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Several New Estimates of the Minimum H-eigenvalue for Nonsingular $\mathcal{M}$-tensors

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Abstract In this paper, several new estimates of the minimum H-eigenvalue for weakly irreducible nonsingular $\mathcal{M}$-tensors, including the new Brauer-type estimates and the new S-type estimates, are derived. It is proved that the new estimates are tighter than some existing ones and numerical examples are given to verify this fact. The other main result of this paper is to provide a sharper Ky Fan-type theorem which is better than the original Ky Fan theorem for the nonsingular $\mathcal{M}$-tensors.

Keywords Nonsingular $\mathcal{M}$-tensors · Minimum H-eigenvalue · Ky Fan theorem

Mathematics Subject Classification 30E25 · 31A10 · 35C15 · 35J55

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1 Introduction

Eigenvalue problems of higher-order tensors have become an important topic of study in a new applied mathematics branch and numerical multilinear algebra, and they have a wide range of practical applications [1–5].

The class of $\mathcal{M}$-tensor introduced in [6,7] is the generalization M-matrices [8]. And some important properties of $\mathcal{M}$-tensors and nonsingular $\mathcal{M}$-tensors have been established in [7,9]. It is noteworthy that some applications of $\mathcal{M}$-tensors [6,7,9,10] are related to the eigenvalue problems of $\mathcal{M}$-tensors. In [11–14], some bounds for the minimum H-eigenvalue of nonsingular $\mathcal{M}$-tensors have been proposed. The main aim of this paper is to present some new bounds for the minimum H-eigenvalue of weakly irreducible nonsingular $\mathcal{M}$-tensors, and these bounds improve some existing ones.

Let $\mathbb{C}(\mathbb{R})$ denote the set of all complex (real) field and $N = \{1, 2, \ldots, n\}$. We consider an m-order n-dimensional tensor $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ consisting of n^m entries, denoted by $\mathcal{A} \in \mathbb{C}^{[m,n]}(\mathbb{R}^{[m,n]})$, if

$$a_{i_1 i_2 \ldots i_m} \in \mathbb{C}(\mathbb{R}),$$

where $i_j = 1, 2, \ldots, n$ for $j = 1, 2, \ldots, m$ [9,15]. Obviously, a vector is a tensor of order 1 and a matrix is a tensor of order 2. Moreover, an m-order n-dimensional tensor $\mathcal{I} = (\delta_{i_1 i_2 \ldots i_m})$ is called the unit tensor [16], if its entries are $\delta_{i_1 \ldots i_m}$ for $i_1, \ldots, i_m \in N$, where

$$\delta_{i_1 i_2 \ldots i_m} = \begin{cases} 1 & \text{if } i_1 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{A} \in \mathbb{C}^{[m,n]}$, if there exist a number $\lambda \in \mathbb{C}$ and a nonzero vector $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{C}^n$ that are solutions of the following homogeneous polynomial equations:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then λ is an eigenvalue of $\mathcal{A}$ and x is the eigenvector of $\mathcal{A}$ associated with λ [1,15,17,18], where $\mathcal{A}x^{m-1}$ and $\lambda x^{[m-1]}$ are vectors, whose ith components are

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\ldots,i_m \in N} a_{i i_2 \ldots i_m} x_{i_2} \ldots x_{i_m}$$

and

$$(x^{[m-1]})_i = x_i^{m-1}.$$

Furthermore, if λ and x are restricted to the real field, then we call λ an H-eigenvalue of $\mathcal{A}$ and x an H-eigenvector of $\mathcal{A}$ associated with λ [15].

Let Γ be a digraph with vertex set V and arc set E. If there exist directed paths from i to j and j to i for any $i, j \in V (i \neq j)$, then Γ is called strongly connected.

For each vertex $i \in V$, if there exists a circuit such that i belong to the circuit, then Γ is called weakly connected. For a tensor $\mathcal{A} = (a_{i_1 \ldots i_m}) \in \mathbb{C}^{[m,n]}$, we associate $\mathcal{A}$ with a digraph $\Gamma_{\mathcal{A}}$ as follows. The vertex set of $\Gamma_{\mathcal{A}}$ is $V(\mathcal{A}) = \{1, \ldots, n\}$, and the arc set of $\Gamma_{\mathcal{A}}$ is $E = \{(i, j) | a_{ii_2 \ldots i_m} \neq 0, j \in \{i_2 \ldots, i_m\} \neq \{i, \ldots, i\}\}$. Let $C(\mathcal{A})$ denote the set of circuits of $\Gamma_{\mathcal{A}}$. A tensor $\mathcal{A}$ is called weakly irreducible if $\Gamma_{\mathcal{A}}$ is strongly connected [19–21]. The tensor $\mathcal{A}$ is called reducible if there exists a nonempty proper index subset $J \in N$ such that $a_{i_1 i_2 \ldots i_m} = 0, \forall i_1 \in J, \forall i_2, \ldots, i_m \notin J$. If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible [22].

Let $\rho(A) = \max\{|\lambda| : \lambda$ is an eigenvalue of $A\}$, where $|\lambda|$ denotes the modulus of λ. We call $\rho(A)$ the spectral radius of tensor $\mathcal{A}$ [23]. An m-order n-dimensional tensor $\mathcal{A}$ is called nonnegative [1,2,16,23,24], if each entry is nonnegative. We call a tensor $\mathcal{A}$ a $\mathcal{Z}$-tensor, if all of its off-diagonal entries are nonpositive, which is equivalent to write $\mathcal{A} = s\mathcal{I} - \mathcal{B}$, where $s > 0$ and $\mathcal{B}$ is a nonnegative tensor ($\mathcal{B} \geq 0$), and the set of m-order and n-dimensional $\mathcal{Z}$-tensors is denoted by $\mathbb{Z}$. A $\mathcal{Z}$-tensor $\mathcal{A} = s\mathcal{I} - \mathcal{B}$ is an $\mathcal{M}$-tensor if $s \geq \rho(\mathcal{B})$, and it is a nonsingular (strong) $\mathcal{M}$-tensor if $s > \rho(\mathcal{B})$ [6,7,9].

Denote by $\tau(\mathcal{A})$ the minimum value of the real part of all eigenvalues of the tensor $\mathcal{A}$. Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}$. For $i, j \in N, j \neq i$, we denote

$$R_i(\mathcal{A}) = \sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\ldots i_m}, \quad R_{\max}(\mathcal{A}) = \max_{i \in N} R_i(\mathcal{A}), \quad R_{\min}(\mathcal{A}) = \min_{i \in N} R_i(\mathcal{A}),$$

$$r_i(\mathcal{A}) = \sum_{\delta_{ii_2\ldots i_m}=0} |a_{ii_2\ldots i_m}|, \quad r_i^j(\mathcal{A}) = \sum_{\substack{\delta_{ii_2\ldots i_m}=0, \\ \delta_{ji_2\ldots i_m}=0}} |a_{ii_2\ldots i_m}| = r_i(\mathcal{A}) - |a_{ij\ldots j}|.$$

In recent years, much literature has focused on the bounds of the minimum H-eigenvalue of nonsingular $\mathcal{M}$-tensors. In [11], He and Huang first proposed the upper and lower bounds for the minimum H-eigenvalue of irreducible nonsingular $\mathcal{M}$-tensors as follows.

Lemma 1.1 [11] *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be an irreducible nonsingular $\mathcal{M}$-tensor. Then*

$$R_{\min}(\mathcal{A}) \leq \tau(\mathcal{A}) \leq R_{\max}(\mathcal{A}).$$

Lemma 1.2 [11] *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be an irreducible nonsingular $\mathcal{M}$-tensor. Then*

$$\min_{\substack{i,j\in N, \\ j\neq i}} \frac{1}{2}\left\{a_{i\ldots i} + a_{j\ldots j} - r_i^j(\mathcal{A}) - \Delta_{ij}^{\frac{1}{2}}(\mathcal{A})\right\} \leq \tau(\mathcal{A})$$

$$\leq \max_{\substack{i,j\in N, \\ j\neq i}} \frac{1}{2}\left\{a_{i\ldots i} + a_{j\ldots j} - r_i^j(\mathcal{A}) - \Delta_{ij}^{\frac{1}{2}}(\mathcal{A})\right\},$$

where

$$\Delta_{ij}(\mathcal{A}) = (a_{i\ldots i} - a_{j\ldots j} + r_i^j(\mathcal{A}))^2 - 4a_{ij\ldots j} r_j(\mathcal{A}).$$

Recently, Zhao and Sang in [13] pointed out that there are some errors in the calculation process of Lemma 1.2, and the correction is as follows:

Lemma 1.3 [13] *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be an irreducible nonsingular $\mathcal{M}$-tensor. Then*

$$\tau(\mathcal{A}) \geq \min_{\substack{i,j\in N,\\ j\neq i}} L_{ij}(A),$$

where

$$L_{ij}(A)=\frac{1}{2}\left\{a_{i\ldots i}+a_{j\ldots j}-r_i^j(\mathcal{A})-\left[(a_{i\ldots i}-a_{j\ldots j}-r_i^j(\mathcal{A}))^2-4a_{ij\ldots j}r_j(\mathcal{A})\right]^{\frac{1}{2}}\right\}.$$

In addition, Wang and Wei presented the upper and lower bounds on $\tau(\mathcal{A})$ for a weakly irreducible nonsingular $\mathcal{M}$-tensor as follows.

Lemma 1.4 [12] *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a weakly irreducible nonsingular $\mathcal{M}$-tensor. Then*

$$\min_{\substack{i,j\in N,\\ j\neq i}} \frac{1}{2}\left\{a_{i\ldots i}+a_{j\ldots j}-\widetilde{r}_i(\mathcal{A})-\widetilde{\Delta}_{ij}^{\frac{1}{2}}(\mathcal{A})\right\} \leq \tau(\mathcal{A})$$
$$\leq \max_{\substack{i,j\in N,\\ j\neq i}} \frac{1}{2}\left\{a_{i\ldots i}+a_{j\ldots j}-\widetilde{r}_i(\mathcal{A})-\widetilde{\Delta}_{ij}^{\frac{1}{2}}(\mathcal{A})\right\},$$

where

$$(M(\mathcal{A}))_{ij}=\begin{cases} r_i(\mathcal{A}) & i=j,\\ |a_{ij\ldots j}|, & i\neq j.\end{cases}$$

is a nonnegative matrix and $\widetilde{\Delta}_{ij}(\mathcal{A})=(a_{i\ldots i}-a_{j\ldots j}-\widetilde{r}_i(\mathcal{A}))^2+4r_i(M(\mathcal{A}))r_j(\mathcal{A})$, with $r_i(M(\mathcal{A}))=\sum_{j\neq i}(M(\mathcal{A}))_{ij}, \widetilde{r}_i(\mathcal{A})=r_i(\mathcal{A})-r_i(M(\mathcal{A}))$.

In this paper, we continue this research on the estimates of the minimum H-eigenvalue for weakly irreducible nonsingular $\mathcal{M}$-tensors; inspired by the ideas of [25,26], we obtain two new estimates of the minimum H-eigenvalue for weakly irreducible nonsingular $\mathcal{M}$-tensors. They are proved to be tighter than Lemmas 1.1 and 1.2 in corrected form. Finally, we derive a sharper bound in Ky Fan theorem for nonsingular $\mathcal{M}$-tensors.

The remainder of the paper is organized as follows. In Sect. 2, we recollect some useful lemmas on tensors which are utilized in the following proofs, then focus on the estimates of $\tau(\mathcal{A})$ and establish some new bounds for $\tau(\mathcal{A})$. In Sect. 3, a sharper bound in Ky Fan theorem is obtained. Finally, some conclusions are given to end this paper in Sect. 4.

2 Several New Estimates of the Minimum H-eigenvalue

In this section, we give several new estimates of the minimum H-eigenvalue for weakly irreducible nonsingular $\mathcal{M}$-tensors.

Lemma 2.1 [12] *If a tensor $\mathcal{A}$ is irreducible, then $\mathcal{A}$ is weakly irreducible.*

Lemma 2.2 [11] *Let $\mathcal{A}$ be a nonsingular $\mathcal{M}$-tensor and denote by $\tau(\mathcal{A})$ the minimum value of the real part of all eigenvalues of $\mathcal{A}$. Then $\tau(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with a nonnegative eigenvector. Moreover, if $\mathcal{A}$ is irreducible, then $\tau(\mathcal{A})$ is the unique eigenvalue with a positive eigenvector.*

Zhang et al. [6] obtained some results similar to those of Lemma 2.2 for weakly irreducible nonsingular $\mathcal{M}$-tensors in the following lemma.

Lemma 2.3 [6] *Let $\mathcal{A}$ be a nonsingular $\mathcal{M}$-tensor and denote by $\tau(\mathcal{A})$ the minimum value of the real part of all eigenvalues of $\mathcal{A}$. Then $\tau(\mathcal{A})$ is an H-eigenvalue of $\mathcal{A}$ with a nonnegative eigenvector. Moreover, if $\mathcal{A}$ is a weakly irreducible $\mathcal{Z}$-tensor, then $\tau(\mathcal{A})$ is the unique eigenvalue with a positive eigenvector.*

Lemma 2.4 [27] *Let $\mathcal{A}$ be a weakly irreducible nonsingular $\mathcal{M}$-tensor. Then $\tau(\mathcal{A}) < \min_{i\in N}\{a_{ii\ldots i}\}$.*

For any given diagonal nonsingular matrix $D = \text{diag}(d_1, \ldots, d_n)$, we define a tensor $\mathcal{A}_D$ as follows:

$$\mathcal{A}_D = \mathcal{A} \times_1 D^{1-m} \times_2 D \times_3 \cdots \times_m D,$$

where $\times_k$ is k-mode tensor-matrix multiplication between $\mathcal{A}$ and D [28]. Here the entries of $\mathcal{A}_D$ are given by [9] as follows:

$$(\mathcal{A}_D)_{i_1 i_2 \ldots i_m} = \mathcal{A}_{i_1 i_2 \ldots i_m} d_1^{1-m} d_2 \ldots d_m, \quad 1 \le i_1, i_2, \ldots, i_m \le n.$$

Lemma 2.5 [23] *The tensors $\mathcal{A}_D$ and $\mathcal{A}$ have the same set of eigenvalues.*

Lemma 2.6 *Let $f(x) = a_1 x^2 + b_1 x + c_1$, $g(x) = a_2 x^2 + b_2 x + c_2$, where $a_1 > 0$ and $a_2 > 0$. Assume that x_1, x_2 and $\widetilde{x}_1, \widetilde{x}_2$ are roots of $f(x) = 0$ and $g(x) = 0$, respectively. Then the solution of $f(x) \le 0$ is $[x_1, x_2]$, and that of $g(x) \le 0$ is $[\widetilde{x}_1, \widetilde{x}_2]$. If $g(x) \le 0$ under the condition $f(x) \le 0$, then $[x_1, x_2] \subseteq [\widetilde{x}_1, \widetilde{x}_2]$.*

Proof It is obvious that $[x_1, x_2]$ and $[\widetilde{x}_1, \widetilde{x}_2]$ are the solutions of $f(x) \le 0$ and $g(x) \le 0$, respectively. Since $g(x) \le 0$ under the condition $f(x) \le 0$, we get $g(x) \le 0$ for any $x \in [x_1, x_2]$. And because $[\widetilde{x}_1, \widetilde{x}_2]$ is the solution of $g(x) \le 0$, we can obtain that $x \in [\widetilde{x}_1, \widetilde{x}_2]$, i.e., $[x_1, x_2] \subseteq [\widetilde{x}_1, \widetilde{x}_2]$. □

Lemma 2.7 ([29], Lemmas 2.2 and 2.3) *Let $a, b, c \ge 0$ and $d > 0$.*

(I) *If $\frac{a}{b+c+d} \le 1$, then*

$$\frac{a-(b+c)}{d} \le \frac{a-b}{c+d} \le \frac{a}{b+c+d}.$$

(II) *If* $\frac{a}{b+c+d} \geq 1$, *then*

$$\frac{a-(b+c)}{d} \geq \frac{a-b}{c+d} \geq \frac{a}{b+c+d}.$$

2.1 The New Brauer-Type Estimates of Minimum H-eigenvalue

In this subsection, we present the new Brauer-type estimates of minimum H-eigenvalue for weakly irreducible nonsingular $\mathcal{M}$-tensors, which are tighter than the results in Lemmas 1.1 and 1.2 in corrected form.

We denote

$$\Delta_i = \{(i_2, i_3, \ldots, i_m) : i_j = i \text{ for some } j \in \{2, \ldots, m\}, \text{ where } i, i_2, \ldots, i_m \in N\},$$
$$\overline{\Delta}_i = \{(i_2, i_3, \ldots, i_m) : i_j \neq i \text{ for any } j \in \{2, \ldots, m\}, \text{ where } i, i_2, \ldots, i_m \in N\},$$

and let

$$r_i^{\Delta_i}(\mathcal{A}) = \sum_{\substack{(i_2,\ldots,i_m)\in\Delta_i,\\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\ldots i_m}|, \quad r_i^{\overline{\Delta}_i}(\mathcal{A}) = \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}_i} |a_{ii_2\ldots i_m}|.$$

Then, $r_i(\mathcal{A}) = r_i^{\Delta_i}(\mathcal{A}) + r_i^{\overline{\Delta}_i}(\mathcal{A})$.

Theorem 2.1 *Let* $\mathcal{A} = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}$ *be a weakly irreducible nonsingular* $\mathcal{M}$-*tensor with* $n \geq 2$. *Then*

$$\Lambda_{\min} \leq \tau(\mathcal{A}) \leq \overline{\Lambda}_{\max},$$

where

$$\Lambda_{\min} = \min\{\widetilde{\Lambda}_{\min}, \overline{\Lambda}_{\min}\}, \quad \widetilde{\Lambda}_{\min} = \min_{i\in N}\{a_{ii\ldots i} - r_i^{\Delta_i}(\mathcal{A})\},$$
$$\overline{\Lambda}_{\min} = \min_{\substack{i,j\in N,\\ j\neq i}} \max\{\frac{1}{2}(a_{ii\ldots i} + a_{jj\ldots j} - r_i^{\Delta_i}(\mathcal{A}) - r_j^{\overline{\Delta}_i}(\mathcal{A}) - \Omega_{i,j}^{\frac{1}{2}}), R_i(\mathcal{A})\},$$
$$\overline{\Lambda}_{\max} = \max_{\substack{i,j\in N,\\ j\neq i}} \min\{\frac{1}{2}(a_{ii\ldots i} + a_{jj\ldots j} - r_i^{\Delta_i}(\mathcal{A}) - r_j^{\overline{\Delta}_i}(\mathcal{A}) - \Omega_{i,j}^{\frac{1}{2}}), R_i(\mathcal{A})\},$$
$$\Omega_{i,j} = (a_{ii\ldots i} - a_{jj\ldots j} - r_i^{\Delta_i}(\mathcal{A}) + r_j^{\overline{\Delta}_i}(\mathcal{A}))^2 + 4r_i^{\overline{\Delta}_i}(\mathcal{A})r_j^{\Delta_i}(\mathcal{A}).$$

Proof Since $\mathcal{A}$ is weakly irreducible nonsingular $\mathcal{M}$-tensor, by Lemma 2.3, there exists $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} > 0$ such that

$$\mathcal{A}x^{m-1} = \tau(\mathcal{A})x^{[m-1]}. \tag{2.1}$$

Now, the proof is proceeded in two steps.

(i) Let $x_t \geq x_l \geq \max\{x_k : k \in N, k \neq t, k \neq l\}$ (where the last term above is defined to be zero if $n = 2$). From (2.1), we have

$$(a_{tt\dots t} - \tau(\mathcal{A}))x_t^{m-1} = -\sum_{\substack{(i_2,\dots,i_m)\in\Delta_t,\\ \delta_{ti_2\dots i_m}=0}} a_{ti_2\dots i_m} x_{i_2} x_{i_3} \dots x_{i_m} - \sum_{(i_2,\dots,i_m)\in\overline{\Delta}_t} a_{ti_2\dots i_m} x_{i_2} x_{i_3} \dots x_{i_m}.$$

Using the inequality technique gives

$$\begin{aligned}(a_{tt\dots t} - \tau(\mathcal{A}))x_t^{m-1} &= \sum_{\substack{(i_2,\dots,i_m)\in\Delta_t,\\ \delta_{ti_2\dots i_m}=0}} |a_{ti_2\dots i_m}| x_{i_2} x_{i_3} \dots x_{i_m} + \sum_{(i_2,\dots,i_m)\in\overline{\Delta}_t} |a_{ti_2\dots i_m}| x_{i_2} x_{i_3} \dots x_{i_m} \\ &\leq \sum_{\substack{(i_2,\dots,i_m)\in\Delta_t,\\ \delta_{ti_2\dots i_m}=0}} |a_{ti_2\dots i_m}| x_t^{m-1} + \sum_{(i_2,\dots,i_m)\in\overline{\Delta}_t} |a_{ti_2\dots i_m}| x_l^{m-1} \\ &= r_t^{\Delta_t}(\mathcal{A}) x_t^{m-1} + r_t^{\overline{\Delta}_t}(\mathcal{A}) x_l^{m-1}.\end{aligned}$$

Equivalently

$$(a_{tt\dots t} - \tau(\mathcal{A}) - r_t^{\Delta_t}(\mathcal{A}))x_t^{m-1} \leq r_t^{\overline{\Delta}_t}(\mathcal{A}) x_l^{m-1}.$$

If $a_{tt\dots t} - \tau(\mathcal{A}) - r_t^{\Delta_t}(\mathcal{A}) \leq 0$, then

$$\tau(\mathcal{A}) \geq a_{tt\dots t} - r_t^{\Delta_t}(\mathcal{A}) \geq \min_{i\in N}\{a_{ii\dots i} - r_i^{\Delta_i}(\mathcal{A})\}. \tag{2.2}$$

Otherwise, we have $a_{tt\dots t} - \tau(\mathcal{A}) - r_t^{\Delta_t}(\mathcal{A}) > 0$, which means that

$$0 < (a_{tt\dots t} - \tau(\mathcal{A}) - r_t^{\Delta_t}(\mathcal{A}))x_t^{m-1} \leq r_t^{\overline{\Delta}_t}(\mathcal{A}) x_l^{m-1}. \tag{2.3}$$

On the other hand, by (2.1) we can get

$$\begin{aligned}(a_{ll\dots l} - \tau(\mathcal{A}))x_l^{m-1} &= \sum_{(i_2,\dots,i_m)\in\Delta_t} |a_{li_2\dots i_m}| x_{i_2} x_{i_3} \dots x_{i_m} + \sum_{\substack{(i_2,\dots,i_m)\in\overline{\Delta}_t,\\ \delta_{li_2\dots i_m}=0}} |a_{li_2\dots i_m}| x_{i_2} x_{i_3} \dots x_{i_m} \\ &\leq \sum_{(i_2,\dots,i_m)\in\Delta_t} |a_{li_2\dots i_m}| x_t^{m-1} + \sum_{\substack{(i_2,\dots,i_m)\in\overline{\Delta}_t,\\ \delta_{li_2\dots i_m}=0}} |a_{li_2\dots i_m}| x_l^{m-1} \\ &= r_l^{\Delta_t}(\mathcal{A}) x_t^{m-1} + r_l^{\overline{\Delta}_t}(\mathcal{A}) x_l^{m-1},\end{aligned}$$

i.e.,

$$(a_{ll\dots l} - \tau(\mathcal{A}) - r_l^{\overline{\Delta}_t}(\mathcal{A}))x_l^{m-1} \leq r_l^{\Delta_t}(\mathcal{A}) x_t^{m-1}. \tag{2.4}$$

Multiplying Inequalities (2.3) and (2.4) yields

$$(a_{tt\ldots t}-\tau(\mathcal{A})-r_t^{\Delta_t}(\mathcal{A}))(a_{ll\ldots l}-\tau(\mathcal{A})-r_l^{\overline{\Delta}_t}(\mathcal{A}))x_t^{m-1}x_l^{m-1} \\ \leq r_t^{\overline{\Delta}_t}(\mathcal{A})r_l^{\Delta_t}(\mathcal{A})x_t^{m-1}x_l^{m-1}.$$

Note that $x_t^{m-1}x_l^{m-1}>0$, thus

$$(a_{tt\ldots t}-\tau(\mathcal{A})-r_t^{\Delta_t}(\mathcal{A}))(a_{ll\ldots l}-\tau(\mathcal{A})-r_l^{\overline{\Delta}_t}(\mathcal{A}))\leq r_t^{\overline{\Delta}_t}(\mathcal{A})r_l^{\Delta_t}(\mathcal{A}),$$

which is equivalent to

$$\tau(\mathcal{A})^2-(a_{tt\ldots t}+a_{ll\ldots l}-r_t^{\Delta_t}(\mathcal{A})-r_l^{\overline{\Delta}_t}(\mathcal{A}))\tau(\mathcal{A}) \\ +(a_{tt\ldots t}-r_t^{\Delta_t}(\mathcal{A}))(a_{ll\ldots l}-r_l^{\overline{\Delta}_t}(\mathcal{A}))-r_t^{\overline{\Delta}_t}(\mathcal{A})r_l^{\Delta_t}(\mathcal{A})\leq 0.$$

This gives the following bounds for $\tau(\mathcal{A})$,

$$\tau(\mathcal{A})\geq\frac{1}{2}\left(a_{tt\ldots t}+a_{ll\ldots l}-r_t^{\Delta_t}(\mathcal{A})-r_l^{\overline{\Delta}_t}(\mathcal{A})-\Omega_{t,l}^{\frac{1}{2}}\right), \tag{2.5}$$

where

$$\Omega_{t,l}=(a_{tt\ldots t}-a_{ll\ldots l}-r_t^{\Delta_t}(\mathcal{A})+r_l^{\overline{\Delta}_t}(\mathcal{A}))^2+4r_t^{\overline{\Delta}_t}(\mathcal{A})r_l^{\Delta_t}(\mathcal{A}).$$

Furthermore, by Inequality (2.3), we can get that

$$a_{tt\ldots t}-\tau(\mathcal{A})-r_t^{\Delta_t}(\mathcal{A})\leq r_t^{\overline{\Delta}_t}(\mathcal{A});$$

consequently,

$$\tau(\mathcal{A})\geq a_{tt\ldots t}-r_t^{\Delta_t}(\mathcal{A})-r_t^{\overline{\Delta}_t}(\mathcal{A})=a_{tt\ldots t}-r_t(\mathcal{A})=R_t(\mathcal{A}). \tag{2.6}$$

Combining Inequalities (2.5) and (2.6), we have

$$\tau(\mathcal{A})\geq\max\left\{\frac{1}{2}(a_{tt\ldots t}+a_{ll\ldots l}-r_t^{\Delta_t}(\mathcal{A})-r_l^{\overline{\Delta}_t}(\mathcal{A})-\Omega_{t,l}^{\frac{1}{2}}),\,R_t(\mathcal{A})\right\} \\ \geq\min_{\substack{i,j\in N,\\ j\neq i}}\max\left\{\frac{1}{2}(a_{ii\ldots i}+a_{jj\ldots j}-r_i^{\Delta_i}(\mathcal{A})-r_j^{\overline{\Delta}_i}(\mathcal{A})-\Omega_{i,j}^{\frac{1}{2}}),\,R_i(\mathcal{A})\right\}. \tag{2.7}$$

The first inequality in Theorem 2.1 follows from Inequalities (2.2) and (2.7).

(ii) Let $x_p\leq x_q\leq\min\{x_k:k\in N,k\neq p,k\neq q\}$. By (2.1), we derive that

$$(a_{pp\ldots p}-\tau(\mathcal{A}))x_p^{m-1}=-\sum_{\substack{(i_2,\ldots,i_m)\in\Delta_p,\\ \delta_{pi_2\ldots i_m=0}}}a_{pi_2\ldots i_m}x_{i_2}x_{i_3}\ldots x_{i_m}$$

$$- \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}_p} a_{pi_2\ldots i_m} x_{i_2} x_{i_3}\ldots x_{i_m}$$

and

$$\begin{aligned}(a_{qq\ldots q} - \tau(\mathcal{A}))x_q^{m-1} = &- \sum_{(i_2,\ldots,i_m)\in\Delta_p} a_{qi_2\ldots i_m} x_{i_2} x_{i_3}\ldots x_{i_m}\\ &- \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta}_p,\\ \delta_{qi_2\ldots i_m}=0}} a_{qi_2\ldots i_m} x_{i_2} x_{i_3}\ldots x_{i_m}.\end{aligned}$$

Using the inequality technique gives

$$\begin{aligned}(a_{pp\ldots p} - \tau(\mathcal{A}))x_p^{m-1} &= \sum_{\substack{(i_2,\ldots,i_m)\in\Delta_p,\\ \delta_{pi_2\ldots i_m}=0}} |a_{pi_2\ldots i_m}| x_{i_2} x_{i_3}\ldots x_{i_m} + \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}_p} |a_{pi_2\ldots i_m}| x_{i_2} x_{i_3}\ldots x_{i_m}\\ &\geq \sum_{\substack{(i_2,\ldots,i_m)\in\Delta_p,\\ \delta_{pi_2\ldots i_m}=0}} |a_{pi_2\ldots i_m}| x_p^{m-1} + \sum_{(i_2,\ldots,i_m)\in\overline{\Delta}_p} |a_{pi_2\ldots i_m}| x_q^{m-1}\\ &= r_p^{\Delta_p}(\mathcal{A})x_p^{m-1} + r_p^{\overline{\Delta}_p}(\mathcal{A})x_q^{m-1} \qquad (2.8)\end{aligned}$$

and

$$\begin{aligned}(a_{qq\ldots q} - \tau(\mathcal{A}))x_q^{m-1} &= \sum_{(i_2,\ldots,i_m)\in\Delta_p} |a_{qi_2\ldots i_m}| x_{i_2} x_{i_3}\ldots x_{i_m} + \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta}_p,\\ \delta_{qi_2\ldots i_m}=0}} |a_{qi_2\ldots i_m}| x_{i_2} x_{i_3}\ldots x_{i_m}\\ &\geq \sum_{(i_2,\ldots,i_m)\in\Delta_p} |a_{qi_2\ldots i_m}| x_p^{m-1} + \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta}_p,\\ \delta_{qi_2\ldots i_m}=0}} |a_{qi_2\ldots i_m}| x_q^{m-1}\\ &= r_q^{\Delta_p}(\mathcal{A})x_p^{m-1} + r_q^{\overline{\Delta}_p}(\mathcal{A})x_q^{m-1}. \qquad (2.9)\end{aligned}$$

Combining Inequalities (2.8) and (2.9) and using the same method as the proof in (i), we can deduce the following result:

$$\begin{aligned}\tau(\mathcal{A}) &\leq \min\left\{\frac{1}{2}\left(a_{pp\ldots p} + a_{qq\ldots q} - r_p^{\Delta_p}(\mathcal{A}) - r_q^{\overline{\Delta}_p}(\mathcal{A}) - \Omega_{p,q}^{\frac{1}{2}}\right), R_p(\mathcal{A})\right\}\\ &\leq \max_{\substack{i,j\in N,\\ j\neq i}} \min\left\{\frac{1}{2}\left(a_{ii\ldots i} + a_{jj\ldots j} - r_i^{\Delta_i}(\mathcal{A}) - r_j^{\overline{\Delta}_i}(\mathcal{A}) - \Omega_{i,j}^{\frac{1}{2}}\right), R_i(\mathcal{A})\right\}.\end{aligned}$$

This completes our proof of Theorem 2.1. □

We now give the following comparison theorem for Theorem 2.1 and Lemma 1.2 in corrected form. First, we prove that the lower bound of Theorem 2.1 is better than that of Lemma 1.2 in corrected form.

Theorem 2.2 *Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m}) \in \mathbb{R}^{[m,n]}$ be a weakly irreducible nonsingular $\mathcal{M}$-tensor with $n \geq 2$. Then*

$$\Lambda_{\min} \geq \min_{\substack{i,j \in N,\\ j \neq i}} L_{ij}(\mathcal{A}).$$

Proof From proof of Lemma 1.3, we can see that $\tau(\mathcal{A}) \geq \min_{\substack{i,j \in N,\\ j \neq i}} L_{ij}(\mathcal{A})$ is obtained by solving the following quadratic inequality

$$(a_{ii\ldots i} - \tau(\mathcal{A}) - r_i^j(\mathcal{A}))(a_{jj\ldots j} - \tau(\mathcal{A})) \leq -a_{ij\ldots j} r_j(\mathcal{A}).$$

Let $g^{ij}(\tau(\mathcal{A})) = (a_{ii\ldots i} - \tau(\mathcal{A}) - r_i^j(\mathcal{A}))(a_{jj\ldots j} - \tau(\mathcal{A})) - (-a_{ij\ldots j}) r_j(\mathcal{A})$, and the left solution of $g^{ij}(\tau(\mathcal{A})) = 0$ is $L_{ij}(\mathcal{A})$. If $\Lambda_{\min} = \widetilde{\Lambda}_{\min} = \min_{i \in N}\{a_{ii\ldots i} - r_i^{\Delta_i}(\mathcal{A})\}$, then there exists $i_0 \in N$ such that

$$\Lambda_{\min} = \widetilde{\Lambda}_{\min} = a_{i_0\ldots i_0} - r_{i_0}^{\Delta_{i_0}}(\mathcal{A}).$$

From Theorem 2.1, we get

$$\tau(\mathcal{A}) \geq \Lambda_{\min} = a_{i_0\ldots i_0} - r_{i_0}^{\Delta_{i_0}}(\mathcal{A}),$$

which together with Lemma 2.4 results in

$$g^{i_0 j}(\tau(\mathcal{A})) = (a_{i_0\ldots i_0} - \tau(\mathcal{A}) - r_{i_0}^j(\mathcal{A}))(a_{jj\ldots j} - \tau(\mathcal{A})) - (-a_{i_0 j\ldots j}) r_j(\mathcal{A}) \leq 0.$$

By Lemma 2.6, we derive that

$$\Lambda_{\min} = a_{i_0\ldots i_0} - r_{i_0}^{\Delta_{i_0}}(\mathcal{A}) \geq L_{i_0 j}(\mathcal{A}) \geq \min_{\substack{i,j \in N,\\ j \neq i}} L_{ij}(\mathcal{A}). \tag{2.10}$$

If $\Lambda_{\min} = \overline{\Lambda}_{\min} = \min_{\substack{i,j \in N,\\ j \neq i}} \max\{\frac{1}{2}(a_{ii\ldots i} + a_{jj\ldots j} - r_i^{\Delta_i}(\mathcal{A}) - r_j^{\overline{\Delta}_i}(\mathcal{A}) - \Omega_{i,j}^{\frac{1}{2}}), R_i(\mathcal{A})\}$, then there exist $i_1, j_1 \in N$ such that

$$\tau(\mathcal{A}) \geq \Lambda_{\min} = \max\left\{\frac{1}{2}\left(a_{i_1\ldots i_1} + a_{j_1\ldots j_1} - r_{i_1}^{\Delta_{i_1}}(\mathcal{A}) - r_{j_1}^{\overline{\Delta}_{i_1}}(\mathcal{A}) - \Omega_{i_1,j_1}^{\frac{1}{2}}\right), R_{i_1}(\mathcal{A})\right\}, \tag{2.11}$$

which means that

$$\tau(\mathcal{A}) \geq R_{i_1}(\mathcal{A}) \tag{2.12}$$

and

$$\tau(\mathcal{A}) \geq \frac{1}{2}\left(a_{i_1\ldots i_1} + a_{j_1\ldots j_1} - r_{i_1}^{\Delta_{i_1}}(\mathcal{A}) - r_{j_1}^{\overline{\Delta}_{i_1}}(\mathcal{A}) - \Omega_{i_1,j_1}^{\frac{1}{2}}\right). \tag{2.13}$$

By proof of Theorem 2.1, we see that $K_{i_1j_1}(\mathcal{A}) := \frac{1}{2}(a_{i_1\ldots i_1} + a_{j_1\ldots j_1} - r_{i_1}^{\Delta_{i_1}}(\mathcal{A}) - r_{j_1}^{\overline{\Delta}_{i_1}}(\mathcal{A}) - \Omega_{i_1,j_1}^{\frac{1}{2}})$ is the left root of the following equation

$$(a_{i_1\ldots i_1} - \tau(\mathcal{A}) - r_{i_1}^{\Delta_{i_1}}(\mathcal{A}))(a_{j_1\ldots j_1} - \tau(\mathcal{A}) - r_{j_1}^{\overline{\Delta}_{i_1}}(\mathcal{A})) - r_{i_1}^{\overline{\Delta}_{i_1}}(\mathcal{A})r_{j_1}^{\Delta_{i_1}}(\mathcal{A}) = 0,$$

so, we let

$$\begin{aligned} f^{i_1j_1}(\tau(\mathcal{A})) := &\left(a_{i_1\ldots i_1} - \tau(\mathcal{A}) - r_{i_1}^{\Delta_{i_1}}(\mathcal{A})\right)\left(a_{j_1\ldots j_1} - \tau(\mathcal{A}) - r_{j_1}^{\overline{\Delta}_{i_1}}(\mathcal{A})\right) \\ &- r_{i_1}^{\overline{\Delta}_{i_1}}(\mathcal{A})r_{j_1}^{\Delta_{i_1}}(\mathcal{A}). \end{aligned}$$

By Lemma 2.6, if $g^{i_1j_1}(\tau(\mathcal{A})) \leq 0$ under the condition $f^{i_1j_1}(\tau(\mathcal{A})) \leq 0$, then $K_{i_1j_1}(\mathcal{A}) \geq L_{i_1j_1}(\mathcal{A}) \geq \min_{i,j\in N, j\neq i} L_{ij}(\mathcal{A})$. Combining with (2.11), we can derive that $\Lambda_{\min} \geq \min\limits_{\substack{i,j\in N,\\ j\neq i}} L_{ij}(\mathcal{A})$. Therefore, now we only need to prove that $g^{i_1j_1}(\tau(\mathcal{A})) \leq 0$ under the condition $f^{i_1j_1}(\tau(\mathcal{A})) \leq 0$.

When $a_{i_1\ldots i_1} - \tau(\mathcal{A}) - r_{i_1}^{\Delta_{i_1}}(\mathcal{A}) \leq 0$, it is not difficult to get the following form

$$g^{i_1j_1}(\tau(\mathcal{A})) = (a_{i_1\ldots i_1} - \tau(\mathcal{A}) - r_{i_1}^{j_1}(\mathcal{A}))(a_{j_1\ldots j_1} - \tau(\mathcal{A})) - (-a_{i_1j_1\ldots j_1})r_{j_1}(\mathcal{A}) \leq 0.$$

Otherwise, we have $a_{i_1\ldots i_1} - \tau(\mathcal{A}) - r_{i_1}^{\Delta_{i_1}}(\mathcal{A}) > 0$. From the condition $f^{i_1j_1}(\tau(\mathcal{A})) \leq 0$, we have

$$(a_{i_1\ldots i_1} - \tau(\mathcal{A}) - r_{i_1}^{\Delta_{i_1}}(\mathcal{A}))(a_{j_1\ldots j_1} - \tau(\mathcal{A}) - r_{j_1}^{\overline{\Delta}_{i_1}}(\mathcal{A})) \leq r_{i_1}^{\overline{\Delta}_{i_1}}(\mathcal{A})r_{j_1}^{\Delta_{i_1}}(\mathcal{A}). \tag{2.14}$$

If $r_{i_1}^{\overline{\Delta}_{i_1}}(\mathcal{A})r_{j_1}^{\Delta_{i_1}}(\mathcal{A}) = 0$, then

$$a_{j_1\ldots j_1} - \tau(\mathcal{A}) - r_{j_1}^{\overline{\Delta}_{i_1}}(\mathcal{A}) \leq 0 \leq r_{j_1}^{\Delta_{i_1}}(\mathcal{A}),$$

which leads to

$$a_{j_1\ldots j_1} - \tau(\mathcal{A}) \leq r_{j_1}(\mathcal{A}). \tag{2.15}$$

In addition, by (2.12) we have

$$\tau(\mathcal{A}) \geq a_{i_1\ldots i_1} - (r_{i_1}^{j_1}(\mathcal{A}) + (-a_{i_1j_1\ldots j_1})),$$

i.e.,

$$a_{i_1\ldots i_1} - \tau(\mathcal{A}) - r_{i_1}^{j_1}(\mathcal{A}) \le -a_{i_1 j_1 \ldots j_1}. \tag{2.16}$$

Note that $\tau(\mathcal{A}) < a_{j_1\ldots j_1}$, then multiplying Inequality (2.15) with Inequality (2.16) gives

$$(a_{i_1\ldots i_1} - \tau(\mathcal{A}) - r_{i_1}^{j_1}(\mathcal{A}))(a_{j_1\ldots j_1} - \tau(\mathcal{A})) \le (-a_{i_1 j_1\ldots j_1}) r_{j_1}(\mathcal{A}),$$

which implies that $g^{i_1 j_1}(\tau(\mathcal{A})) \le 0$.

If $r_{i_1}^{\overline{\Delta}_{i_1}}(\mathcal{A}) r_{j_1}^{\Delta_{i_1}}(\mathcal{A}) > 0$, then by dividing Inequality (2.14) by $r_{i_1}^{\overline{\Delta}_{i_1}}(\mathcal{A}) r_{j_1}^{\Delta_{i_1}}(\mathcal{A})$, we get

$$\frac{a_{i_1\ldots i_1} - \tau(\mathcal{A}) - r_{i_1}^{\Delta_{i_1}}(\mathcal{A})}{r_{i_1}^{\overline{\Delta}_{i_1}}(\mathcal{A})} \frac{a_{j_1\ldots j_1} - \tau(\mathcal{A}) - r_{j_1}^{\overline{\Delta}_{i_1}}(\mathcal{A})}{r_{j_1}^{\Delta_{i_1}}(\mathcal{A})} \le 1. \tag{2.17}$$

By (2.12), we have

$$\frac{a_{i_1\ldots i_1} - \tau(\mathcal{A}) - r_{i_1}^{\Delta_{i_1}}(\mathcal{A})}{r_{i_1}^{\overline{\Delta}_{i_1}}(\mathcal{A})} \le 1. \tag{2.18}$$

Then it follows from Inequality (2.17) that

$$\frac{a_{j_1\ldots j_1} - \tau(\mathcal{A}) - r_{j_1}^{\overline{\Delta}_{i_1}}(\mathcal{A})}{r_{j_1}^{\Delta_{i_1}}(\mathcal{A})} \ge 1,$$

or

$$\frac{a_{j_1\ldots j_1} - \tau(\mathcal{A}) - r_{j_1}^{\overline{\Delta}_{i_1}}(\mathcal{A})}{r_{j_1}^{\Delta_{i_1}}(\mathcal{A})} \le 1.$$

When $-a_{i_1 j_1\ldots j_1} > 0$, from the part (I) in Lemma 2.7 and Inequality (2.18) we have

$$\frac{a_{i_1\ldots i_1} - \tau(\mathcal{A}) - r_{i_1}^{j_1}(\mathcal{A})}{-a_{i_1 j_1\ldots j_1}} \le \frac{a_{i_1\ldots i_1} - \tau(\mathcal{A}) - r_{i_1}^{\Delta_{i_1}}(\mathcal{A})}{r_{i_1}^{\overline{\Delta}_{i_1}}(\mathcal{A})}. \tag{2.19}$$

Furthermore, if $\frac{a_{j_1\ldots j_1} - \tau(\mathcal{A}) - r_{j_1}^{\overline{\Delta}_{i_1}}(\mathcal{A})}{r_{j_1}^{\Delta_{i_1}}(\mathcal{A})} \ge 1$, it follows from the part (II) in Lemma 2.7 that

$$\frac{a_{j_1\ldots j_1}-\tau(\mathcal{A})}{r_{j_1}(\mathcal{A})}\leq\frac{a_{j_1\ldots j_1}-\tau(\mathcal{A})-r_{j_1}^{\overline{\Delta}_{i_1}}(\mathcal{A})}{r_{j_1}^{\Delta_{i_1}}(\mathcal{A})}. \tag{2.20}$$

Then multiplying Inequality (2.19) with Inequality (2.20), together with (2.17), gives

$$\begin{aligned}&\frac{a_{i_1\ldots i_1}-\tau(\mathcal{A})-r_{i_1}^{j_1}(\mathcal{A})}{-a_{i_1 j_1\ldots j_1}}\frac{a_{j_1\ldots j_1}-\tau(\mathcal{A})}{r_{j_1}(\mathcal{A})}\\ &\leq\frac{a_{i_1\ldots i_1}-\tau(\mathcal{A})-r_{i_1}^{\Delta_{i_1}}(\mathcal{A})}{r_{i_1}^{\overline{\Delta}_{i_1}}(\mathcal{A})}\frac{a_{j_1\ldots j_1}-\tau(\mathcal{A})-r_{j_1}^{\overline{\Delta}_{i_1}}(\mathcal{A})}{r_{j_1}^{\Delta_{i_1}}(\mathcal{A})}\leq 1,\end{aligned}$$

equivalently,

$$(a_{i_1\ldots i_1}-\tau(\mathcal{A})-r_{i_1}^{j_1}(\mathcal{A}))(a_{j_1\ldots j_1}-\tau(\mathcal{A}))\leq(-a_{i_1 j_1\ldots j_1})r_{j_1}(\mathcal{A}),$$

that is $g^{i_1 j_1}(\tau(\mathcal{A}))\leq 0$. And if $\frac{a_{j_1\ldots j_1}-\tau(\mathcal{A})-r_{j_1}^{\overline{\Delta}_{i_1}}(\mathcal{A})}{r_{j_1}^{\Delta_{i_1}}(\mathcal{A})}\leq 1$, then

$$a_{j_1\ldots j_1}-\tau(\mathcal{A})\leq r_{j_1}(\mathcal{A}).$$

Inequality (2.18) implies

$$a_{i_1\ldots i_1}-\tau(\mathcal{A})-r_{i_1}^{j_1}(\mathcal{A})\leq -a_{i_1 j_1\ldots j_1}.$$

The above two inequalities lead to

$$(a_{i_1\ldots i_1}-\tau(\mathcal{A})-r_{i_1}^{j_1}(\mathcal{A}))(a_{j_1\ldots j_1}-\tau(\mathcal{A}))\leq(-a_{i_1 j_1\ldots j_1})r_{j_1}(\mathcal{A}),$$

i.e., $g^{i_1 j_1}(\tau(\mathcal{A}))\leq 0$.

When $a_{i_1 j_1\ldots j_1}=0$, from (2.18), we easily get

$$a_{i_1\ldots i_1}-\tau(\mathcal{A})-r_{i_1}^{j_1}(\mathcal{A})\leq 0=-a_{i_1 j_1\ldots j_1}.$$

Hence,

$$(a_{i_1\ldots i_1}-\tau(\mathcal{A})-r_{i_1}^{j_1}(\mathcal{A}))(a_{j_1\ldots j_1}-\tau(\mathcal{A}))\leq 0=(-a_{i_1 j_1\ldots j_1})r_{j_1}(\mathcal{A}),$$

i.e., $g^{i_1 j_1}(\tau(\mathcal{A}))\leq 0$. □

By using the technique in the proof of Theorem 2.2, we can get $\overline{\Lambda}_{\max}\leq\max_{i,j\in N,\, j\neq i}L_{ij}(\mathcal{A})$. Combining with Theorem 5 in [13], we can easily obtain the bounds in Theorem 2.1 are shaper than Lemmas 1.1 and 1.2 in corrected form.

Now we take an example to show the efficiency of the bounds established in Theorem 2.1.

Example 2.1 Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{[3,3]}$ be a weakly irreducible $\mathcal{M}$-tensor with entries defined as follows:

$$\mathcal{A} = [A(1,:,:), A(2,:,:), A(3,:,:)] \in \mathbb{R}^{[3,3]},$$

where

$$A(1,:,:) = \begin{pmatrix} 15 & 0 & 0 \\ 0 & -0.5 & -0.2 \\ 0 & -1 & -2 \end{pmatrix}, A(2,:,:) = \begin{pmatrix} -1 & -5.8 & -2 \\ 0 & 55 & 0 \\ 0 & 0 & -0.5 \end{pmatrix},$$
$$A(3,:,:) = \begin{pmatrix} -1 & -2 & 0 \\ 0 & -1 & -3 \\ 0 & -3 & 15 \end{pmatrix}.$$

We compare the results derived in Theorem 2.1 with those of Lemmas 1.1, 1.2 in the correct form and Lemma 1.4. By Lemma 1.1, we have

$$5 \leq \tau(\mathcal{A}) \leq 45.7.$$

By Lemma 1.2 in the corrected form, we get

$$5.4256 \leq \tau(\mathcal{A}) \leq 14.8406.$$

By Lemma 1.4, we obtain

$$5.8038 \leq \tau(\mathcal{A}) \leq 14.7458.$$

By Theorem 2.1 , we have

$$8.4610 \leq \tau(\mathcal{A}) \leq 10.4580.$$

This example shows that the upper and lower bounds in Theorem 2.1 are better than those in Lemmas 1.1, 1.2 and 1.4.

2.2 The New *S*-type Estimates of Minimum *H*-eigenvalue

In this subsection, the new S-type estimates of minimum H-eigenvalue for weakly irreducible nonsingular $\mathcal{M}$-tensor are derived, which are better than the ones in Lemmas 1.1 and 1.2 in corrected form.

Given a nonempty proper subset S of N, we denote

$$\Delta^N = \{(i_2, i_3, \ldots, i_m) : \text{ each } i_j \in N, \text{ for } j \in 2, 3, \ldots, m\},$$
$$\Delta^S = \{(i_2, i_3, \ldots, i_m) : \text{ each } i_j \in S, \text{ for } j \in 2, 3, \ldots, m\}, \overline{\Delta^S} = \Delta^N \backslash \Delta^S.$$

This implies that for $i \in S$, we have

$$r_i(\mathcal{A}) = r_i^{\Delta^S}(\mathcal{A}) + r_i^{\overline{\Delta^S}}(\mathcal{A}),$$

where

$$r_i^{\Delta^S}(\mathcal{A}) = \sum_{\substack{(i_2,\ldots,i_m)\in\Delta^S,\\ \delta_{ii_2\ldots i_m}=0}} |a_{ii_2\ldots i_m}|, \quad r_i^{\overline{\Delta^S}}(\mathcal{A}) = \sum_{(i_2,\ldots,i_m)\in\overline{\Delta^S}} |a_{ii_2\ldots i_m}|.$$

Theorem 2.3 *Let $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in \mathbb{R}^{[m,n]}$ be a weakly irreducible nonsingular $\mathcal{M}$-tensor with $n \geq 2$, and S be a nonempty proper subset of N. Then*

$$\Upsilon_{\min}(\mathcal{A}) \leq \tau(\mathcal{A}) \leq \Upsilon_{\max}(\mathcal{A}),$$

where

$$\Upsilon_{\min}(\mathcal{A}) = \min\{\overline{\Upsilon}^S(\mathcal{A}), \overline{\Upsilon}^{\overline{S}}(\mathcal{A})\}, \ \Upsilon_{\max}(\mathcal{A}) = \max\{\widetilde{\Upsilon}^S(\mathcal{A}), \widetilde{\Upsilon}^{\overline{S}}(\mathcal{A})\},$$
$$\overline{\Upsilon}^S(\mathcal{A}) = \min_{\substack{i\in S,\\ j\in\overline{S}}} \max\{\frac{1}{2}(a_{jj\ldots j} + a_{ii\ldots i} - r_j^{\overline{\Delta^S}}(\mathcal{A}) - (\Psi_{i,j}^S)^{\frac{1}{2}}), R_j(\mathcal{A})\},$$
$$\widetilde{\Upsilon}^S(\mathcal{A}) = \max_{\substack{i\in S,\\ j\in\overline{S}}} \min\{\frac{1}{2}(a_{ii\ldots i} + a_{jj\ldots j} - r_j^{\overline{\Delta^S}}(\mathcal{A}) - (\Psi_{i,j}^S)^{\frac{1}{2}}), R_j(\mathcal{A})\},$$
$$\Psi_{i,j}^S = (a_{jj\ldots j} - a_{ii\ldots i} - r_j^{\overline{\Delta^S}}(\mathcal{A}))^2 + 4r_j^{\Delta^S}(\mathcal{A})r_i(\mathcal{A}).$$

Proof Since $\mathcal{A}$ is a weakly irreducible nonsingular $\mathcal{M}$-tensor, by Lemma 2.3, there exists $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} > 0$ such that

$$\mathcal{A}x^{m-1} = \tau(\mathcal{A})x^{[m-1]}. \tag{2.21}$$

(i) Let $x_l = \max_{i\in S} x_i$ and $x_t = \max_{i\in\overline{S}} x_i$. Next, we divide into two cases to prove. **Case I** $x_t \geq x_l$, that is, $x_t = \max_{i\in N} x_i$. From (2.21), we have

$$(\tau(\mathcal{A}) - a_{tt\ldots t})x_t^{m-1} = \sum_{(i_2,\ldots,i_m)\in\Delta^S} a_{ti_2\ldots i_m}x_{i_2}x_{i_3}\ldots x_{i_m} + \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^S},\\ \delta_{ti_2\ldots i_m}=0}} a_{ti_2\ldots i_m}x_{i_2}x_{i_3}\ldots x_{i_m}.$$

Using the inequality technique, together with $\tau(\mathcal{A}) < a_{tt\ldots t}$, gives

$$(a_{tt\ldots t} - \tau(\mathcal{A}))x_t^{m-1} = \sum_{(i_2,\ldots,i_m)\in\Delta^S} |a_{ti_2\ldots i_m}|x_{i_2}x_{i_3}\ldots x_{i_m}$$

$$
\begin{aligned}
&+\sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^S},\\ \delta_{ti_2\ldots i_m}=0}}|a_{ti_2\ldots i_m}|x_{i_2}x_{i_3}\ldots x_{i_m}\\
&\leq \sum_{(i_2,\ldots,i_m)\in\Delta^S}|a_{ti_2\ldots i_m}|x_l^{m-1}+\sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^S},\\ \delta_{ti_2\ldots i_m}=0}}|a_{ti_2\ldots i_m}|x_t^{m-1}\\
&=r_t^{\Delta^S}(\mathcal{A})x_l^{m-1}+r_t^{\overline{\Delta^S}}(\mathcal{A})x_t^{m-1};
\end{aligned}
$$

hence,

$$
(a_{tt\ldots t}-\tau(\mathcal{A})-r_t^{\overline{\Delta^S}}(\mathcal{A}))x_t^{m-1}\leq r_t^{\Delta^S}(\mathcal{A})x_l^{m-1}. \tag{2.22}
$$

On the other hand, by (2.21), we also get that

$$
\begin{aligned}
(a_{ll\ldots l}-\tau(\mathcal{A}))x_l^{m-1}&=\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{li_2\ldots i_m}=0}}|a_{li_2\ldots i_m}|x_{i_2}x_{i_3}\ldots x_{i_m}\\
&\leq r_l(\mathcal{A})x_t^{m-1}.
\end{aligned} \tag{2.23}
$$

Multiplying (2.22) with (2.23) gives

$$
(a_{tt\ldots t}-\tau(\mathcal{A})-r_t^{\overline{\Delta^S}}(\mathcal{A}))(a_{ll\ldots l}-\tau(\mathcal{A}))\leq r_t^{\Delta^S}(\mathcal{A})r_l(\mathcal{A}).
$$

Solving the above quadratic inequality yields

$$
\tau(\mathcal{A})\geq\frac{1}{2}(a_{tt\ldots t}+a_{ll\ldots l}-r_t^{\overline{\Delta^S}}(\mathcal{A})-(\Psi_{l,t}^S)^{\frac{1}{2}}), \tag{2.24}
$$

with

$$
\Psi_{l,t}^S=(a_{tt\ldots t}-a_{ll\ldots l}-r_t^{\overline{\Delta^S}}(\mathcal{A}))^2+4r_t^{\Delta^S}(\mathcal{A})r_l(\mathcal{A}).
$$

Furthermore, by Inequality (2.22), we can get that

$$
a_{tt\ldots t}-\tau(\mathcal{A})-r_t^{\overline{\Delta^S}}(\mathcal{A})\leq r_t^{\Delta^S}(\mathcal{A}),
$$

i.e.,

$$
\tau(\mathcal{A})\geq a_{tt\ldots t}-r_t^{\Delta^S}(\mathcal{A})-r_t^{\overline{\Delta^S}}(\mathcal{A})=a_{tt\ldots t}-r_t(\mathcal{A})=R_t(\mathcal{A}). \tag{2.25}
$$

It follows from Inequalities (2.24) and (2.25) that

$$
\begin{aligned}
\tau(\mathcal{A})&\geq\max\{\frac{1}{2}(a_{tt\ldots t}+a_{ll\ldots l}-r_t^{\overline{\Delta^S}}(\mathcal{A})-(\Psi_{l,t}^S)^{\frac{1}{2}}),\,R_t(\mathcal{A})\}\\
&\geq\min_{\substack{i\in S,\\ j\in\overline{S}}}\max\{\frac{1}{2}(a_{ii\ldots i}+a_{jj\ldots j}-r_j^{\overline{\Delta^S}}(\mathcal{A})-(\Psi_{i,j}^S)^{\frac{1}{2}}),\,R_j(\mathcal{A})\}.
\end{aligned} \tag{2.26}
$$

Case II $x_l \ge x_t$, that is, $x_l = \max_{i\in N} x_i$. In a similar manner to the proof of Case I, we have

$$(a_{ll\ldots l} - \tau(\mathcal{A}) - r_l^{\overline{\Delta^{\overline{S}}}}(\mathcal{A}))x_l^{m-1} \le r_l^{\Delta^{\overline{S}}}(\mathcal{A})x_t^{m-1}$$

and

$$(a_{tt\ldots t} - \tau(\mathcal{A}))x_t^{m-1} \le r_t(\mathcal{A})x_l^{m-1}.$$

Note that $x_t x_l > 0$. Thus,

$$(a_{ll\ldots l} - \tau(\mathcal{A}) - r_l^{\overline{\Delta^{\overline{S}}}}(\mathcal{A}))(a_{tt\ldots t} - \tau(\mathcal{A})) \le r_l^{\Delta^{\overline{S}}}(\mathcal{A})r_t(\mathcal{A})$$

and

$$\tau(\mathcal{A}) \ge a_{ll\ldots l} - r_l^{\Delta^{\overline{S}}}(\mathcal{A}) - r_l^{\overline{\Delta^{\overline{S}}}}(\mathcal{A}) = R_l(\mathcal{A}).$$

Then, solve for $\tau(\mathcal{A})$,

$$\begin{aligned}\tau(\mathcal{A}) &\ge \max\left\{\frac{1}{2}\left(a_{tt\ldots t} + a_{ll\ldots l} - r_l^{\overline{\Delta^{\overline{S}}}}(\mathcal{A}) - (\Psi_{t,l}^{\overline{S}})^{\frac{1}{2}}\right), R_l(\mathcal{A})\right\}\\ &\ge \min_{\substack{i\in\overline{S},\\ j\in S}} \max\left\{\frac{1}{2}\left(a_{ii\ldots i} + a_{jj\ldots j} - r_j^{\overline{\Delta^{\overline{S}}}}(\mathcal{A}) - (\Psi_{i,j}^{\overline{S}})^{\frac{1}{2}}\right), R_j(\mathcal{A})\right\}. \quad (2.27)\end{aligned}$$

Combining (2.26) and (2.27) yields the first inequality of Theorem 2.3.

(ii) Let $x_p = \min_{i\in S} x_i$ and $x_q = \min_{i\in\overline{S}} x_i$. Dividing into two cases to prove: $x_p \ge x_q$ and $x_q \ge x_p$ and by the analogical proof as (i), we can prove the second inequality of Theorem 2.3. □

Next, we show the bounds of Theorem 2.3 are sharper than those of Lemma 1.2 in corrected form. We first proof that the lower bound of Theorem 2.3 is greater than or equal to than that of Lemma 1.2 in corrected form.

Theorem 2.4 *Let $\mathcal{A} = (a_{i_1 i_2\ldots i_m}) \in \mathbb{R}^{[m,n]}$ be a weakly irreducible nonsingular $\mathcal{M}$-tensor with $n \ge 2$. Then*

$$\Upsilon_{\min}(\mathcal{A}) \ge \min_{\substack{i,j\in N,\\ j\ne i}} L_{ij}(\mathcal{A}).$$

Proof By Theorem 2.3, we have $\Upsilon_{\min}(\mathcal{A}) = \overline{\Upsilon}^S(\mathcal{A})$ or $\Upsilon_{\min}(\mathcal{A}) = \overline{\Upsilon}^{\overline{S}}(\mathcal{A})$. Without loss of generality, we suppose that $\Upsilon_{\min}(\mathcal{A}) = \overline{\Upsilon}^S(\mathcal{A})$ (we can prove it similarly if $\Upsilon_{\min}(\mathcal{A}) = \overline{\Upsilon}^{\overline{S}}(\mathcal{A})$). Then there are $i_2 \in S$, $j_2 \in \overline{S}$ such that

$$\Upsilon_{\min}(\mathcal{A}) = \overline{\Upsilon}^S(\mathcal{A}) = \max\left\{\frac{1}{2}\left(a_{j_2\ldots j_2} + a_{i_2\ldots i_2} - r_{j_2}^{\overline{\Delta^S}}(\mathcal{A}) - (\Psi_{i_2,j_2}^S)^{\frac{1}{2}}\right), R_{j_2}(\mathcal{A})\right\},$$

which leads to

$$\tau(\mathcal{A}) \geq R_{j_2}(\mathcal{A}) \tag{2.28}$$

and

$$\tau(\mathcal{A}) \geq \frac{1}{2}\left(a_{j_2\ldots j_2} + a_{i_2\ldots i_2} - r_{j_2}^{\overline{\Delta^S}}(\mathcal{A}) - (\Psi_{i_2,j_2}^S)^{\frac{1}{2}}\right). \tag{2.29}$$

From proof of Theorem 2.3, Inequality (2.29) is derived by solving the following quadratic inequality

$$\left(a_{j_2\ldots j_2} - \tau(\mathcal{A}) - r_{j_2}^{\overline{\Delta^S}}(\mathcal{A})\right)\left(a_{i_2\ldots i_2} - \tau(\mathcal{A})\right) \leq r_{j_2}^{\Delta^S}(\mathcal{A}) r_{i_2}(\mathcal{A}).$$

So we let $h^{i_2 j_2}(\tau(\mathcal{A})) = (a_{j_2\ldots j_2} - \tau(\mathcal{A}) - r_{j_2}^{\overline{\Delta^S}}(\mathcal{A}))(a_{i_2\ldots i_2} - \tau(\mathcal{A})) - r_{j_2}^{\Delta^S}(\mathcal{A}) r_{i_2}(\mathcal{A})$ and $W_{i_2 j_2}(\mathcal{A}) := \frac{1}{2}(a_{j_2\ldots j_2} + a_{i_2\ldots i_2} - r_{j_2}^{\overline{\Delta^S}}(\mathcal{A}) - (\Psi_{i_2,j_2}^S)^{\frac{1}{2}})$ is the left root of the equation $h^{i_2 j_2}(\tau(\mathcal{A})) = 0$. By Lemma 2.6, if $g^{j_2 i_2}(\tau(\mathcal{A})) \leq 0$ under the condition $h^{i_2 j_2}(\tau(\mathcal{A})) \leq 0$, then $W_{i_2 j_2}(\mathcal{A}) \geq L_{j_2 i_2}(\mathcal{A}) \geq \min_{i,j\in N, j\neq i} L_{ij}(\mathcal{A})$, that is, $\Upsilon_{\min}(\mathcal{A}) \geq \min_{i,j\in N, j\neq i} L_{ij}(\mathcal{A})$. We now prove that $g^{j_2 i_2}(\tau(\mathcal{A})) \leq 0$ under the condition $h^{i_2 j_2}(\tau(\mathcal{A})) \leq 0$. From the condition $h^{i_2 j_2}(\tau(\mathcal{A})) \leq 0$, we have

$$\left(a_{j_2\ldots j_2} - \tau(\mathcal{A}) - r_{j_2}^{\overline{\Delta^S}}(\mathcal{A})\right)\left(a_{i_2\ldots i_2} - \tau(\mathcal{A})\right) \leq r_{j_2}^{\Delta^S}(\mathcal{A}) r_{i_2}(\mathcal{A}). \tag{2.30}$$

If $r_{j_2}^{\Delta^S}(\mathcal{A}) r_{i_2}(\mathcal{A}) = 0$, then $r_{j_2}^{\Delta^S}(\mathcal{A}) = 0$ or $r_{i_2}(\mathcal{A}) = 0$. When $r_{j_2}^{\Delta^S}(\mathcal{A}) = 0$, we get $-a_{j_2 i_2\ldots i_2} = 0$, $r_{j_2}^{\overline{\Delta^S}}(\mathcal{A}) = r_{j_2}^{i_2}(\mathcal{A})$. Therefore,

$$\begin{aligned}
\left(a_{j_2\ldots j_2} - \tau(\mathcal{A}) - r_{j_2}^{i_2}(\mathcal{A})\right)\left(a_{i_2\ldots i_2} - \tau(\mathcal{A})\right) &= \left(a_{j_2\ldots j_2} - \tau(\mathcal{A}) - r_{j_2}^{\overline{\Delta^S}}(\mathcal{A})\right)\left(a_{i_2\ldots i_2} - \tau(\mathcal{A})\right)\\
&\leq r_{j_2}^{\Delta^S}(\mathcal{A}) r_{i_2}(\mathcal{A})\\
&= 0\\
&= (-a_{j_2 i_2\ldots i_2}) r_{i_2}(\mathcal{A});
\end{aligned}$$

consequently, $g^{i_2 j_2}(\tau(\mathcal{A})) \leq 0$. When $r_{i_2}(\mathcal{A}) = 0$,

$$\begin{aligned}
\left(a_{j_2\ldots j_2} - \tau(\mathcal{A}) - r_{j_2}^{i_2}(\mathcal{A})\right)\left(a_{i_2\ldots i_2} - \tau(\mathcal{A})\right) &\leq \left(a_{j_2\ldots j_2} - \tau(\mathcal{A}) - r_{j_2}^{\overline{\Delta^S}}(\mathcal{A})\right)\left(a_{i_2\ldots i_2} - \tau(\mathcal{A})\right)\\
&\leq r_{j_2}^{\Delta^S}(\mathcal{A}) r_{i_2}(\mathcal{A})
\end{aligned}$$

$$= 0$$
$$= \left(-a_{j_2i_2\ldots i_2}\right) r_{i_2}(\mathcal{A}).$$

This leads to $g^{j_2i_2}(\tau(\mathcal{A})) \leq 0$.

If $r_{j_2}^{\Delta^S}(\mathcal{A}) r_{i_2}(\mathcal{A}) > 0$, then we can equivalently express Inequality (2.30) as

$$\frac{a_{j_2\ldots j_2} - \tau(\mathcal{A}) - r_{j_2}^{\overline{\Delta^S}}(\mathcal{A})}{r_{j_2}^{\Delta^S}(\mathcal{A})} \frac{a_{i_2\ldots i_2} - \tau(\mathcal{A})}{r_{i_2}(\mathcal{A})} \leq 1. \tag{2.31}$$

By (2.28), we have $\frac{a_{j_2\ldots j_2} - \tau(\mathcal{A}) - r_{j_2}^{\overline{\Delta^S}}(\mathcal{A})}{r_{j_2}^{\Delta^S}(\mathcal{A})} \leq 1$, and when $a_{j_2i_2\ldots i_2} > 0$, from the part (I) in Lemma 2.7 we have

$$\frac{a_{j_2\ldots j_2} - \tau(\mathcal{A}) - r_{j_2}^{i_2}(\mathcal{A})}{-a_{j_2i_2\ldots i_2}} \leq \frac{a_{j_2\ldots j_2} - \tau(\mathcal{A}) - r_{j_2}^{\overline{\Delta^S}}(\mathcal{A})}{r_{j_2}^{\Delta^S}(\mathcal{A})},$$

together with Inequality (2.31), we can derive that

$$\frac{a_{j_2\ldots j_2} - \tau(\mathcal{A}) - r_{j_2}^{i_2}(\mathcal{A})}{-a_{j_2i_2\ldots i_2}} \frac{a_{i_2\ldots i_2} - \tau(\mathcal{A})}{r_{i_2}(\mathcal{A})} \leq \frac{a_{j_2\ldots j_2} - \tau(\mathcal{A}) - r_{j_2}^{\overline{\Delta^S}}(\mathcal{A})}{r_{j_2}^{\Delta^S}(\mathcal{A})} \frac{a_{i_2\ldots i_2} - \tau(\mathcal{A})}{r_{i_2}(\mathcal{A})} \leq 1.$$

i.e., $g^{j_2i_2}(\tau(\mathcal{A})) \leq 0$. When $a_{j_2i_2\ldots i_2} = 0$, by (2.28) we easily get

$$a_{j_2\ldots j_2} - \tau(\mathcal{A}) - r_{j_2}^{i_2}(\mathcal{A}) \leq 0 = -a_{j_2i_2\ldots i_2}.$$

Hence,

$$(a_{j_2\ldots j_2} - \tau(\mathcal{A}) - r_{j_2}^{i_2}(\mathcal{A}))(a_{i_2\ldots i_2} - \tau(\mathcal{A})) \leq 0 = -a_{j_2i_2\ldots i_2} r_{i_2}(\mathcal{A}).$$

This also implies $g^{j_2i_2}(\tau(\mathcal{A})) \leq 0$. This completes our proof of Theorem 2.4. □

By using the technique in the proof of Theorem 2.4, we can get $\Upsilon_{\max}(\mathcal{A}) \leq \max_{i,j\in N, j\neq i} L_{ij}(\mathcal{A})$. Together with Theorem 5 in [13], we can easily see the bounds in Theorem 2.3 are better than Lemmas 1.1 and 1.2 in corrected form.

Let us show the advantage of Theorem 2.3 over the results in Lemma 1.1, 1.2 which are corrected, Lemma 1.4 and newly derived by Huang et al. [14] by a simple example as follows.

Example 2.2 Let $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{[3,4]}$ be a weakly irreducible $\mathcal{M}$-tensor with entries defined as follows:

$$\mathcal{A} = [A(1, :, :), A(2, :, :), A(3, :, :), A(4, :, :)] \in \mathbb{R}^{[3,4]},$$

where

$$A(1,:,:)=\begin{pmatrix}37&-2&-1&-4\\-1&-3&-3&-2\\-1&-1&-3&-2\\-2&-3&-3&-3\end{pmatrix},\ A(2,:,:)=\begin{pmatrix}-2&-4&-2&-3\\-1&39&-2&-1\\-3&-3&-4&-2\\-2&-3&-1&-4\end{pmatrix},$$
$$A(3,:,:)=\begin{pmatrix}-4&-1&-1&-1\\-1&0&-2&-3\\-1&-1&35&-1\\-2&-2&-4&-3\end{pmatrix},\ A(4,:,:)=\begin{pmatrix}-2&-4&0&1\\-4&-4&-2&-4\\-3&0&-3&-3\\-3&-3&-4&49\end{pmatrix}.$$

We now compute the bounds for $\tau(\mathcal{A})$. Let $S=\{1,2\}$, then $\overline{S}=\{3,4\}$. By Lemma 1.1, we have

$$2 \le \tau(\mathcal{A}) \le 9.$$

By Lemma 1.2 in the corrected form, we get

$$2.0541 \le \tau(\mathcal{A}) \le 8.8969.$$

By Lemma 1.4, we obtain

$$2.2233 \le \tau(\mathcal{A}) \le 8.7447.$$

By Theorem 3.5 in [14], we get

$$2.6604 \le \tau(\mathcal{A}) \le 8.1955.$$

By Theorem 2.3, we have

$$3.5550 \le \tau(\mathcal{A}) \le 7.1629.$$

Obviously, the bounds given in Theorem 2.3 are sharper than the aforementioned existing results.

3 Ky Fan Theorem

In [11], He and Huang gave the Ky Fan theorem for nonsingular $\mathcal{M}$-tensors as follows:

Lemma 3.1 [11] *Let $\mathcal{A}$, $\mathcal{B}$ be of order m dimension n, suppose that $\mathcal{B}$ is a nonsingular $\mathcal{M}$-tensor and $|b_{i_1,\ldots,i_m}| \ge |a_{i_1,\ldots,i_m}|$ for any $i_1,\ldots,i_m \in N$ and $\delta_{i_1,\ldots,i_m} \ne 0$. Then, for any eigenvalue λ of $\mathcal{A}$, there exists $i \in N$ such that*

$$|\lambda - a_{i\ldots i}| \le b_{i\ldots i} - \tau(\mathcal{B}). \tag{3.1}$$

In [19], Bu et al. derived the following Brualdi-type eigenvalue inclusion sets of tensors.

Lemma 3.2 [19] *Let* $\mathcal{A} = (a_{i_1,\ldots,i_m}) \in \mathbb{C}^{[m,n]}$ *be a tensor such that* $\Gamma_{\mathcal{A}}$ *is weakly connected. Then,*

$$\sigma(\mathcal{A}) \subseteq \bigcup_{\gamma \in C(\mathcal{A})} \left\{ z \in \mathbb{C} : \prod_{i \in \gamma} |z - a_{ii\ldots i}| \leq \prod_{i \in \gamma} r_i(\mathcal{A}) \right\}.$$

Based on Lemma 3.2, we derive a new set in Ky Fan theorem, which is sharper than the one in (3.1).

Theorem 3.1 *Let* $\mathcal{A}, \mathcal{B}$ *be* m*-order* n*-dimensional tensors such that* $\Gamma_{\mathcal{A}}$ *is weakly connected and* $\mathcal{B}$ *be a nonsingular* $\mathcal{M}$*-tensor, and* $|b_{i_1\ldots i_m}| \geq |a_{i_1\ldots i_m}|$ *for all* $i_1 \neq \ldots \neq i_m$*. Then, there exists a circuit* $\gamma \in C(\mathcal{A})$*, such that*

$$\prod_{i \in \gamma} |\lambda - a_{ii\ldots i}| \leq \prod_{i \in \gamma} (b_{ii\ldots i} - \tau(\mathcal{B})).$$

Proof We first suppose that $\mathcal{B}$ is irreducible, by Lemma 2.2, there exists $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} > 0$ such that

$$\mathcal{B}x^{m-1} = \tau(\mathcal{B})x^{[m-1]}. \tag{3.2}$$

Let $D = \mathrm{diag}(x_1, \ldots, x_n)$, $\mathcal{A}_D = \mathcal{A}D^{1-m} \overbrace{D \ldots D}^{m-1}$, $y = (y_1, \ldots, y_n)^{\mathrm{T}}$ be an eigenvector of $\mathcal{A}_D$ corresponding to λ. Then

$$\mathcal{A}_D y^{m-1} = \lambda y^{[m-1]}.$$

By Lemma 2.5, we have

$$\lambda(\mathcal{A}) = \lambda(\mathcal{A}_D).$$

Equation (3.2) implies that for any i,

$$(b_{i\ldots i} - \tau(\mathcal{B}))x_i^{m-1} = - \sum_{\delta_{ii_2\ldots i_m}=0} b_{ii_2\ldots i_m} x_{i_2} \ldots x_{i_m} = \sum_{\delta_{ii_2\ldots i_m}=0} |b_{ii_2\ldots i_m}| x_{i_2} \ldots x_{i_m},$$

which is equivalent to

$$b_{i\ldots i} - \tau(\mathcal{B}) = \sum_{\delta_{ii_2\ldots i_m}=0} |b_{ii_2\ldots i_m}| x_i^{1-m} x_{i_2} \ldots x_{i_m}.$$

Since $\Gamma_{\mathcal{A}}$ is weakly connected, so is $\Gamma_{\mathcal{A}_D}$. From Lemma 3.2 and the above equation, for any eigenvalue λ of $\mathcal{A}_D$, there exists a circuit $\gamma \in C(\mathcal{A})$, such that

$$\begin{aligned}\prod_{i\in\gamma}|\lambda - a_{ii\ldots i}| \leq \prod_{i\in\gamma} r_i(\mathcal{A}_D)&\\ &= \prod_{i\in\gamma}\left(\sum_{\delta_{ii_2\ldots i_m=0}} |a_{ii_2\ldots i_m}| x_i^{1-m} x_{i_2}\ldots x_{i_m}\right)\\ &\leq \prod_{i\in\gamma}\left(\sum_{\delta_{ii_2\ldots i_m=0}} |b_{ii_2\ldots i_m}| x_i^{1-m} x_{i_2}\ldots x_{i_m}\right)\\ &= \prod_{i\in\gamma}(b_{i\ldots i} - \tau(\mathcal{B})).\end{aligned}$$

When the tensor $\mathcal{B}$ is reducible, by replacing the zero entries of $\mathcal{B}$ with $-\frac{1}{k}$, where k is a positive integer, we see that the Z-tensor $\mathcal{B}_k$ is irreducible and $|(\mathcal{B}_k)_{i_1\ldots i_m}| \geq |\mathcal{A}_{i_1\ldots i_m}|$. Then there exists a circuit $\gamma \in C(\mathcal{A})$ such that

$$\prod_{i\in\gamma}|\lambda - a_{ii\ldots i}| \leq \prod_{i\in\gamma}(b_{ii\ldots i} - \tau(\mathcal{B}_k)). \tag{3.3}$$

From the proof process of Theorem 3.6 in [14], we have

$$\lim_{k\to\infty} \tau(\mathcal{B}_k) = \tau(\mathcal{B}).$$

In Inequality (3.3), letting $k \to \infty$ results in

$$\prod_{i\in\gamma}|\lambda - a_{ii\ldots i}| \leq \prod_{i\in\gamma}(b_{ii\ldots i} - \tau(\mathcal{B})).$$

This completes our proof of Theorem 3.1. □

Denote

$$G(\mathcal{A}) = \bigcup_{i\in N}\left\{z \in \mathbb{C} : |z - a_{ii\ldots i}| \leq (b_{i\ldots i} - \tau(\mathcal{B}))\right\},$$

$$S(\mathcal{A}) = \bigcup_{\gamma\in C(\mathcal{A})}\left\{z \in \mathbb{C} : \prod_{i\in\gamma}|z - a_{ii\ldots i}| \leq \prod_{i\in\gamma}(b_{i\ldots i} - \tau(\mathcal{B}))\right\}.$$

It follows from Lemma 3.1 and Theorem 3.1 that $\sigma(\mathcal{A}) \subseteq G(\mathcal{A})$ and $\sigma(\mathcal{A}) \subseteq S(\mathcal{A})$. Next, we compare the sets $S(\mathcal{A})$ and $G(\mathcal{A})$ in the following theorem, showing that Theorem 3.1 is better than the Ky Fan theorem.

Theorem 3.2 *Let $\mathcal{A}, \mathcal{B}$ be m-order n-dimensional tensors such that $\Gamma_{\mathcal{A}}$ is weakly connected, $\mathcal{B}$ be a nonsingular $\mathcal{M}$-tensor, and $|b_{i_1\ldots i_m}| \geq |a_{i_1\ldots i_m}|$ for all $i_1 \neq \ldots \neq i_m$. Then*

$$S(\mathcal{A}) \subseteq G(\mathcal{A}).$$

Proof For any $z \in S(\mathcal{A})$, if $z \notin G(\mathcal{A})$, then $|z - a_{ii\ldots i}| > b_{ii\ldots i} - \tau(\mathcal{B})$ $(i = 1, 2, \ldots, n)$. In this case, $\prod_{i\in\gamma} |z - a_{ii\ldots i}| > \prod_{i\in\gamma}(b_{ii\ldots i} - \tau(\mathcal{B}))$ for any $\gamma \in C(\mathcal{A})$, a contradiction to $z \in S(\mathcal{A})$. Hence $z \in G(\mathcal{A})$, i.e., $S(\mathcal{A}) \subseteq G(\mathcal{A})$. □

4 Conclusions

In this paper, several new estimates of the minimum H-eigenvalue for weakly irreducible nonsingular $\mathcal{M}$-tensors are presented, which are proved to be sharper than those of [11,12]. On the other hand, we have studied a new Ky Fan-type theorem. It should be noted that the new Ky Fan theorem is based on the condition that $\Gamma_{\mathcal{A}}$ is weakly connected and $\mathcal{B}$ is a nonsingular $\mathcal{M}$-tensor, and the new Ky Fan-type theorem improves the one in [11].

However, the new S-type estimates for minimum H-eigenvalue depend on the set S. Then an interesting problem is how to pick S to make the bounds exhibited in Theorem 2.3 as tight as possible. But it is very difficult when the dimension of the tensor $\mathcal{A}$ is large. Therefore, future work will include numerical or theoretical studies for finding the best choice for S.

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Compliance with Ethical Standards

Conflict of interest The authors declare that they have no competing interests.

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