Bull. Malays. Math. Sci. Soc. (2019) 42:1329–1348
https://doi.org/10.1007/s40840-017-0551-3

BULLETIN of the MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Existence and Multiplicity of Solutions for Semilinear Elliptic Systems with Periodic Potential

Guofeng Che[1] · Haibo Chen[1] · Liu Yang[2]

Received: 28 February 2017 / Revised: 28 August 2017 / Published online: 20 September 2017

Abstract In this paper, we consider the following semilinear elliptic systems:

$$\begin{cases} -\Delta u + V(x)u = F_u(x, u, v), & \text{in } \mathbb{R}^N, \\ -\Delta v + V(x)v = F_v(x, u, v), & \text{in } \mathbb{R}^N, \end{cases}$$

where $V : \mathbb{R}^N \to \mathbb{R}$, $F_u(x, u, v)$ and $F_v(x, u, v)$ are periodic in x. We assume that 0 is a right boundary point of the essential spectrum of $-\triangle + V$. Under appropriate assumptions on $F_u(x, u, v)$ and $F_v(x, u, v)$, we prove the above system has a ground-state solution by using the Nehari-type technique in a strongly indefinite setting. Furthermore, the existence of infinitely many geometrically distinct solutions is obtained via variational methods. Recent results from the literature are improved and extended.

Communicated by Norhashidah Hj. Mohd. Ali.

This work is partially supported by National Natural Science Foundation of China 11671403, by the Fundamental Research Funds for the Central Universities of Central South University 2017zzts058, and by the Mathematics and Interdisciplinary Sciences Project of CSU.

✉ Haibo Chen
math_chb@163.com

Guofeng Che
cheguofeng222@163.com

1 School of Mathematics and Statistics, Central South University, Changsha 410083, Hunan, People's Republic of China

2 School of Mathematics and Statistics, Hengyang Normal University, Hengyang 421002, People's Republic of China

Keywords Semilinear elliptic systems · Strongly indefinite functional · Ground state · Nehari–Pankov manifold · Variational methods

Mathematics Subject Classification 35B38 · 35J20

1 Introduction

In this paper, we consider the existence and multiplicity of nontrivial solutions to the following semilinear elliptic systems:

$$\begin{cases} -\Delta u + V(x)u = F_u(x, u, v), & \text{in } \mathbb{R}^N, \\ -\Delta v + V(x)v = F_v(x, u, v), & \text{in } \mathbb{R}^N. \end{cases} \tag{1.1}$$

When Ω is a bounded domain of $\mathbb{R}^N$, the problem

$$\begin{cases} -\Delta u = \lambda\big(a(x)u + b(x)v\big) + F_u(x, u, v), & \text{in } \Omega, \\ -\Delta v = \lambda\big(b(x)u + c(x)v\big) + F_v(x, u, v), & \text{in } \Omega, \\ u(x) = v(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

which is related to reaction–diffusion systems that appear in chemical and biological phenomena, including the steady- and unsteady-state situation (see [1]), has been extensively investigated in recent years. For the results on existence, multiple solutions, and positive solutions to problem (1.2), we refer the reader to [1–5] and the references therein. In [3], Qu and Tang obtained the existence and multiplicity of weak solutions for problem (1.2) by using the Ekeland variational principle together with variational methods, and some new existence theorems of weak solutions were obtained in Duan et al. [2]. Lots of work has been done when Ω is an unbounded domain of $\mathbb{R}^N$, and we refer the reader to [6–16] and the references therein.

Recall that the spectrum $\sigma(-\Delta + V)$ of $-\Delta + V$ is purely continuous and may contain gaps, i.e., open intervals free of spectrum (see [17]). In [18], Szulkin and Weth considered the following Schrödinger equation:

$$-\Delta u + V(x)u = f(x, u), \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

Assuming that $0 \notin \sigma(-\Delta + V)$, they proved that problem (1.3) possesses a ground-state solution, which is just a minimizer of the energy functional associated with problem (1.3) on the Nehari–Pankov manifold [19]. Later, Mederski [20] considered the ground-state solutions to the system of coupled Schrödinger equations as follows:

$$-\Delta u_i + V_i(x)u_i = \partial_{u_i} F(x, u), \ \text{ on } \mathbb{R}^N, \ \ i = 1, 2, \ldots, K, \tag{1.4}$$

where F and V_i are periodic in x, $0 \notin \sigma(-\Delta + V_i)$, $i = 1, 2, \ldots, K$. Moreover, they made use of a new linking-type result involving the Nehari–Pankov manifold and assumed that F satisfies the following conditions:

1. $f_i : \mathbb{R}^N \times \mathbb{R}^K \to \mathbb{R}$ is measurable, $\mathbb{Z}^N$-periodic in $x \in \mathbb{Z}^N$ and continuous in $u \in \mathbb{R}^K$ for $a.e.\ x \in \mathbb{R}^N$. Moreover, $f = (f_1, f_2, \ldots, f_K) = \partial_u F$, where $F : \mathbb{R}^N \times \mathbb{R}^K \to \mathbb{R}$ is differentiable with respect to the second variable $u \in \mathbb{R}^K$ and $F(x, 0) = 0$ for a.e. $x \in \mathbb{R}^N$.
2. There are $a > 0$ and $2 < p < 2^*$ such that

$$|f(x,u)| \le a\left(1 + |u|^{p-1}\right), \quad \text{for all } u \in \mathbb{R}^K \text{and a.e. } x \in \mathbb{R}^N.$$

3. $f(x, u) = o(u)$ uniformly with respect to x as $|u| \to 0$.
   In [21], Guo and Mederski considered the existence and nonexistence of ground-state solutions of system (1.4) with $K = 1$ and $V(x) = V_1(x) - \frac{\mu}{|x|^2}$, where $V_1 \in L^\infty(\mathbb{R}^N)$, V_1 is $\mathbb{Z}^N$-periodic in $x \in \mathbb{R}^N$, and $0 \notin \sigma(-\Delta + V)$. Moreover, they assumed that $f(x, u)$ satisfies (1)–(3) and (4)–(5) as follows:
4. $\frac{F(x,u)}{u^2} \to \infty$ uniformly in x as $|u| \to \infty$, where F is the primitive of f with respect to u, that is, $F(x, u) = \int_0^u f(x, s)\mathrm{d}s$.
5. $u \mapsto \frac{f(x,u)}{|u|}$ is nondecreasing on $(-\infty, 0)$ and $(0, +\infty)$.

When 0 is a right boundary point of the essential spectrum of $-\Delta + V$ and $f(x, u)$ is superlinear and subcritical, Mederski [22] obtained the existence of ground-state solutions and multiple solutions of system (1.3) with $u(x) \to 0$, as $|x| \to \infty$.

Inspired by the above facts, more precisely by [21–23], the aim of this paper is to study the existence and multiplicity of nontrivial solutions to problem (1.1) via variational methods. To the best of our knowledge, there have been few works concerning this case up to now.

We assume that $V(x)$ and $F(x, u, v)$ satisfy the following hypotheses:

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$, V is 1-periodic in x_i, $i = 1, 2, \ldots, N$, $0 \in \sigma(-\Delta + V)$, and there exists $\alpha > 0$ such that $(0, \alpha] \cap \sigma(-\Delta + V) = \emptyset$.

(F_1) $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2)$, $F_u(x, u, v)$ and $F_v(x, u, v)$ are measurable, 1-periodic in x_i, $i = 1, 2, \ldots, N$.

(F_2) There exist $c > 0$ and $2 < \gamma \le p < 2^*$ such that $|F_u(x, u, v)| \le c(|(u, v)|^{\gamma-1} + |(u, v)|^{p-1})$ and $|F_v(x, u, v)| \le c(|(u, v)|^{\gamma-1} + |(u, v)|^{p-1})$ for all $(u, v) \in \mathbb{R} \times \mathbb{R}$ and $x \in \mathbb{R}^N$, where $|(u, v)| = (u^2 + v^2)^{\frac{1}{2}}$.

(F_3) There exists $d > 0$ such that

$$F(x, u, v) \ge d|(u, v)|^\gamma, \quad \text{for } |(u, v)| \le 1,\ x \in \mathbb{R}^N.$$

(F_4) $\frac{F(x,u,v)}{|(u,v)|^2} \to \infty$ uniformly in $x \in \mathbb{R}^N$ as $|(u, v)| \to \infty$.

(F_5) If $F_u(x, u_2, v_2)u_1 > 0$ or $F_v(x, u_2, v_2)v_1 > 0$ for any (u_2, v_2), $(u_1, v_1) \in \mathbb{R}^2$, then we have

$$F(x, u_2, v_2) - F(x, u_1, v_1) \le \frac{(F_u(x, u_2, v_2)u_2)^2 - (F_u(x, u_2, v_2)u_1)^2}{2F_u(x, u_2, v_2)u_2} + \frac{(F_v(x, u_2, v_2)v_2)^2 - (F_v(x, u_2, v_2)v_1)^2}{2F_v(x, u_2, v_2)v_2}.$$

(F_6) $F_u(x,u,v)u \geq 0$ and $F_v(x,u,v)v \geq 0$ and $F_u(x,u,v)u + F_v(x,u,v)v \geq 2F(x,u,v)$ for any $(x,u,v) \in \mathbb{R}^N \times \mathbb{R}^2$.

(F_7) $F(x,-u,-v) = F(x,u,v)$ for any $(x,u,v) \in \mathbb{R}^N \times \mathbb{R}^2$.

Assumptions (V) and $(F_1)-(F_6)$ allow us to find a function space $E_{2,\gamma}$ (see Sect. 2) on which the energy functional associated with (1.1) is given by

$$J(u,v) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)|u|^2\right) \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 + V(x)|v|^2\right) \mathrm{d}x - \int_{\mathbb{R}^N} F(x,u,v)\mathrm{d}x, \tag{1.5}$$

and

$$\langle J'(u,v),(\varphi,\psi)\rangle = \int_{\mathbb{R}^N} \nabla u \nabla\varphi \mathrm{d}x + \int_{\mathbb{R}^N} V(x)u\varphi\mathrm{d}x - \int_{\mathbb{R}^N} F_u(x,u,v)\varphi\mathrm{d}x + \int_{\mathbb{R}^N} \nabla v\nabla\psi\mathrm{d}x + \int_{\mathbb{R}^N} V(x)v\psi\mathrm{d}x - \int_{\mathbb{R}^N} F_v(x,u,v)\psi\mathrm{d}x. \tag{1.6}$$

Now we state our main results.

Theorem 1.1 *Suppose that (V) and $(F_1)-(F_6)$ hold, then problem (1.1) has a ground-state solution $(u,v) \in \mathcal{N}$ such that $J(u,v) = \inf_{\mathcal{N}} J > 0$. Furthermore, $(u,v) \in \left(H^2_{loc}(\mathbb{R}^N) \times H^2_{loc}(\mathbb{R}^N)\right) \cap \left(L^t(\mathbb{R}^N) \times L^t(\mathbb{R}^N)\right)$ for $\gamma \leq t \leq 2^*$.*

Theorem 1.2 *Suppose that (V) and $(F_1)-(F_7)$ hold, then problem (1.1) has infinitely many geometrically distinct solutions which lie in $\left(H^2_{loc}(\mathbb{R}^N) \times H^2_{loc}(\mathbb{R}^N)\right) \cap \left(L^t(\mathbb{R}^N) \times L^t(\mathbb{R}^N)\right)$ for $\gamma \leq t \leq 2^*$.*

Notation Throughout this paper, we shall denote by $\|\cdot\|_r$ the L^r-norm and C various positive generic constants, which may vary from line to line. $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent. Also if we take a subsequence of a sequence $\{(u_n,v_n)\}$ we shall denote it again by $\{(u_n,v_n)\}$.

This paper is organized as follows: In Sect. 2, some preliminary results are presented. In Sect. 3, we introduce the Nehari–Pankov manifold $\mathcal{N} \subset E_{2,\gamma}$ on which we minimize J to find a ground state and we prove Theorem 1.1. Eventually, in Sect. 4, the multiplicity result is obtained and we prove Theorem 1.2.

2 Variational Setting and Preliminaries

In this section we outline the variational framework for problem (1.1) and give some preliminary lemmas.

It follows from condition (V) that there exists a decomposition of $H^1(\mathbb{R}^N) = X^+ \oplus X^-$ corresponding to the decomposition of the spectrum of $\sigma(S)$ into $\sigma(S)\cap[\alpha,+\infty)$ and $\sigma(S)\cap(-\infty,0]$, where $S = -\triangle + V$ with the domain $D(S) = H^2(\mathbb{R}^N)$. We introduce a new norm $\|\cdot\|_X$ on X^+ (resp. X^-) by setting

$$\|u^+\|_X^2 = \int_{\mathbb{R}^N} \left(|\nabla u^+|^2 + V(x)|u^+|^2\right) \mathrm{d}x$$

and

$$\|u^-\|_X^2 = -\int_{\mathbb{R}^N} \left(|\nabla u^-|^2 + V(x)|u^-|^2\right) \mathrm{d}x$$

for $u^+ \in X^+$ and $u^- \in X^-$. Then, $\|\cdot\|_X$ is equivalent to $\|\cdot\|_{H^1}$ on X^+ and is weaker than $\|\cdot\|_{H^1}$ on X^- (see [24])

$$\langle u, v\rangle_X = \int_{\mathbb{R}^N} \left(\nabla u^+ \nabla v^+ + V(x)u^+v^+\right) \mathrm{d}x - \int_{\mathbb{R}^N} \left(\nabla u^- \nabla v^- + V(x)u^-v^-\right) \mathrm{d}x$$

and a norm given by

$$\|u\|_X^2 = \|u^+\|_X^2 + \|u^-\|_X^2,$$

which is equivalent to the usual Sobolev norm in $H^1(\mathbb{R}^N)$, that is

$$\|u\|_{H^1} = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2)\mathrm{d}x\right)^{\frac{1}{2}}.$$

Therefore, X^+ and X^- are orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle_X$ as well.

As usual, for $1 \le p < +\infty$, we let

$$\|u\|_p = \left(\int_{\mathbb{R}^N} |u(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}, \quad u \in L^p\left(\mathbb{R}^N\right),$$

and

$$\|u\|_\infty = ess \sup_{x\in\mathbb{R}^N} |u(x)|, \quad u \in L^\infty\left(\mathbb{R}^N\right).$$

Then, $E = X \times X = E^+ \oplus E^-$, where $E^+ = X^+ \times X^+$, $E^- = X^- \times X^-$, is a Hilbert space with the following inner product

$$\langle(u, v), (\varphi, \psi)\rangle = \langle u, \varphi\rangle_X + \langle v, \psi\rangle_X \;\; (u, v),\; (\varphi, \psi) \in X \times X,$$

and the norm

$$\|(u, v)\|^2 = \langle(u, v), (u, v)\rangle = \|u\|_X^2 + \|v\|_X^2, \;\; (u, v) \in X \times X.$$

Note that the energy functional J can be written as follows:

$$J(u,v)=\frac{1}{2}\big(\|(u^+,v^+)\|^2-\|(u^-,v^-)\|^2\big)-\int_{\mathbb{R}^N}F(x,u,v)\mathrm{d}x$$
$$=\frac{1}{2}\|(u^+,v^+)\|^2-I(u,v),$$

where

$$I(u,v)=\frac{1}{2}\|(u^-,v^-)\|^2+\int_{\mathbb{R}^N}F(x,u,v)\mathrm{d}x$$

for any $(u,v)=(u^+,v^+)+(u^-,v^-)\in E^+\oplus E^-$. We do not know whether J has critical points in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ since $0\in\sigma(S)$. Furthermore, I is not defined on E owing to our assumptions on $F(x,u,v)$. Hence, similar to [24], we are going to define a space $E_{2,\lambda}$ such that there are continuous embeddings

$$H^1\left(\mathbb{R}^N\right)\times H^1\left(\mathbb{R}^N\right)\hookrightarrow E_{2,\lambda}\hookrightarrow E,$$

where I is well defined on $E_{2,\lambda}$ and J admits critical points on $E_{2,\lambda}$.

Let $\big(P_\lambda : L^2(\mathbb{R}^N)\to L^2(\mathbb{R}^N)\big)_{\lambda\in\mathbb{R}}$ denote the spectral family of S. Let $L^-=P_0\big(L^2(\mathbb{R}^N)\big)$ and $L^+=(id-P_0)\big(L^2(\mathbb{R}^N)\big)$. Thus, we have the orthogonal decomposition $L^2(\mathbb{R}^N)=L^++L^-$, $E^+=\big(H^1(\mathbb{R}^N\cap L^+)\big)\times\big(H^1(\mathbb{R}^N\cap L^+)\big)$, and $E^-=\big(H^1(\mathbb{R}^N\cap L^-)\big)\times\big(H^1(\mathbb{R}^N L^-\big)$ (see[17,25]). Furthermore,

$$\|u\|_X^2=\int_{-\infty}^{+\infty}|\lambda|d\|P_\lambda u\|_2^2.$$

Let us suppose that $2\le\mu\le\gamma$. $L^{\mu,\gamma}=L^\mu(\mathbb{R}^N)+L^\gamma(\mathbb{R}^N)$ denotes the Banach space of all vector fields of the form $u=u_1+u_2$, where $u_1\in L^\mu(\mathbb{R}^N)$ and $u_2\in L^\gamma(\mathbb{R}^N)$, endowed with the following norm

$$\|u\|_{\mu,\gamma}=\inf\left\{\|u_1\|_\mu+\|u_2\|_\gamma\right\}.$$

It follows from Proposition 2.5 in [26] that the infimum in $\|\cdot\|_{\mu,\gamma}$ is attained. Furthermore, there is a continuous embedding

$$L^t\left(\mathbb{R}^N\right)\hookrightarrow L^{\mu,\gamma}\left(\mathbb{R}^N\right)$$

for any $\mu\le t\le\gamma$, and if $\mu=\gamma$, then the norms $\|\cdot\|_{\mu,\gamma}$ and $\|\cdot\|_\gamma$ are equivalent. Let $X^-_{\mu,\gamma}$ and X^-_γ be the completions of X^- with the respect to the norms

$$|||\cdot|||_{\mu,\gamma}=\left(\|\cdot\|_X^2+\|\cdot\|_{\mu,\gamma}\right)^{\frac{1}{2}}$$

and

$$|||\cdot|||_\gamma = \left(\|\cdot\|_X^2 + \|\cdot\|_\gamma\right)^{\frac{1}{2}},$$

respectively. Therefore, we have the following continuous embeddings

$$X^- \hookrightarrow X_\gamma^- \hookrightarrow X_{\mu,\gamma}^- \hookrightarrow X.$$

Space X_γ^- has been introduced in [24], and note that if $\mu = \gamma$, then $X_{\mu,\gamma}^- = X_\gamma^-$ and the norms $|||\cdot|||_{\mu,\gamma}$ and $|||\cdot|||_\gamma$ are equivalent. In this paper, space $E_{\mu,\gamma}^- = X_{\mu,\gamma}^- \times X_{\mu,\gamma}^-$ with $\mu = 2$ is crucial due to the superlinear growth conditions (F_3) and (F_4). Moreover, we have the following lemmas from [22,24].

Lemma 2.1 *$E_{\mu,\gamma}^- = E_\gamma^-$ and norms $\|\cdot\|_{\mu,\gamma}$, $\|\cdot\|_\gamma$ are equivalent for any $2 \le \mu \le \gamma \le 2^*$.*

Lemma 2.2 *If $2 \le \mu \le \gamma \le 2^*$, then $E_{\mu,\gamma}^-$ embeds continuously into $H_{loc}^2(\mathbb{R}^N) \times H_{loc}^2(\mathbb{R}^N)$ and $L^t(\mathbb{R}^N) \times L^t(\mathbb{R}^N)$ for $\gamma \le t \le 2^*$ and compactly embeds into $L_{loc}^t(\mathbb{R}^N) \times L_{loc}^t(\mathbb{R}^N)$ for $2 \le t \le 2^*$.*

Note that we have the following continuous embeddings

$$H^1\left(\mathbb{R}^N\right) \times H^1\left(\mathbb{R}^N\right) \hookrightarrow E_{\mu,\gamma} = E^+ \oplus E_{\mu,\gamma}^- \hookrightarrow E,$$

where $E_{\mu,\gamma}$ is endowed with the norm

$$\|(u,v)\| = \left(\|(u^+,v^+)\|^2 + |||(u^-,v^-)|||_{\mu,\gamma}^2\right)^{\frac{1}{2}}$$

for $(u,v) = (u^+,v^+) + (u^-,v^-) \in E^+ \oplus E_{\mu,\gamma}^-$. Since $\|\cdot\|_{\mu,\gamma}$ is uniformly convex (see Proposition 2.6 in [26]), then $E_{\mu,\gamma}$ is reflective and bounded sequences in $E_{\mu,\gamma}$ are relatively weakly compact. Then, it follows from the Sobolev embedding that Lemma 2.2 holds and $J : E_{\mu,\gamma} \to \mathbb{R}$ given by (1.5) is a well defined C^1-map. Furthermore, it follows from Lemma 2.1 and Corollary 2.3 in [24] that a solution to problem (1.1) in $E_{\mu,\gamma}$ vanishes at infinity. Analogously, we have the following corollary.

Corollary 2.3 *If $(u,v) \in E_{\mu,\gamma}$ solves problem (1.1), then $(u(x),v(x)) \to (0,0)$ as $|x| \to \infty$.*

In order to ensure that a unit sphere in E^+

$$S^+ = \left\{(u,v) \in E^+ \middle| \|(u,v)\| = 1\right\}$$

is a C^1-submanifold of E^+, we suppose that E^+ is a Hilbert space with the scalar product $\langle\cdot,\cdot\rangle$ such that $\langle(u,v),(u,v)\rangle = \|(u,v)\|^2$ for any $(u,v) \in E^+$. In addition to the norm topology, we need the topology $\mathcal{T}$ on E, which is the product of the norm topology in E^+ and the weak topology in E^-. In particular, $(u_n, v_n) \xrightarrow{\mathcal{T}} (u, v)$ provided that $(u_n^+, v_n^+) \to (u^+, v^+)$ and $(u_n^-, v_n^-) \rightharpoonup (u^-, v^-)$.

We define Nehari–Pankov manifold [22] as follows:

$$\mathcal{N}=\left\{(u,v) \in E\backslash\{(0,0)\} : \langle J'(u,v),(u,v)\rangle=0,\ \langle J'(u,v),(\omega,s)\rangle=0,\ \forall\,(\omega,s)\in E^-\right\}.$$

We say that J satisfies the $(PS)_c^{\mathcal{T}}$-condition in $\mathcal{N}$ if every $(PS)_c$ sequence in $\mathcal{N}$ has a subsequence which converges in $\mathcal{T}$:

$$(u_n, v_n) \in \mathcal{N},\ J'(u_n, v_n) \to 0,\ J(u_n, v_n) \to c \Rightarrow (u_n, v_n) \xrightarrow{\mathcal{T}} (u, v) \in E.$$

Lemma 2.4 ([22,27]) *Let $J \in C^1(E, \mathbb{R})$ be a map of the form*

$$J(u, v) = \frac{1}{2}\|(u^+, v^+)\|^2 - I(u, v)$$

for any $(u, v) = (u^+, v^+) + (u^-, v^-) \in E^+ \oplus E^-$ such that

(J_1) $I(u, v) \geq I(0, 0)$ for any $(u, v) \in E$ and I is $\mathcal{T}$-sequentially lower semicontinuous, i.e., if $(u_n, v_n) \xrightarrow{\mathcal{T}} (u_0, v_0)$, then $\liminf_{n\to\infty} I(u_n, v_n) \geq I(u_0, v_0)$.
(J_2) If $(u_n, v_n) \xrightarrow{\mathcal{T}} (u_0, v_0)$ and $I(u_n, v_n) \to I(u_0, v_0)$, then $(u_n, v_n) \to (u_0, v_0)$.
(J_3) If $(u, v) \in \mathcal{N}$, then $J(u, v) > J(tu + \omega, tv + s)$ for $t \geq 0$, $(\omega, s) \in E^-$ such that $(tu + \omega, tv + s) \neq (u, v)$.
(J_4) $0 < \inf_{(u,v)\in E^+, \|(u,v)\|=r} J(u, v)$.
(J_5) $\|(u^+, v^+)\| + I(u, v) \to \infty$ as $\|(u, v)\| \to \infty$.
(J_6) $I(t_n u_n, t_n v_n)/t_n^2 \to \infty$ if $t_n \to \infty$ and $(u_n^+, v_n^+) \to (u_0^+, v_0^+)$ for some $(u_0^+, v_0^+) \neq (0, 0)$, $n \to \infty$.

Then,

(a) $c = \inf_{\mathcal{N}} J > 0$ and there exists a $(PS)_c$-sequence $(u_n, v_n)_{n\in\mathbb{N}}$, i.e., $J(u_n, v_n) \to c$ and $J'(u_n, v_n) \to 0,\ n \to \infty$. If J satisfies the $(PS)_c^{\mathcal{T}}$-condition in $\mathcal{N}$, then c is achieved by a critical point of J.
(b) There is a homeomorphism $m : S^+ \to \mathcal{N}$ such that $m^{-1}(u, v) = (\frac{u^+}{\|(u^+,v^+)\|}, \frac{v^+}{\|(u^+,v^+)\|})$, $m(u, v)$ is the unique maximum point of J on $(\mathbb{R}^+u, \mathbb{R}^+v) \oplus E^-$ for $(u, v) \in E$, and $J \circ m : S^+ \to \mathbb{R}$ is of class C^1. Furthermore, a sequence $(u_n, v_n) \subset S^+$ is $J \circ m$, if and only if $m(u_n, v_n)$ is a Palais-Smale sequence for J, and $(u, v) \in S^+$ is a critical point of $J \circ m$ if and only if $m(u, v)$ is a critical point of J.

3 Ground-State Solutions

In order to look for critical points of $J : E_{2,\gamma} \to \mathbb{R}$, we define the following Nehari–Pankov manifold:

$$\mathcal{N} = \Big\{(u, v) \in E_{2,\gamma} \backslash E^{-}_{2,\gamma} : \langle J'(u, v), (u, v)\rangle = 0, \langle J'(u, v), (\omega, s)\rangle = 0, \ \forall\, (\omega, s) \in E^{-}_{2,\gamma}\Big\}. \tag{3.1}$$

Lemma 3.1 *There exists a constant $a > 0$ such that for any $(u, v) \in E_{2,\gamma}$*

$$\int_{\mathbb{R}^N} F(x, u, v)\mathrm{d}x \geq a \min\left\{\|(u, v)\|^2_{2,\gamma}, \|(u, v)\|^{\gamma}_{2,\gamma}\right\}. \tag{3.2}$$

Proof It follows from (F_3) and (F_4) that there exists $b > 0$ such that

$$F(x, u, v) \geq b \min\left\{|(u, v)|^2, |(u, v)|^{\gamma}\right\} \tag{3.3}$$

for all $(u, v) \in \mathbb{R}^2$ and $x \in \mathbb{R}^N$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^N} F(x, u, v)\mathrm{d}x &\geq b\left(\int_{\Omega_{(u,v)}} |(u, v)|^2\mathrm{d}x + \int_{\Omega^c_{(u,v)}} |(u, v)|^{\gamma}\mathrm{d}x\right)\\ &= b\left(\|(u, v)\chi_{\Omega_{(u,v)}}\|^2_2 + \|(u, v)\chi_{\Omega^c_{(u,v)}}\|^{\gamma}_{\gamma}\right)\\ &\geq a \min\left\{\|(u, v)\|^2_{2,\gamma,\infty}, \|(u, v)\|^{\gamma}_{2,\gamma,\infty}\right\},\end{aligned}$$

for some constant $a > 0$, where χ denotes the characteristic function, $\Omega_{(u,v)} = \{x \in \mathbb{R}^N \,\|\,(u, v)\| > 1\}$, and

$$\begin{aligned}\|(u, v)\|_{2,\gamma,\infty} = \inf\Big\{&\max\left\{\|(u_1, v_1)\|_2, \|(u_2, v_2)\|_{\gamma}\right\} \Big| (u, v) = (u_1, v_1) + (u_2, v_2),\\ &(u_1, v_1) \in L^2\left(\mathbb{R}^N\right) \times L^2\left(\mathbb{R}^N\right), \ (u_2, v_2) \in L^{\gamma}\left(\mathbb{R}^N\right) \times L^{\gamma}\left(\mathbb{R}^N\right)\Big\}\end{aligned}$$

defines a norm on $L^{2,\gamma}(\mathbb{R}^N) \times L^{2,\gamma}(\mathbb{R}^N)$, which is equivalent to $\|.\|_{2,\gamma}$ (see Proposition 2.4 in [26]). The proof is complete. □

The following lemma shows that $(J_4) - (J_6)$ hold for J.

Lemma 3.2 *The following conditions hold:*

(a) $\inf\limits_{(u,v)\in E^+, \|(u,v)\|=r} J(u, v) > 0.$

(b) $\|(u^+, v^+)\| + I(u, v) \to \infty$ *as* $\|(u, v)\| \to \infty$.

(c) $\frac{I(t_n u_n, t_n v_n)}{t_n^2} \to \infty$ *if* $(u_n^+, v_n^+) \to (u_0^+, v_0^+) \neq (0, 0)$ *and* $t_n \to \infty,$ *as* $n \to \infty$.

Proof (a) If $(u, v) \in E^+$, then it follows from (F_2) that

$$J(u, v) \geq \frac{1}{2}\|(u, v)\|^2 - \frac{c}{\gamma}\|(u, v)\|_\gamma^\gamma - \frac{c}{p}\|(u, v)\|_p^p.$$

Since the embeddings from E^+ into $L^\gamma(\mathbb{R}^N) \times L^\gamma(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ are continuous, then we have

$$J(u, v) \geq \frac{1}{2}\|(u, v)\|^2 - C\big(\|(u, v)\|^\gamma + \|(u, v)\|^p\big).$$

Therefore, the inequality in (a) holds.

(b) Suppose that $\|(u_n, v_n)\| \to \infty,\ n \to \infty$, and $(\|(u_n^+, v_n^+)\|)_{n\in\mathbb{N}}$ is bounded. Then, $(\|(u_n^+, v_n^+)\|_{2,\gamma})_{n\in\mathbb{N}}$ is bounded and

$$\|(u_n^-, v_n^-)\|_{2,\gamma}^2 = \|(u_n^-, v_n^-)\|^2 + \|(u_n^-, v_n^-)\|_{2,\gamma}^2 \to \infty,\ n \to \infty.$$

If $\|(u_n^-, v_n^-)\| \to \infty$, then $I(u_n, v_n) \to \infty$. Suppose that $(\|(u_n^-, v_n^-)\|)_{n\in\mathbb{N}} \to \infty$ as $n \to \infty$, then $\|(u_n^-, v_n^-)\|_{2,\gamma} \to \infty$ and by (3.2), we have $I(u_n, v_n) \to \infty,\ n \to \infty$.

(c) Suppose that, up to a subsequence, $I(t_n u_n, t_n v_n)/t_n^2$ is bounded, $(u_n^+, v_n^+) \to (u_0^+, v_0^+) \neq (0, 0)$, and $t_n \to \infty$, then by (3.2), we have

$$\frac{I(t_n u_n, t_n v_n)}{t_n^2} \geq \frac{1}{2}\|(u_n^-, v_n^-)\|^2 + c \min\left\{\|(u_n, v_n)\|_{2,\gamma}^2, t_n^{\gamma-2}\|(u_n, v_n)\|_{2,\gamma}^\gamma\right\}$$

and then $(||(u_n^-, v_n^-)||_{2,\gamma})_{n\in\mathbb{N}}$ is bounded. It follows from Lemma 2.2 that $(u_n^-, v_n^-) \rightharpoonup (u_0^-, v_0^-)$ in $E_{2,\gamma}^-$ and $(u_n^-(x), v_n^-(x)) \to (u_0^-(x), v_0^-(x))$ a.e. on $\mathbb{R}^N$. If the Lebesgue measure $|\Omega| > 0$, where $\Omega = \left\{x \in \mathbb{R}^N | (u_0^+(x) + u_0^-(x), v_0^+(x) + v_0^-(x)) \neq (0, 0)\right\}$, then it follows from (F_4) and Fatou's lemma that

$$\int_{\mathbb{R}^N} \frac{F(x, t_n u_n, t_n v_n)}{t_n^2} \mathrm{d}x \to \infty,\ n \to \infty.$$

Therefore, we obtain $I(t_n u_n, t_n v_n)/t_n^2 \to \infty$ as $n \to \infty$, which is a contradiction. Hence, $|\Omega| = 0$ and $(u_0^-, v_0^-) = -(u_0^+, v_0^+)$. Since $\langle (u_0^-, v_0^-), (u_0^+, v_0^+)\rangle = 0$, then $(u_0^+, v_0^+) = (0, 0)$. This is a contradiction; then, $I(t_n u_n, t_n v_n)/t_n^2 \to \infty$, as $n \to \infty$.

Lemma 3.3 *The following conditions hold:*

(a) $I(u, v) \geq 0$ *for any* $(u, v) \in E_{2,\gamma}$ *and* I *is* $\mathcal{T}$*-sequentially lower semicontinuous.*

(b) *If* $(u_n, v_n) \xrightarrow{\mathcal{T}} (u_0, v_0)$ *and* $I(u_n, v_n) \to I(u_0, v_0)$*, then* $(u_n, v_n) \to (u_0, v_0)$.

(c) *If* $(u, v) \in \mathcal{N}$*, then* $J(u, v) > J(tu + \omega, tv + s)$ *for* $t \geq 0$*,* $(\omega, s) \in E^-$ *such that* $(tu + \omega, tv + s) \neq (u, v)$.

Proof (*a*) Let $(u_n, v_n) \xrightarrow{\mathcal{T}} (u_0, v_0)$, since $E_{2,\gamma}$ is compactly embedded in $L^2_{loc}(\mathbb{R}^N) \times L^2_{loc}(\mathbb{R}^N)$ and $(u_n(x), v_n(x)) \to (u_0(x), v_0(x))$ a.e. in $\mathbb{R}^N$. Then, it follows from Fatou's lemma and the weakly sequentially lower semicontinuity of the map $E^- \ni (u^-, v^-) \to \frac{1}{2}\|(u^-, v^-)\|^2$ that $\liminf_{n\to\infty} I(u_n, v_n) \geq I(u_0, v_0)$.

(*b*) Let $(u_n, v_n) \xrightarrow{\mathcal{T}} (u_0, v_0)$ and $I(u_n, v_n) \to I(u_0, v_0)$. Since $E_{2,\gamma} \ni (u, v) \to \int_{\mathbb{R}^N} F(x, u, v)\mathrm{d}x$ is $\mathcal{T}$-sequentially lower semicontinuous, then

$$\lim_{n\to\infty} \| \left(u_n^-, v_n^-\right) \|^2 = \| \left(u_0^-, v_0^-\right) \|^2$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(x, u_n, v_n)\mathrm{d}x = \int_{\mathbb{R}^N} F(x, u_0, v_0)\mathrm{d}x. \tag{3.4}$$

Observe that

$$\begin{aligned}\|(u_n^- - u_0^-, v_n^- - v_0^-)\|^2 &= \|(u_n^-, v_n^-)\|^2 - \|(u_0^-, v_0^-)\|^2 \\ &\quad - 2\langle(u_n^- - u_0^-, v_n^- - v_0^-), (u_0^-, v_0^-)\rangle \\ &\to 0, \ n \to \infty.\end{aligned}$$

Therefore, $(u_n, v_n) = (u_n^+ + u_n^-, v_n^+ + v_n^-) \to (u_0, v_0) = (u_0^+ + u_0^-, v_0^+ + v_0^-)$. Since we need to prove that $(u_n^-, v_n^-) \to (u_0^-, v_0^-)$ a.e. in $\mathbb{R}^N$, let us consider the function $L : \mathbb{R}^N \times [0, 1] \to \mathbb{R}$ given by $L(x, t) = F(x, u_n - tu_0, v_n - tv_0)$. Then,

$$F(x, u_n - u_0, v_n - v_0) - F(x, u_n, v_n) = L(x, 1) - L(x, 0) = \int_0^1 \frac{\partial L}{\partial s}(x, s)\mathrm{d}s.$$

Furthermore,

$$\begin{aligned}&\int_{\mathbb{R}^N} \Big[F(x, u_n - u_0, v_n - v_0) + F(x, u_0, v_0) - F(x, u_n, v_n)\Big]\mathrm{d}x \\ &\quad= \int_{\mathbb{R}^N} \Big[\int_0^1 \frac{\partial L}{\partial s}(x, s)\mathrm{d}s + F(x, u_0, v_0)\Big]\mathrm{d}x \\ &\quad= \int_{\mathbb{R}^N} \int_0^1 \frac{\partial L}{\partial s}(x, s)\mathrm{d}s\mathrm{d}x + \int_{\mathbb{R}^N} F(x, u_0, v_0)\mathrm{d}x \\ &\quad= \int_0^1 \int_{\mathbb{R}^N} \Big[- F_u(x, u_n - su_0, v_n - sv_0)u_0 \\ &\qquad - F_v(x, u_n - su_0, v_n - sv_0)v_0\Big]\mathrm{d}x\mathrm{d}s + \int_{\mathbb{R}^N} F(x, u_0, v_0)\mathrm{d}x.\end{aligned} \tag{3.5}$$

Let $E \subset \mathbb{R}^N$ be a measurable set, then it follows from the Hölder inequality that

$$\begin{aligned}
&\int_E |F_u(x, u_n - su_0, v_n - sv_0)u_0|\mathrm{d}x \\
&\quad \le \varepsilon \int_E |(u_n - su_0, v_n - sv_0)||u_0|\mathrm{d}x \\
&\qquad + C(\varepsilon) \int_E |(u_n - su_0, v_n - sv_0)|^{p-1}|u_0|\mathrm{d}x \\
&\qquad + C \int_E |(u_n - su_0, v_n - sv_0)|^{\gamma-1}|u_0|\mathrm{d}x \\
&\quad \le \varepsilon \|(u_n - su_0, v_n - sv_0)\chi_E\|_2 \|u_0\chi_E\|_2 \\
&\qquad + C(\varepsilon)\|(u_n - su_0, v_n - sv_0)\chi_E\|_p^{p-1} \|u_0\chi_E\|_p \\
&\qquad + C\|(u_n - su_0, v_n - sv_0)\chi_E\|_\gamma^{\gamma-1} \|u_0\chi_E\|_\gamma.
\end{aligned}$$

Therefore, $F_u(x, u_n - su_0, v_n - sv_0)u_0$ is uniformly integrable and by the Vitali convergence theorem, we derive

$$\begin{aligned}
&\int_0^1 \int_{\mathbb{R}^N} -F_u(x, u_n - su_0, v_n - sv_0)u_0 \mathrm{d}x\mathrm{d}s \to \\
&\int_0^1 \int_{\mathbb{R}^N} -F_u(x, u_0 - su_0, v_0 - sv_0)u_0 \mathrm{d}x\mathrm{d}s
\end{aligned}$$

as $n \to \infty$. Analogously,

$$\begin{aligned}
&\int_0^1 \int_{\mathbb{R}^N} -F_v(x, u_n - su_0, v_n - sv_0)v_0 \mathrm{d}x\mathrm{d}s \to \\
&\int_0^1 \int_{\mathbb{R}^N} -F_v(x, u_0 - su_0, v_0 - sv_0)v_0 \mathrm{d}x\mathrm{d}s
\end{aligned}$$

as $n \to \infty$. Furthermore,

$$\begin{aligned}
&\int_0^1 \int_{\mathbb{R}^N} \Big[- F_u(x, u_n - su_0, v_n - sv_0)u_0 - F_v(x, u_n - su_0, v_n - sv_0)v_0 \Big] \mathrm{d}x\mathrm{d}s \\
&\quad = \int_{\mathbb{R}^N} \int_0^1 \Big[- F_u(x, u_n - su_0, v_n - sv_0)u_0 - F_v(x, u_n - su_0, v_n - sv_0)v_0 \Big] \mathrm{d}s\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^N} \int_0^1 -\frac{\partial}{\partial s}\Big[F(x, u_n - su_0, v_n - sv_0) \Big] \mathrm{d}s\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^N} \big(F(x, 0, 0) - F(x, u_0, v_0)\big)\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^N} -F(x, u_0, v_0)\mathrm{d}x.
\end{aligned}$$

Then, from (3.4) and (3.5), we derive

$$\lim_{n\to\infty} F(x, u_n - u_0, v_n - v_0)\mathrm{d}x = 0.$$

It follows from (3.2) that $(u_n, v_n) \to (u_0, v_0)$ in $L^{2,\gamma}(\mathbb{R}^N) \times L^{2,\gamma}(\mathbb{R}^N)$.
(c) Let $(u, v) \in \mathcal{N}$. Observe that for any $t \geq 0$ and $(\omega, s) \in E^-_{2,\gamma}$

$$J(tu + \omega, tv + s) - J(u, v) = -\frac{1}{2}\|(\omega, s)\|^2 - \int_{\mathbb{R}^N} \varphi(t, x)\mathrm{d}x,$$

where

$$\begin{aligned}\varphi(t, x) = &F_u(x, u, v)\left(\frac{t^2-1}{2}u + t\omega\right) + F_v(x, u, v)\left(\frac{t^2-1}{2}v + ts\right)\\ &+ F(x, u, v) - F(x, tu + \omega, tv + s).\end{aligned}$$

Suppose that $(u(x), v(x)) \neq (0, 0)$. Similarly as in [20,21], we show that $\varphi(t, x) \leq 0$. In fact, it follows from (F_6) that $\varphi(0, x) \leq 0$. By (F_4), we obtain $\varphi(t, x) \to -\infty$ as $t \to \infty$. Let $t_0 > 0$ be such that

$$\varphi(t_0, x) = \max_{t\geq 0} \varphi(t, x).$$

We may assume that $t_0 > 0$ and $\partial_t\varphi(t_0, x) = 0$; therefore,

$$\begin{aligned}&F_u(x, u, v)\big(t_0u + \omega\big) + F_v(x, u, v)\big(t_0v + s\big)\\ &\quad = F_u(x, t_0u + \omega, t_0v + s)u + F_v(x, t_0u + \omega, t_0v + s)v.\end{aligned}$$

If $F_u(x, u, v)(t_0u + \omega) \leq 0$, $F_v(x, u, v)(t_0v + s) \leq 0$, then by (F_6), we have

$$\begin{aligned}\varphi(t_0, x) \leq &\frac{-t_0^2-1}{2}F_u(x, u, v)u + \frac{-t_0^2-1}{2}F_v(x, u, v)v + F(x, u, v)\\ &- F(x, t_0u + \omega, t_0v + s) \leq 0.\end{aligned}$$

Otherwise, (F_5) implies that

$$\begin{aligned}\varphi(t_0, x) \leq &F_u(x, u, v)\left(\frac{t_0^2-1}{2}u + t_0\omega\right) + F_v(x, u, v)\left(\frac{t_0^2-1}{2}v + t_0s\right)\\ &+ \frac{\big(F_u(x, u, v)u\big)^2 - \big(F_u(x, u, v)(t_0u + \omega)\big)^2}{2F_u(x, u, v)u}\\ &+ \frac{\big(F_v(x, u, v)v\big)^2 - \big(F_v(x, u, v)(t_0v + s)\big)^2}{2F_v(x, u, v)v}\end{aligned}$$

$$= -\frac{\left(F_u(x,u,v)\omega\right)^2}{2F_u(x,u,v)u} - \frac{\left(F_v(x,u,v)s\right)^2}{2F_v(x,u,v)v}$$
$$\leq 0.$$

The proof is complete. □

Lemma 3.4 *J is coercive on $\mathcal{N}$, i.e., $J(u,v) \to \infty$ as $\|(u,v)\| \to \infty$, $(u,v) \in \mathcal{N}$.*

Proof Suppose that $\|(u_n, v_n)\| \to \infty$ as $n \to \infty$, $(u_n, v_n) \in \mathcal{N}$ and $J(u_n, v_n) \leq c_1$ for some constant $c_1 > 0$. Let $(\omega_n, s_n) = \frac{(u_n, v_n)}{\|(u_n, v_n)\|}$, since $E_{2,\gamma}$ is reflective and compactly embedded in $L^2_{loc}(\mathbb{R}^N) \times L^2_{loc}(\mathbb{R}^N)$. Then, there exists $(\omega, s) \in E_{2,\gamma}$ such that $(\omega_n, s_n) \rightharpoonup (\omega, s)$ and $(\omega_n(x), s_n(x)) \to (\omega(x), s(x))$ a.e. on $\mathbb{R}^N$. Furthermore, there exists a sequence $(y_n)_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B(y_n,1)} \left|(\omega_n^+, s_n^+)\right|^2 \mathrm{d}x > 0. \tag{3.6}$$

Otherwise, it follows from Lions lemma (see Lemma 1.21 in [28]) that $(\omega_n^+, s_n^+) \to (0,0)$ in $L^t(\mathbb{R}^N) \times L^t(\mathbb{R}^N)$ for $t \in (2, 2^*)$. Then, $\int_{\mathbb{R}^N} F(x, t\omega_n^+, ts_n^+)\mathrm{d}x \to 0$ for any $t > 0$. For any $t > 0$, in view of Lemma 3.3 (c), we obtain

$$c_1 \geq \limsup_{n\to\infty} J(u_n, v_n) \geq \limsup_{n\to\infty} J\left(t\omega_n^+, ts_n^+\right) = \frac{t^2}{2} \left\|(\omega_n^+, s_n^+)\right\|^2_{\mu_n}. \tag{3.7}$$

It follows from Lemma 2.4 (a) and (3.2) that

$$\frac{1}{2}\left(\left\|(u_n^+, v_n^+)\right\|^2 - \left\|(u_n^-, v_n^-)\right\|^2\right) - c\min\left\{\|(u_n, v_n)\|^2_{2,\gamma}, \|(u_n, v_n)\|^{\gamma}_{2,\gamma}\right\}$$
$$\geq J(u_n, v_n) \geq c_{inf} = \inf_N J > 0.$$

If $\liminf_{n\to\infty} \|(u_n, v_n)\|_{2,\gamma} = 0$, then up to a subsequence, $\|(u_n, v_n)\|_{2,\gamma} \to 0$, $n \to \infty$, and for sufficiently large n

$$\begin{aligned} 2\|(u_n^+, v_n^+)\|^2 &\geq \|(u_n^+, v_n^+)\|^2 + \|(u_n^-, v_n^-)\|^2 + 2c_{inf} \\ &\quad + 2c\min\left\{\|(u_n, v_n)\|^2_{2,\gamma}, \|(u_n, v_n)\|^{\gamma}_{2,\gamma}\right\} \\ &\geq \|(u_n^+, v_n^+)\|^2 + \|(u_n^-, v_n^-)\|^2 + \|(u_n, v_n)\|^{\gamma}_{2,\gamma} \\ &= \|(u_n, v_n)\|^2. \end{aligned}$$

If $\liminf_{n\to\infty} \|(u_n, v_n)\|_{2,\gamma} > 0$, then there exists $c_2 \in (0,1)$ such that for sufficiently large n

$$
\begin{aligned}
2\|(u_n^+, v_n^+)\|^2 &\geq \|(u_n^+, v_n^+)\|^2 + \|(u_n^-, v_n^-)\|^2 + 2c_{inf} \\
&\quad + 2c \min\left\{\|(u_n, v_n)\|_{2,\gamma}^2, \|(u_n, v_n)\|_{2,\gamma}^\gamma\right\} \\
&\geq c_2 \left(\|(u_n^+, v_n^+)\|^2 + \|(u_n^-, v_n^-)\|^2 + \|(u_n, v_n)\|_{2,\gamma}^\gamma\right) \\
&= c_2\|(u_n, v_n)\|^2.
\end{aligned}
$$

Hence, passing to a subsequence if necessary, $c_3 = \inf_{n\in\mathbb{N}} \|(\omega_n^+, s_n^+)\|^2 > 0$ and by (3.7), we have

$$
c_1 \geq \frac{t^2}{2} c_3
$$

for any $t \geq 0$. Then, we get a contradiction. Therefore, we may assume that $(y_n)_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and

$$
\liminf_{n\to\infty} \int_{B(y_n,1)} |(\omega_n^+, s_n^+)|^2 \mathrm{d}x > 0.
$$

Since J and $\mathcal{N}$ are invariant under translations of the form $(u, v) \mapsto (u(\cdot - k), v(\cdot - k))$, $k \in \mathbb{Z}^N$, then we may assume that $(\omega_n^+, s_n^+) \to (\omega^+, s^+)$ in $L^2_{loc}(\mathbb{R}^N) \times L^2_{loc}(\mathbb{R}^N)$ and $(\omega^+, s^+) \neq (0, 0)$. Note that $|(\omega^+, s^+)| \neq 0$, then $(u_n(x), v_n(x)) = (\omega_n(x), s_n(x))\|(u_n, v_n)\| \to \infty$, and by (F_4), we have

$$
\frac{F(x, u_n(x), v_n(x))}{\|(u_n, v_n)\|^2} = \frac{F(x, u_n(x), v_n(x))}{|(u_n, v_n)|^2} |(\omega_n(x), s_n(x))|^2 \to \infty, \; n \to \infty.
$$

Hence, it follows from Fatou's lemma that

$$
\begin{aligned}
\frac{J_{\mu_n}(u_n, v_n)}{\|(u_n, v_n)\|^2} &= \limsup_{n\to\infty} \left(\frac{1}{2}\left(\|(\omega_n^+, s_n^+)\|^2 - \|(\omega_n^-, s_n^-)\|^2\right) - \int_{\mathbb{R}^N} \frac{F(x, u_n, v_n)}{\|(u_n, v_n)\|^2} \mathrm{d}x\right) \\
&\to -\infty, \; n \to \infty,
\end{aligned}
$$

which is a contradiction. The proof is complete. □

Proof of Theorem 1.1. It follows from Lemma 2.4 (a) that $c_{inf} = \inf_{\mathcal{N}} J > 0$ and there exists a $(PS)_{c_{inf}}$-sequence $(u_n, v_n)_{n\in\mathbb{N}}$. By Lemma 3.4. we get that $(u_n, v_n)_{n\in\mathbb{N}}$ is bounded and passing to a subsequence $(u_n, v_n) \rightharpoonup (u, v)$ in $E_{2,\gamma}$. Then, there exists a subsequence $(y_n)_{n\in\mathbb{N}}$ such that

$$
\liminf_{n\to\infty} \int_{B(y_n,1)} |(u_n^+, v_n^+)|^2 \mathrm{d}x > 0. \tag{3.8}
$$

Otherwise, it follows from Lions lemma (see Lemma 1.21 in [28]) that $(u_n^+, v_n^+) \to (0, 0)$ in $L^t(\mathbb{R}^N) \times L^t(\mathbb{R}^N)$ for $t \in (2, 2^*)$. From (F_2), we have

$$\|(u_n^+, v_n^+)\|^2 = \langle J'(u_n, v_n), (u_n^+, v_n^+)\rangle + \int_{\mathbb{R}^N} \left(F_u(x, u_n, v_n)u_n^+ + F_v(x, u_n, v_n)v_n^+\right)\mathrm{d}x$$
$$\to 0, \text{ as } n \to \infty.$$

Therefore,

$$0 < c_{\inf} = \lim_{n\to\infty} J(u_n, v_n) \leq \frac{1}{2} \lim_{n\to\infty} \|(u_n^+, v_n^+)\|^2 = 0,$$

which is a contradiction. Therefore, (3.8) holds and we may assume that there is a subsequence $(y_n)_{n\in\mathbb{N}}$ such that

$$\liminf_{n\to\infty} \int_{B(y_n,r)} |(u_n^+, v_n^+)|^2 \mathrm{d}x > 0, \tag{3.9}$$

for some $r > 1$. Since $\|(u_n(\cdot + y_n), v_n(\cdot + y_n))\| = \|(u_n, v_n)\|$, then there exists $(u, v) \in E_{2,\gamma}$ such that $(u_n(\cdot + y_n), v_n(\cdot + y_n)) \rightharpoonup (u, v)$ in $E_{2,\gamma}$ and $(u_n^+(\cdot + y_n), v_n^+(\cdot + y_n)) \rightharpoonup (u^+, v^+)$ in $L^2_{loc}(\mathbb{R}^N) \times L^2_{loc}(\mathbb{R}^N)$. It follows from (3.9) that $(u^+, v^+) \neq (0, 0)$ and then $(u, v) \neq (0, 0)$. Since J and $\mathcal{N}$ are invariant under translations of the form $(u, v) \mapsto (u(\cdot+y), v(\cdot+y))$ for $y \in \mathbb{Z}^N$, then $J'(u, v) = 0$. Hence, $(u, v) \in \mathcal{N}$ and $J(u, v) = c_{\inf}$. Since $(u, v) \in E_{2,\gamma}$ is a solution to problem (1.1), then by Corollary 2.3, we get $(u(x), v(x)) \to (0, 0)$ as $|x| \to \infty$. The proof is complete.

□

4 Multiple Solutions

Observe that if $(u, v) \in E_{2,\gamma}$ is a critical point of J, then the orbit under the action of $\mathbb{Z}^N$, $O(u, v) = \left\{(u(\cdot - k), v(\cdot - k)) | k \in \mathbb{Z}^N\right\}$ consists of critical points. Two critical points $(u_1, v_1),\ (u_2, v_2) \in E_{2,\gamma}$ are said to be geometrically distinct if $O(u_1, v_1) \cap O(u_2, v_2) = \emptyset$. It follows from Lemma 2.4(b) that $\psi = J \circ m : S^+ \to \mathbb{R}$ is a C^1 map. In order to prove Theorem 1.2, we just need to prove that ψ has infinitely many geometrically distinct critical points (Lemma 2.4(b)). The following lemma is crucial in the consideration of the multiplicity of critical points (see Lemma 2.14 in [18]).

Lemma 4.1 *Let $d \geq c_{\inf}$, if $(u_n^1, v_n^1),\ (u_n^2, v_n^2) \subset \psi^d = \{(u, v) \in S^+ | \psi(u, v) \leq d\}$ are two Palais–Smale sequences for ψ, then either $\|(u_n^1 - u_n^2, v_n^1 - v_n^2)\| \to 0,\ n \to \infty$, or*

$$\limsup_{n\to\infty} \|(u_n^1 - u_n^2, v_n^1 - v_n^2)\|^2 \geq \rho(d) \inf \Big\{ \|(u_1 - u_2, v_1 - v_2)\| \big| \psi'(u^1, v^1) = \psi'(u^2, v^2) = 0,\ (u^1, v^1) \neq (u^2, v^2) \in S^+ \Big\}, \tag{4.1}$$

where $\rho(d)$ depends on d but not on the particular choice of Palais–Smale sequences.

Proof Let $(u_n^1, v_n^1),\ (u_n^2, v_n^2) \subset \psi^d = \left\{(u, v) \in S^+ | \psi(u, v) \leq d\right\}$ be two Palais–Smale sequences for ψ. Let $(\omega_n^i, s_n^i) = (m(u_n^i, v_n^i)),\ i = 1, 2$, and we consider two cases:

Case 1: $\|((\omega_n^1)^+ - (\omega_n^2)^+, (s_n^1)^+ - (s_n^2)^+)\|_\gamma \to 0$, and by (F_2), we have

$$
\begin{aligned}
&\left\|\left(\left(\omega_n^1\right)^+ - \left(\omega_n^2\right)^+, \left(s_n^1\right)^+ - \left(s_n^2\right)^+\right)\right\|^2 \\
&= \left\langle J'\left(\omega_n^1, s_n^1\right), \left(\left(\omega_n^1\right)^+ - \left(\omega_n^2\right)^+, \left(s_n^1\right)^+ - \left(s_n^2\right)^+\right)\right\rangle \\
&\quad - \left\langle J'\left(\omega_n^2, s_n^2\right), \left(\left(\omega_n^1\right)^+ - \left(\omega_n^2\right)^+, \left(s_n^1\right)^+ - \left(s_n^2\right)^+\right)\right\rangle \\
&\quad + \int_{\mathbb{R}^N} \left(F_u\left(x, \omega_n^1, s_n^1\right) - F_u\left(x, \omega_n^2, s_n^2\right)\right)\left(\left(\omega_n^1\right)^+ - \left(\omega_n^2\right)^+\right) \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \left(F_v\left(x, \omega_n^1, s_n^1\right) - F_v\left(x, \omega_n^2, s_n^2\right)\right)\left(\left(s_n^1\right)^+ - \left(s_n^2\right)^+\right) \mathrm{d}x \\
&\le \left\langle J'\left(\omega_n^1, s_n^1\right), \left(\left(\omega_n^1\right)^+ - \left(\omega_n^2\right)^+, \left(s_n^1\right)^+ - \left(s_n^2\right)^+\right)\right\rangle \\
&\quad - \left\langle J'\left(\omega_n^2, s_n^2\right), \left(\left(\omega_n^1\right)^+ - \left(\omega_n^2\right)^+, \left(s_n^1\right)^+ - \left(s_n^2\right)^+\right)\right\rangle \\
&\quad + C\left(\left\|\left(\omega_n^1, s_n^1\right)\right\|_q^{p-1} + \left\|\left(\omega_n^1, s_n^1\right)\right\|_\gamma^{\gamma-1} + \left\|\left(\omega_n^2, s_n^2\right)\right\|_q^{p-1}\right. \\
&\quad \left. + \left\|\left(\omega_n^2, s_n^2\right)\right\|_\gamma^{\gamma-1}\right)\left\|\left(\left(\omega_n^1\right)^+ - \left(\omega_n^2\right)^+, \left(s_n^1\right)^+ - \left(s_n^2\right)^+\right)\right\|_\gamma,
\end{aligned}
$$

where $q = \frac{(p-1)\gamma}{\gamma-1}$ and $\gamma \le q \le p$. It follows from Lemma 2.4(*b*) that $(\omega_n^1, s_n^1)_{n\in\mathbb{N}}$ and $(\omega_n^2, s_n^2)_{n\in\mathbb{N}}$ are Palais–Smale sequences for J and by Lemma 3.4, we know that they are bounded in $E_{2,\gamma}$. Since the embeddings from $E_{2,\gamma}$ into $L^\gamma(\mathbb{R}^N) \times L^\gamma(\mathbb{R}^N)$ and into $L^q(\mathbb{R}^N) \times L^q(\mathbb{R}^N)$ are continuous, then we derive

$$
\left\|\left(\left(\omega_n^1\right)^+ - \left(\omega_n^2\right)^+, \left(s_n^1\right)^+ - \left(s_n^2\right)^+\right)\right\| \to 0, \ n \to \infty.
$$

Note that if $(u, v) = (u^+ + u^-, v^+ + v^-) \in \mathcal{N}$, then $J(u, v) \ge c_{inf}$ and

$$
\left\|\left(u^+, v^+\right)\right\| \ge \left\{\sqrt{2c_{\mathrm{inf}}}\|(u^-, v^-)\|\right\}. \tag{4.2}
$$

Similarly as in Lemma 2.13 in [18], we obtain

$$
\begin{aligned}
\left\|\left(u_n^1 - u_n^2, v_n^1 - v_n^2\right)\right\| &= \left\|\frac{\left((\omega_n^1)^+, (s_n^1)^+\right)}{\left\|\left((\omega_n^1)^+, (s_n^1)^+\right)\right\|} - \frac{\left((\omega_n^2)^+, (s_n^2)^+\right)}{\left\|\left((\omega_n^2)^+, (s_n^2)^+\right)\right\|}\right\| \\
&\le \sqrt{\frac{2}{c_{\mathrm{inf}}}}\left\|\left(\left(\omega_n^1\right)^+ - \left(\omega_n^2\right)^+, \left(s_n^1\right)^+ - \left(s_n^2\right)^+\right)\right\|.
\end{aligned}
$$

Therefore,

$$\left\|\left(u_n^1-u_n^2, v_n^1-v_n^2\right)\right\| \to 0 \; n \to \infty.$$

Case 2: $\|((\omega_n^1)^+ - (\omega_n^2)^+, (s_n^1)^+ - (s_n^2)^+)\|_\gamma \nrightarrow 0$. It follows from Lemma 1.21 in [18] that there exists a subsequence $(y_n) \in \mathbb{Z}^N$ and $r > 1$ such that

$$\liminf_{n\to\infty} \int_{B(y_n,1)} \left| \left(\left(\omega_n^1\right)^+ - \left(\omega_n^2\right)^+, \left(s_n^1\right)^+ - \left(s_n^2\right)^+ \right) \right|^2 \mathrm{d}x > 0. \tag{4.3}$$

Then, we may assume that up to a subsequence,

$$\begin{aligned}
&\left(\omega_n^1(\cdot+y_n), s_n^1(\cdot+y_n)\right) \rightharpoonup \left(\omega^1, s^1\right), \text{ in } E_{2,\gamma},\\
&\left(\omega_n^2(\cdot+y_n), s_n^2(\cdot+y_n)\right) \rightharpoonup \left(\omega^2, s^2\right), \text{ in } E_{2,\gamma},\\
&\left(\left(\omega_n^1\right)^+(\cdot+y_n), \left(s_n^1\right)^+(\cdot+y_n)\right) \rightharpoonup \left(\left(\omega^1\right)^+, \left(s^1\right)^+\right), \text{ in } L^2_{loc}\left(\mathbb{R}^N\right) \times L^2_{loc}\left(\mathbb{R}^N\right),\\
&\left(\left(\omega_n^2\right)^+(\cdot+y_n), \left(s_n^2\right)^+(\cdot+y_n)\right) \rightharpoonup \left(\left(\omega^2\right)^+, \left(s^2\right)^+\right), \text{ in } L^2_{loc}\left(\mathbb{R}^N\right) \times L^2_{loc}\left(\mathbb{R}^N\right),
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\left(\left(\omega_n^1\right)^+(\cdot+y_n), \left(s_n^1\right)^+(\cdot+y_n)\right)\right\| \to \alpha_1,\\
&\left\|\left(\left(\omega_n^2\right)^+(\cdot+y_n), \left(s_n^2\right)^+(\cdot+y_n)\right)\right\| \to \alpha_2,
\end{aligned}$$

for $\alpha_1,\ \alpha_2 \geq \sqrt{2c_{inf}}$. It follows from (4.3) that $((\omega^1)^+, (s^1)^+) \neq ((\omega^2)^+, (s^2)^+)$ and then $(\omega^1, s^1) \neq (\omega^2, s^2)$. Since $m,\ m^-,\ J^-,\ (J \circ m)^-$ are equivariant with respect to $\mathbb{Z}^N$-action, then $J^-(\omega^1, s^1) = J^-(\omega^2, s^2) = 0$. Observe that if $(\omega^1, s^1) \neq (0,0)$ and $(\omega^2, s^2) \neq (0,0)$, then $(\omega^1, s^1),\ (\omega^2, s^2) \in \mathcal{N}$ and

$$\begin{aligned}
&\liminf_{n\to\infty} \left\|\left(u_n^1-u_n^2, v_n^1-v_n^2\right)\right\|\\
&\quad= \liminf_{n\to\infty} \left\|\left(\left(u_n^1-u_n^2\right)(\cdot+y_n), \left(v_n^1-v_n^2\right)(\cdot+y_n)\right)\right\|\\
&\quad= \liminf_{n\to\infty} \left\| \frac{\left(\left(\omega_n^1\right)^+, \left(s_n^1\right)^+\right)(\cdot+y_n)}{\left\|\left(\left(\omega_n^1\right)^+, \left(s_n^1\right)^+\right)(\cdot+y_n)\right\|} - \frac{\left(\left(\omega_n^2\right)^+, \left(s_n^2\right)^+\right)(\cdot+y_n)}{\left\|\left(\left(\omega_n^2\right)^+, \left(s_n^2\right)^+\right)(\cdot+y_n)\right\|} \right\|\\
&\quad\geq \left\| \frac{\left(\left(\omega_n^1\right)^+, \left(s_n^1\right)^+\right)}{\alpha_1} - \frac{\left(\left(\omega_n^2\right)^+, \left(s_n^2\right)^+\right)}{\alpha_2} \right\|
\end{aligned}$$

$$\geq \min\left\{\frac{\left\|\left(\left(\omega_n^1\right)^+,\left(s_n^1\right)^+\right)\right\|}{\alpha_1},\frac{\left\|\left(\left(\omega_n^2\right)^+,\left(s_n^2\right)^+\right)\right\|}{\alpha_2}\right\}\left\|\frac{\left(\left(\omega_n^1\right)^+,\left(s_n^1\right)^+\right)}{\left\|\left(\left(\omega_n^1\right)^+,\left(s_n^1\right)^+\right)\right\|}\right.$$
$$\left.-\frac{\left(\left(\omega_n^2\right)^+,\left(s_n^2\right)^+\right)}{\left\|\left(\left(\omega_n^2\right)^+,\left(s_n^2\right)^+\right)\right\|}\right\|$$
$$\geq \frac{\sqrt{2c_{\inf}}}{s\left(d\right)}\left\|m^-\left(\omega^1,s^1\right)-m^-\left(\omega^2,s^2\right)\right\|,$$

where $s(d) = \sup\{\|(u^+, v^+)\| \,|\, (u, v) \in \mathcal{N},\ J(u, v) \leq d\}$. It follows from Lemma 2.4 that $m - (\omega^1, s^1), m - (\omega^2, s^2)$ are critical points and we have (4.1). Note that if $(\omega^1, s^1) = \{0, 0\}$ or $(\omega^2, s^2) = \{0, 0\}$, then similarly as above, we prove that

$$\liminf_{n\to\infty}\left\|\left(u_n^1-u_n^2,v_n^1-v_n^2\right)\right\|\geq\frac{\sqrt{2c_{\inf}}}{s(d)}$$

and again (4.1) holds. The proof is complete. □

Proof of Theorem 1.2. It follows from Lemma 2.4(b) that (u, v) is equivariant with respect to the $\mathbb{Z}^N$-action given by $(u, v) \mapsto (u(\cdot - k), v(\cdot - k))$ for $k \in \mathbb{Z}^N$. Furthermore, J is even and (u, v) is odd. Thus, ψ is even and invariant with respect to the $\mathbb{Z}^N$-action. Let $\mathcal{F}$ be the set of geometrically distinct critical points of ψ and assume that $\mathcal{F}$ is finite. Then, similar to Lemma 2.13 in [18], we prove that

$$\inf\left\{\|(u_1-u_2,v_1-v_2)\|\,\middle|\,\psi'(u_1,v_1)=\psi'(u_2,v_2)=0,\ (u_1,v_1)\neq(u_2,v_2)\in S^+\right\}$$
$$>0.$$

The discreteness of the Palais–Smale sequences in Lemma 4.1 allows us to repeat the following arguments: Lemma 2.15, Lemma 2.16 and proof of Theorem 1.2 in [18]. In fact, we show that for any $k \in \mathbb{N}$, there exists $(u, v) \in S^+$ such that $\psi'(u, v) = 0$ and $\psi(u, v) = c_k$, where

$$c_k = \inf\left\{d \in \mathbb{R} | \gamma(\psi^d) \geq k\right\}$$

and γ denotes the usual Krasnoselskii genus (see [29]). Furthermore, $c_k < c_{k+1}$ for $k \in \mathbb{Z}^N$ and thus we get a contradiction (see [18] for detailed arguments). It follows from Lemma 2.4(b) that we get the existence of infinitely many geometrically distinct solutions to problem (1.1). The proof is complete. □

References

1. Silva, E.: Existence and multiplicity of solutions for semilinear elliptic systems. NoDEA **1**, 339–363 (1994)

2. Duan, S., Wu, X.: The existence of solutions for a class of semilinear elliptic systems. Nonlinear Anal. **73**, 2842–2854 (2010)
3. Qu, Z., Tang, C.: Existence and multiplicity results for some elliptic systems at resonance. Nonlinear Anal. **71**, 2660–2666 (2009)
4. Shi, H.X., Chen, H.B.: Ground state solutions for resonant cooperative elliptic systems with general superlinear terms. Mediterr. J. Math. **13**, 2897–2909 (2016)
5. Zhang, J., Zhang, Z.: Existence results for some nonlinear elliptic systems. Nonlinear Anal. **71**, 2840–2846 (2009)
6. Li, G., Tang, X.: Nehari-type state solutions for Schrödinger equations including critical exponent. Appl. Math. Lett. **37**, 101–106 (2014)
7. Che, G.F., Chen, H.B.: Multiplicity of small negative-energy solutions for a class of semilinear elliptic systems. Bound. Value. Probl. **2016**, 1–12 (2016)
8. Liao, F.F., Tang, X.H., Qin, D.D.: Super-quadratic conditions for periodic elliptic system on $\mathbb{R}^N$. Electron. J. Differ. Equ. **127**, 1–11 (2015)
9. Maia, L., Montefusco, E., Pellacci, B.: Positive solutions for a weakly coupled nonlinear Schrödinger system. J. Differ. Equ. **229**, 743–767 (2006)
10. Zhang, J., Tang, X.H., Zhang, W.: Ground-state solutions for superquadratic Hamiltonian elliptic systems with gradient terms. Nonlinear Anal. **95**, 1–10 (2014)
11. Zhang, J., Qin, W.P., Zhao, F.K.: Existence and multiplicity of solutions for asymptotically linear nonperiodic Hamiltonian elliptic system. J. Math. Anal. Appl. **399**, 433–441 (2013)
12. Chen, G.W., Ma, S.W.: Nonexistence and multiplicity of solutions for nonlinear elliptic systems in $\mathbb{R}^N$. Nonlinear Anal. **36**, 233–248 (2017)
13. Zhang, W., Zhang, J., Zhao, F.K.: Multiple solutions for asymptotically quadratic and superquadratic elliptic system of Hamiltonian type. Appl. Math. Comput. **263**, 36–46 (2015)
14. Shi, H.X., Chen, H.B.: Ground state solutions for asymptotically periodic coupled Kirchhoff-type systems with critical growth. Math. Methods Appl. Sci. **39**, 2193–2201 (2016)
15. Che, G.F., Chen, H.B.: Existence and multiplicity of systems of Kirchhoff-type equations with general potentials. Math. Methods Appl. Sci. **40**, 775–785 (2017)
16. Tang, X.H.: Ground state solutions of Nehari–Pankov type for a superlinear Hamiltonian elliptic system on $\mathbb{R}^N$. Canad. Math. Bull. **58**(3), 651–663 (2015)
17. Reed, M., Simon, B.: Methods of Modern Mathematical Physics, Analysis of Operators, vol. IV. Academic Press, New York (1978)
18. Szulkin, A., Weth, T.: Ground state solutions for some indefinite variational problems. J. Funct. Anal. **257**(12), 3802–3822 (2009)
19. Pankov, A.: On decay of solutions to nonlinear Schrödinger equations. Proc. Am. Math. Soc. **136**, 2565–2570 (2008)
20. Mederski, J.: Ground states of a system of nonlinear Schrödinger equations with periodic potentials. Commun. Partial Diff. Equ. **41**(9), 1426–1440 (2016)
21. Guo, Q., Mederski, J.: Ground states of nonlinear Schrödinger equations with sum of periodic and inverse square potentials. J. Differ. Equ. **260**, 4180–4202 (2016)
22. Mederski, J.: Solutions to a nonlinear Schrödinger equation with periodic potential and zero on the boundary of the spectrum. Topol. Methods Nonlinear Anal. **46**(2), 755–771 (2015)
23. Bieganowski, B., Mederski, J.: Nonlinear Schrödinger equations with sum of periodic and vanishing potentials and sign-changing nonlinearities. arXiv preprint arXiv:1602.05078
24. Bartsch, T., Ding, Y.: On a nonlinear Schrödinger equation with periodic potential. Math. Ann. **313**(1), 15–37 (1999)
25. Pankov, A.: Periodic nonlinear Schröinger equation with application to photonic crystals. Milan J. Math. **73**, 259–287 (2005)
26. Badiale, M., Pisani, L., Rolando, S.: Sum of weighted Lebesgue spaces and nonlinear elliptic equations. Nodea Nonlinear Differ. Equ. Appl. **18**, 369–405 (2011)
27. Bartsch, T., Mederski, J.: Ground and bound state solutions of semilinear time-harmonic Maxwell equations in a bounded domain. Arch. Ration. Mech. Anal. **215**(1), 283–306 (2015)
28. Willem, M.: Minimax Theorems. Birkhäuser Verlag, Basel (1996)
29. Struwe, M.: Variational Methods. Springer, Berlin (2008)