \in P_{h,e}.$$

According to Definition 2.3, we know that A is a $\varphi - (h, e)$-concave operator.

In the following, we prove that $A : P_{h,e} \to E$ is increasing. For $u \in P_{h,e}$, we have $u + e \in P_h$, so there exists $\mu > 0$ such that $u(t) + e(t) \geq \mu h(t)$; thus, we obtain

$$u(t) \geq \mu h(t) - e(t) \geq -e(t) \geq -\hat{e}, \ t \in [0, 1].$$

From (H_1), we know $A : P_{h,e} \to E$ is increasing.

As follows, we prove that $Ah \in P_{h,e}$, so we need to prove $Ah + e \in P_h$. By Lemma 3.2 and (H_1), we have

$$\begin{aligned}
Ah(t) + e(t) &= \frac{\beta}{[\alpha-1]_q} t^{\alpha-1} + \int_0^1 G(t, qs) f(s, h(s)) d_q s \\
&= \frac{\beta}{[\alpha-1]_q} t^{\alpha-1} + \int_0^1 G(t, qs) f\left(s, Hs^{\alpha-1}\right) d_q s \\
&\leq \frac{\beta(1-q)}{1-q^{\alpha-1}} t^{\alpha-1} + \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-2)} t^{\alpha-1} f(s, H) d_q s \\
&\leq \frac{\beta}{1-q^{\alpha-1}} t^{\alpha-1} + \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-2)} f(s, H) d_q s \cdot t^{\alpha-1} \\
&= \frac{\beta}{(1-q^{\alpha-1})H} \cdot h(t) + \frac{1}{H\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-2)} f(s, H) d_q s \cdot h(t)
\end{aligned}$$

and

$$\begin{aligned} Ah(t) + e(t) \\ &= \frac{\beta}{[\alpha-1]_q} t^{\alpha-1} + \int_0^1 G(t, qs) f\left(s, Hs^{\alpha-1}\right) d_q s \\ &\geq \frac{\beta(1-q)}{1-q^\alpha} t^{\alpha-1} + \frac{1}{\Gamma_q(\alpha)} \int_0^1 \left[(1-qs)^{(\alpha-2)} - (1-qs)^{(\alpha-1)}\right] t^{\alpha-1} f(s, 0) d_q s \\ &= \frac{\beta(1-q)}{1-q^\alpha} t^{\alpha-1} + \frac{1}{\Gamma_q(\alpha)} \int_0^1 \left[(1-qs)^{(\alpha-2)} - (1-qs)^{(\alpha-1)}\right] f(s, 0) d_q s \cdot t^{\alpha-1} \\ &= \frac{\beta(1-q)}{(1-q^\alpha)H} \cdot h(t) + \frac{1}{H\Gamma_q(\alpha)} \int_0^1 \left[(1-qs)^{(\alpha-2)} - (1-qs)^{(\alpha-1)}\right] \\ &\quad f(s, 0) d_q s \cdot h(t). \end{aligned}$$

Let

$$\mu = \frac{\beta}{(1-q^{\alpha-1})H} + \frac{1}{H\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-2)} f(s, H) d_q s,$$

$$\nu = \frac{\beta(1-q)}{(1-q^\alpha)H} + \frac{1}{H\Gamma_q(\alpha)} \int_0^1 \left[(1-qs)^{(\alpha-2)} - (1-qs)^{(\alpha-1)}\right] f(s, 0) d_q s.$$

Since $\beta \geq 0$, $\Gamma_q(\alpha) > 0$ and $\frac{1-q}{1-q^\alpha} < \frac{1}{1-q^{\alpha-1}}$, and from (H_1), (H_3),

$$\int_0^1 (1-qs)^{(\alpha-2)} f(s, H) d_q s \geq \int_0^1 \left[(1-qs)^{(\alpha-2)} - (1-qs)^{(\alpha-1)}\right] f(s, 0) d_q s > 0,$$

hence, we have $\mu \geq \nu > 0$. It follows that $Ah + e \in P_h$.
In the last, by using Lemma 2.4, the operator A has a unique fixed point u^* in $P_{h,e}$, and

$$u^*(t) = \frac{\beta}{[\alpha-1]_q} t^{\alpha-1} + \int_0^1 G(t, qs) f(s, u^*(s)) d_q s - e(t), \quad t \in [0, 1].$$

Evidently, $u^*(t) \not\equiv 0$, $t \in [0, 1]$. Therefore, $u^*(t)$ is a nontrivial solution. Moreover, for any $v_0 \in P_{h,e}$, the sequence $v_n = Av_{n-1}$, $n = 1, 2, \ldots$, satisfies $v_n \to u^*$ as $n \to \infty$. That is,

$$\begin{aligned} v_n(t) = \frac{\beta}{[\alpha-1]_q} t^{\alpha-1} &+ \int_0^1 G(t, qs) f(s, v_{n-1}(s)) d_q s \\ &- \frac{b(1-q)^2}{(1-q^{\alpha-1})^2 \Gamma_q(\alpha-1)} t^{\alpha-1} \\ &+ \frac{b(1-q)^2}{(1-q^\alpha)(1-q^{\alpha-1}) \Gamma_q(\alpha-1)} t^\alpha, n = 1, 2, \ldots, \end{aligned}$$

and $v_n(t) \to u^*(t)$ as $n \to \infty$. □

Next we consider a special case of problem (1.1) with $\beta = 0$, that is, the following boundary value problem

$$\begin{cases} D_q^\alpha u(t) + f(t, u(t)) = b, & t \in (0, 1), \\ u(0) = D_q u(0) = D_q u(1) = 0. \end{cases} \tag{3.3}$$

Theorem 3.2 *Assume* (H_1), (H_3) *hold and satisfy*

$(H_2)'$ *for any* $\lambda \in (0, 1)$, *and there is* $\varphi(\lambda) > \lambda$ *such that*

$$f(t, \lambda x + (\lambda - 1)y) \geq \varphi(\lambda) f(t, x), \forall\, t \in [0, 1], x \in (-\infty, +\infty), y \in [0, \hat{e}];$$

then, problem (3.3) *has a unique nontrivial solution* u^* *in* $P_{h,e}$, *where* h, e *are given as in* (3.1), (3.2). *Further, making a monotone iterative sequence*

$$v_n(t) = \int_0^1 G(t, qs) f(s, v_{n-1}(s)) d_q s - \frac{b(1-q)^2}{(1-q^{\alpha-1})^2 \Gamma_q(\alpha-1)} t^{\alpha-1}$$
$$+ \frac{b(1-q)^2}{(1-q^\alpha)(1-q^{\alpha-1})\Gamma_q(\alpha-1)} t^\alpha, n = 1, 2, \ldots .$$

For any $v_0 \in P_{h,e}$, *we have* $v_n(t) \to u^*(t)$ *as* $n \to \infty$.

Proof In view of Lemma 3.1, the solution $u(t)$ of problem (3.3) can be expressed as

$$\begin{aligned} u(t) &= \int_0^1 G(t, qs) f(s, u(s)) d_q s - b \int_0^1 G(t, qs) d_q s \\ &= \int_0^1 G(t, qs) f(s, u(s)) d_q s \\ &\quad - \frac{b(1-q)^2}{\Gamma_q(\alpha-1)} \left[\frac{t^{\alpha-1}}{(1-q^{\alpha-1})^2} - \frac{t^\alpha}{(1-q^\alpha)(1-q^{\alpha-1})} \right] \\ &= \int_0^1 G(t, qs) f(s, u(s)) d_q s - e(t). \end{aligned}$$

For any $u \in P_{h,e}$, we consider the following operator

$$Au(t) = \int_0^1 G(t, qs) f(s, u(s)) d_q s - e(t), \quad t \in [0, 1].$$

So $u(t)$ is the solution of problem (3.3) if and only if u is the fixed point of A.

Firstly, we show that $A : P_{h,e} \to E$ is a $\varphi-(h,e)$-concave operator. By the condition $(H_2)'$, for any $u \in P_{h,e}$, $\lambda \in (0,1)$, we obtain

$$\begin{aligned} A(\lambda u + (\lambda-1)e)(t) &= \int_0^1 G(t,qs) f(s, \lambda u(s) + (\lambda-1)e(s)) d_q s - e(t) \\ &\geq \varphi(\lambda) \int_0^1 G(t,qs) f(s,u(s)) d_q s - e(t) \\ &= \varphi(\lambda) \left[\int_0^1 G(t,qs) f(s,u(s)) d_q s - e(t) \right] + [\varphi(\lambda)-1]e(t) \\ &= \varphi(\lambda) Au(t) + [\varphi(\lambda)-1]e(t). \end{aligned}$$

Hence, we have

$$A(\lambda u + (\lambda-1)e) \geq \varphi(\lambda) Au + [\varphi(\lambda)-1]e, \quad \lambda \in (0,1).$$

Therefore, A is $\varphi-(h,e)$-concave operator. In addition, it is known that $A : P_{h,e} \to E$ is increasing.

Next, we prove that $Ah \in P_{h,e}$, so we only prove $Ah + e \in P_h$. From Lemma 3.2 and (H_1), (H_3),

$$\begin{aligned} Ah(t) + e(t) &= \int_0^1 G(t,qs) f\left(s, Hs^{\alpha-1}\right) d_q s \\ &\leq \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-2)} f(s,H) d_q s \cdot t^{\alpha-1} \\ &= \frac{1}{H\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-2)} f(s,H) d_q s \cdot h(t) \\ &= \mu' \cdot h(t), \end{aligned}$$

where

$$\mu' = \frac{1}{H\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-2)} f(s,H) d_q s.$$

In addition,

$$\begin{aligned} Ah(t) + e(t) &= \int_0^1 G(t,qs) f\left(s, Hs^{\alpha-1}\right) d_q s \\ &\geq \frac{1}{\Gamma_q(\alpha)} \int_0^1 \left[(1-qs)^{(\alpha-2)} - (1-qs)^{(\alpha-1)}\right] f(s,0) d_q s \cdot t^{\alpha-1} \\ &= \frac{1}{H\Gamma_q(\alpha)} \int_0^1 \left[(1-qs)^{(\alpha-2)} - (1-qs)^{(\alpha-1)}\right] f(s,0) d_q s \cdot h(t) \\ &= \nu' \cdot h(t), \end{aligned}$$

where

$$\nu' = \frac{1}{H\Gamma_q(\alpha)} \int_0^1 \left[(1-qs)^{(\alpha-2)} - (1-qs)^{(\alpha-1)}\right] f(s,0) d_q s.$$

Since $\Gamma_q(\alpha) > 0$, $H > 0$ and from (H_1), (H_3), we have $\mu' \geq \nu' > 0$, and this implies that $Ah + e \in P_h$.

In the last, by using Lemma 2.4, the operator A has a unique fixed point u^* in $P_{h,e}$, so

$$u^*(t) = \int_0^1 G(t,qs) f(s,u^*(s)) d_q s - e(t), \quad t \in [0,1].$$

Obviously, $u^*(t) \not\equiv 0$, $t \in [0,1]$. Therefore, $u^*(t)$ is a nontrivial solution. Moreover, for any $v_0 \in P_{h,e}$, the sequence $v_n = Av_{n-1}, n = 1, 2, \ldots$, satisfies $v_n \to u^*$ as $n \to \infty$. That is,

$$v_n(t) = \int_0^1 G(t,qs) f(s,v_{n-1}(s)) d_q s - \frac{b(1-q)^2}{(1-q^{\alpha-1})^2 \Gamma_q(\alpha-1)} t^{\alpha-1} + \frac{b(1-q)^2}{(1-q^{\alpha})(1-q^{\alpha-1})\Gamma_q(\alpha-1)} t^{\alpha}, n = 1, 2, \ldots,$$

and $v_n(t) \to u^*(t)$ as $n \to \infty$. □

If $b = 0$, we can get the uniqueness of positive solutions for problems (1.1) and (3.3) by using Lemma 2.5. The proofs are similar to Theorems 3.1 and 3.2.

Corollary 3.3 *Assume that*

(H_4) $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ *is continuous with* $f(t,0) \not\equiv 0$;

(H_5) *for each* $t \in [0,1]$, $f(t,x)$ *is increasing with respect to the second variable;*

(H_6) *for any* $\lambda \in (0,1)$, *there exists* $\varphi(\lambda) \in (\lambda, 1)$ *such that*

$$f(t,\lambda x) \geq \varphi(\lambda) f(t,x), \quad \forall t \in [0,1], x \in [0,+\infty).$$

Then the following fractional q-difference equation boundary value problem

$$\begin{cases} D_q^{\alpha} u(t) + f(t,u(t)) = 0, & t \in (0,1), \\ u(0) = D_q u(0), & D_q u(1) = \beta \geq 0, \end{cases}$$

where $0 < q < 1, 2 < \alpha \leq 3$, *has a unique positive solution* u^* *in* P_h, *where* $h(t) = t^{\alpha-1}$, $t \in [0,1]$. *Moreover, for any initial value* $v_0 \in P_h$, *constructing the sequence*

$$v_n(t) = \frac{\beta}{[\alpha-1]_q} t^{\alpha-1} + \int_0^1 G(t,qs) f(s,v_{n-1}(s)) d_q s, \ n = 1, 2, \ldots,$$

one has $v_n(t) \to u^*(t)$ *as* $n \to \infty$.

Corollary 3.4 *Assume that* (H_4)–(H_6) *hold. Then the following fractional q-difference equation boundary value problem*

$$\begin{cases} D_q^{\alpha} u(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = D_q u(0) = D_q u(1) = 0, \end{cases}$$

where $0 < q < 1, 2 < \alpha \leq 3$, *has a unique positive solution* u^* *in* P_h, *where* $h(t) = t^{\alpha-1},\ t \in [0,1]$. *Moreover, for any initial value* $v_0 \in P_h$, *constructing the sequence*

$$v_n(t) = \int_0^1 G(t, qs) f(s, v_{n-1}(s)) d_q s,\ n = 1, 2, \ldots,$$

one has $v_n(t) \to u^*(t)$ *as* $n \to \infty$.

4 An Example

In this section, we give an example to illustrate our main results.

Example 4.1 Consider the following boundary value problem:

$$\begin{cases} D_q^{\frac{5}{2}} u(t) + f(t, u(t)) = 1,\ t \in (0,1), \\ u(0) = D_q u(0) = 0, \qquad D_q u(1) = 2, \end{cases} \tag{4.1}$$

where

$$f(t,u) = \left\{ \left(u + \frac{18 + 8\sqrt{2}}{49 \Gamma_q\left(\frac{3}{2}\right)} \right) t^{\frac{3}{2}} - \left(\frac{45570 - 6076\sqrt{2}}{47089} u + \frac{68 + 24\sqrt{2}}{217 \Gamma_q\left(\frac{3}{2}\right)} \right) t^{\frac{5}{2}} \right\}^{\frac{1}{5}},$$
$$t \in [0,1],$$

$\alpha = \frac{5}{2},\ q = \frac{1}{2},\ b = 1,\ \beta = 2$. Set

$$e(t) = \frac{18 + 8\sqrt{2}}{49 \Gamma_q\left(\frac{3}{2}\right)} t^{\frac{3}{2}} - \frac{68 + 24\sqrt{2}}{217 \Gamma_q\left(\frac{3}{2}\right)} t^{\frac{5}{2}}, \quad h(t) = H t^{\frac{3}{2}} \text{ with } H \geq \frac{72 + 32\sqrt{2}}{49 \Gamma_q\left(\frac{3}{2}\right)},$$
$$t \in [0,1].$$

Then we have

$$e(t) = \frac{t^{\frac{3}{2}}}{\Gamma_q\left(\frac{3}{2}\right)} \left(\frac{18 + 8\sqrt{2}}{49} - \frac{68 + 24\sqrt{2}}{217} t \right) \geq \frac{82 + 80\sqrt{2}}{1519 \Gamma_q\left(\frac{3}{2}\right)} t^{\frac{3}{2}} \geq 0$$

and

$$e(t) \leq \frac{72 + 32\sqrt{2}}{49 \Gamma_q\left(\frac{3}{2}\right)} t^{\frac{3}{2}} \leq H t^{\frac{3}{2}} = h(t);$$

further, $\hat{e}(t)=\frac{18+8\sqrt{2}}{49\Gamma_q\left(\frac{3}{2}\right)}$ for $t\in[0,1]$. One can see that $f:[0,1]\times\left[-\frac{18+8\sqrt{2}}{49\Gamma_q\left(\frac{3}{2}\right)},+\infty\right)\to(-\infty,+\infty)$ is continuous and increasing with respect to the second variable,

$$f(t,0)=\left\{\frac{18+8\sqrt{2}}{49\Gamma_q\left(\frac{3}{2}\right)}t^{\frac{3}{2}}-\frac{68+24\sqrt{2}}{217\Gamma_q\left(\frac{3}{2}\right)}t^{\frac{5}{2}}\right\}^{\frac{1}{5}}=[e(t)]^{\frac{1}{5}}\geq 0$$

with $f(t,0)\not\equiv 0$, and then the conditions (H_1), (H_3) are satisfied.

Clearly, it can be represented as $f(t,u(t))=\left[\frac{e(t)}{\hat{e}}u(t)+e(t)\right]^{\frac{1}{5}}$ and $\frac{e(t)}{\hat{e}}=t^{\frac{3}{2}}-\frac{45570-6076\sqrt{2}}{47089}t^{\frac{5}{2}}<1$, for $t\in[0,1]$. In view of the Remark 4 in [27], we have

$$f(t,\lambda x+(\lambda-1)y)\geq\varphi(\lambda)f(t,x),\ \text{for } \lambda\in(0,1), x\in(-\infty,+\infty), y\in[0,\hat{e}];$$

here $\varphi(\lambda)=\lambda^{\frac{1}{5}}>\lambda$, $\lambda\in(0,1)$, and the condition (H_2) is satisfied. Thence, Theorem 3.1 implies that problem (4.1) has a unique solution $u^*\in C[0,1]$. For $u\in P_{h,e}$, we construct a sequence

$$\begin{aligned}v_n(t)=&\int_0^1 G(t,qs)\left\{\left(v_{n-1}(s)+\frac{18+8\sqrt{2}}{49\Gamma_q(\frac{3}{2})}\right)s^{\frac{3}{2}}\right.\\&\left.-\left(\frac{45570-6076\sqrt{2}}{47089}v_{n-1}(s)+\frac{68+24\sqrt{2}}{217\Gamma_q(\frac{3}{2})}\right)s^{\frac{5}{2}}\right\}^{\frac{1}{5}}d_qs\\&+\left(\frac{8+\sqrt{2}}{7}-\frac{18+8\sqrt{2}}{49\Gamma_q(\frac{3}{2})}\right)t^{\frac{3}{2}}+\frac{68+24\sqrt{2}}{217\Gamma_q(\frac{3}{2})}t^{\frac{5}{2}},\ n=1,2,\ldots,\end{aligned}$$

and we have $\lim_{n\to\infty}v_n=u^*$.

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