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Asymptotic Behavior, Regularity Criterion and Global Existence for the Generalized Navier–Stokes Equations

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Abstract This paper deals with the asymptotic behavior, regularity criterion and global existence for the generalized Navier–Stokes equations. Firstly, an upper bound for the difference between the solution of our equation and the generalized heat equation in L^2 space is proved. We optimize the upper bound of decay for the solutions and obtain the algebraic lower bound by using Fourier splitting method. Then, a new scaling invariant regularity criterion on the fractional derivative is established. Finally, global existence is obtained provided that the initial data are small enough.

Keywords Generalized Navier–Stokes equations · Asymptotic behavior · Regularity criterion · Global existence

Mathematics Subject Classification 35Q35 · 35B65 · 76D05

1 Introduction

We consider the following incompressible generalized Navier–Stokes equations:

$$u_t + u \cdot \nabla u + \nu(-\Delta)^{\alpha} u + \nabla P = 0, \tag{1.1}$$

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$$\operatorname{div} u = 0, \tag{1.2}$$

here $u = u(x, t) \in \mathbb{R}^n$, $p = p(x, t) \in \mathbb{R}$ represent the unknown velocity field and the pressure, respectively. $\nu > 0$ is the kinematic viscosity. For simplicity, we set $\nu = 1$ in the sequel. $(-\Delta)^\alpha$ is defined in terms of Fourier transform by $\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha} \hat{f}(\xi)$.

The existence of weak solutions was investigated by Jiu and Yu [12] (see also [10,16] for the classical Navier–Stokes equations ($\alpha = 1$)). Some decay estimates were shown as follows:

Theorem 1.1 [12] *Let $0 < \alpha \leq \frac{5}{4}$. Then for divergence-free vector field $u_0 \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ with $\max\left\{\frac{1}{3-2\alpha}, 1\right\} \leq p < 2$, system (1.1)–(1.2) admits a weak solution such that*

$$\|u(t)\|_{L^2}^2 \leq C(t+1)^{-\frac{3}{2\alpha}\left(\frac{2}{p}-1\right)},$$

where the constant C depends on α, the L^p and L^2 norms of the initial data.

From [31] (or [3] for the Navier–Stokes equations), we know that if u is a solution to system (1.1)–(1.2), then u_λ with any $\lambda > 0$ is also a solution, where $u_\lambda(x, t) = \lambda^{2\alpha-1} u(\lambda x, \lambda^{2\alpha} t)$. By direct calculation, we obtain the norms $\|u\|_{L^{p,q}}$ and $\|\Lambda^\gamma u\|_{L^{p,q}}$ are scaling invariant for $\frac{2\alpha}{p} + \frac{3}{q} = 2\alpha - 1$ and $\frac{2\alpha}{p} + \frac{3}{q} = 2\alpha + \gamma - 1$, respectively.

Very recently, the local solution for the generalized MHD system was investigated by Jiang and Zhou [11].

$$u_t + u \cdot \nabla u + \Lambda^{2\alpha} u + \nabla P = B \cdot \nabla B \tag{1.3}$$

$$B_t + u \cdot \nabla B - \Lambda^{2\beta} B = B \cdot \nabla u \tag{1.4}$$

$$\operatorname{div} u = \operatorname{div} B = 0. \tag{1.5}$$

If $B \equiv 0$, generalized MHD systems (1.3)–(1.5) reduce to this generalized Navier–Stokes equations. The main result in [11] reduces to

Theorem 1.2 [11] *For $s > \max\{\frac{n}{2} + 1 - \alpha, 1\}$, and the initial data $u_0 \in H^s(\mathbb{R}^n)$ with $\operatorname{div} u_0 = 0$, there exists a time T_* such that (1.1)–(1.2) have a unique solution $u \in C(0, T_*; H^s(\mathbb{R}^n))$.*

It is shown that if $\alpha \geq \frac{1}{2} + \frac{N}{4}$, then the solution $u(x, t)$ remains smooth for all time (refer [17,29] for details). When $\alpha = 1$, systems (1.1)–(1.2) reduce to the classical Navier–Stokes equations. The existence of a weak solution to the three-dimensional Navier–Stokes equations is well known by Leray [16] and Hopf [10]. However, its uniqueness and global regularity are still major challenging open problems. On the other hand, many sufficient conditions ensuring the smoothness of a weak solution are known. The classical Prodi–Serrin's-type criteria (see [20,24], and for the case $s = 3$, see [7]) say that if a weak solution u additionally belongs to $L^t(0; T; L^s(\mathbb{R}^3))$, with $\frac{2}{t} + \frac{3}{s} = 1$, $s \in [3; +\infty]$, then it is regular and unique. Analogous result in terms of the gradient of velocity, i.e., $\nabla u \in L^t(0, T; L^s(\mathbb{R}^3))$, with $\frac{2}{t} + \frac{3}{s} = 2$, $s \in (\frac{3}{2}; +\infty]$ is established by Beirão da Veiga (see [2]).

In this paper, we deal with the asymptotic behavior of solutions to the generalized Navier–Stokes equations by using Fourier splitting method. The Fourier splitting method [23] was first applied to the parabolic conservation laws to obtain algebraic energy decay rates. Then, it is used in the study of the classical Navier–Stokes equations [9,13,21,22] and the references therein. It is worth to point out that Zhou used a new method to get the famous result in [30]. A new regularity criterion which almost consists with the results in [2,20,24] and global existence with small initial data will also be established to the generalized Navier–Stokes equations.

The rest of this paper is organized as follows. In Sect. 2, we collect some elementary facts and inequalities that will be needed in later analysis. Section 3 is devoted to the decay results for the generalized Navier–Stokes equation. A new regularity criterion will be established in Sect. 4. Finally, global existence with small initial data will be studied in Sect. 5.

2 Preliminaries

In this section, we will recall some known facts and elementary inequalities that will be used frequently later.

Now, we list some notations that will be used in our paper. Use $\|u\|_{L^p}$ to denote the $L^p(\mathbb{R}^n)$ norm. Throughout this paper, C denotes a generic positive constant (generally large); it may be different from line to line. Use $\hat{f}$ and $\check{f}$(or $\mathcal{F}^{-1}$) to denote the Fourier transform $\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)e^{-ix\xi}\mathrm{d}x$ and the inverse Fourier transform $\check{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)e^{ix\xi}\mathrm{d}x$. $H^s(\mathbb{R}^n)$ and $\dot{H}^s(\mathbb{R}^n)$ denote the nonhomogeneous Sobolev spaces $\|u\|^2_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2\mathrm{d}\xi$ and homogeneous Sobolev spaces$\|u\|^2_{\dot{H}^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\xi|^{2s}|\hat{u}(\xi)|^2\mathrm{d}\xi$. We introduce the norm $L^{p,q}$

$$\|f\|_{L^{p,q}} = \begin{cases} \left(\int_0^t \|f(\cdot,\tau)\|^p_{L^q}\mathrm{d}\tau\right)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty, \\ \operatorname{esssup}_{0<\tau<t} \|f\|_{L^q}, & \text{if } p = \infty. \end{cases}$$

A fractional power of the Laplace transform $(-\Delta)^\alpha$ is defined through the Fourier transform

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha}\widehat{f}(\xi).$$

In particular, $\Lambda = (-\Delta)^{\frac{1}{2}}$ is defined in terms of Fourier transform by $\widehat{\Lambda f}(\xi) = |\xi|\hat{f}(\xi)$. More details on $(-\Delta)^\alpha$ can be found in Chapter 5 of Stein's book [25] (or see [6]).

Lemma 2.1 (Plancherel's theorem) *Assume* $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. *Then* $\hat{f}, \check{f} \in L^2(\mathbb{R}^n)$ *and*

$$\|\hat{f}\|_{L^2(\mathbb{R}^n)} = \|\check{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}.$$

Lemma 2.2 (Gagliardo–Nirenberg inequality [18,19]) *Let u belong to L^q in $\mathbb{R}^n$ and its derivatives of order m, $\Lambda^m u$, belong to L^r, $1 \leq q, r \leq \infty$. For the derivatives $\Lambda^j u$, $0 \leq j < m$, the following inequalities hold*

$$\|\Lambda^j u\|_{L^p} \leq C\|\Lambda^m u\|_{L^r}^{\alpha}\|u\|_{L^q}^{1-\alpha}, \tag{2.1}$$

where

$$\frac{1}{p} = \frac{j}{n} + \alpha\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\alpha)\frac{1}{q},$$

for all α in the interval

$$\frac{j}{m} \leq \alpha \leq 1$$

(the constant depending only on n, m, j, q, r, α), with the following exceptional cases

1. *If $j = 0$, $rm < n$, $q = \infty$, then we make the additional assumption that either u tends to zero at infinity or $u \in L_{\tilde{q}}$ for some finite $\tilde{q} > 0$.*
2. *If $1 < r < \infty$, and $m - j - n/r$ is a nonnegative integer, then (2.1) holds only for a satisfying $j/m \leq \alpha < 1$.*

By applying the Coifman–Meyer multiplier theorem [5] and Stein's complex interpolation theorem for analytic families [27], they [14,15] proved the following calculus inequalities in the Sobolev spaces.

Lemma 2.3 (Kato–Ponce inequality [14,15]) *Let $s > 0$, $1 < p < \infty$, if $f \in W^{1,p_1} \cap W^{s,q_2}$, $g \in L_{p_2} \cap W^{s,q_1}$, then*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{q_1}} + \|g\|_{L^{p_2}}\|\Lambda^s f\|_{L^{q_2}})$$

and

$$\|\Lambda^s(fg)\|_{L^p} \leq C(\|f\|_{L^{p_1}}\|\Lambda^s g\|_{L^{q_1}} + \|g\|_{L^{p_2}}\|\Lambda^s f\|_{L^{q_2}})$$

with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$.

3 Decay Results

As the assumption, $v(x,t)$ is the solution of

$$\begin{cases} v_t + \Lambda^{2\alpha} v = 0; \\ v(x,0) = u_0(x). \end{cases}$$

By directly computation, we have

$$\hat{v}(\xi,t) = e^{-|\xi|^{2\alpha}t}\hat{u}_0(\xi).$$

By the Plancherel's theorem, we know that

$$\|v\|_{L^2}^2 = \|\hat{v}\|_{L^2}^2 = \|e^{-|\xi|^{2\alpha}t}\hat{u}_0(\xi)\|_{L^2}^2.$$

Our main results are as follows. Firstly, we establish the upper bound for the weak solution of system (1.1)–(1.2) in L^2 space.

Theorem 3.1 *Assume v is the solution to the generalized heat equation $v_t + \Lambda^{2\alpha}v = 0$ with the same initial data $u_0 \in L^2(\mathbb{R}^n)$, and*

$$\|v(t)\|_{L^2}^2 \le C(1+t)^{-\theta}$$

for some $\theta > 0$. Then, for $n \ge 2$ and $\alpha \in (0, \frac{n+2}{4}]$, there exists a weak solution $u(x,t)$ such that

$$\|u(t)\|_{L^2}^2 \le C(1+t)^{-\theta_0}, \quad t \ge 0$$

with $\theta_0 = \min\{\theta, \frac{n+2}{2\alpha}\}$.

Proof Multiplying u on (1.1), integration by parts, we get the following energy equality:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 + \|\Lambda^\alpha u\|_{L^2}^2 = 0. \tag{3.1}$$

By Lemma 2.1 (Plancherel's theorem), we have

$$\begin{aligned}\|\Lambda^\alpha u\|_{L^2}^2 &= \int_{\mathbb{R}^n}|\xi|^{2\alpha}|\widehat{u}|^2\mathrm{d}\xi \ge (2\pi)^n|r(t)|^{2\alpha}\int_{|\xi|\ge r(t)}|\widehat{u}|^2\mathrm{d}\xi\\ &\ge \left(|r(t)|^{2\alpha}\int_{\mathbb{R}^n}|\widehat{u}|^2\mathrm{d}\xi - |r(t)|^{2\alpha}\int_{|\xi|\le r(t)}|\widehat{u}|^2\right)\mathrm{d}\xi.\end{aligned} \tag{3.2}$$

Combining (3.2) to (3.1), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{u}\|_{L^2}^2(t) + 2r(t)^{2\alpha}\|\hat{u}\|_{L^2}^2 \le 2r(t)^{2\alpha}\int_{|\xi|\le r(t)}|\hat{u}|^2\mathrm{d}\xi. \tag{3.3}$$

From the generalized Navier–Stokes equations, we obtain

$$|\hat{u}| \le |\hat{v}(\xi,t)| + |\xi|\int_0^t \|u\|_{L^2}^2\mathrm{d}\tau.$$

Therefore, it follows from (3.3) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{u}\|_{L^2}^2(t) + 2r(t)^{2\alpha}\|\hat{u}\|_{L^2}^2 \le Cr(t)^{2\alpha}\left[\|\hat{v}\|_{L^2}^2 + r(t)^{2+n}\left(\int_0^t\|u\|_{L^2}^2\mathrm{d}\tau\right)^2\right]. \tag{3.4}$$

Let $r(t)^{2\alpha} = \frac{1}{2(t+e)\ln(t+e)}$, it yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\ln(t+e)\|\hat{u}\|_{L^2}^2(t)\right] \leq C(t+e)^{-1-\theta} + C(t+e)^{-1-\frac{n+2}{2\alpha}}\left(\int_0^t \|u\|_{L^2}^2 \mathrm{d}\tau\right)^2. \tag{3.5}$$

Now, we claim that $\|u(s)\|_{L^2}^2 \leq C(1+s)^\beta$ for some $\beta > 0$ when $\alpha = \frac{n+2}{4}$. In order to prove the claim, we need to show that $\ln(t+e)^2\|u\|_{L^2}^2 \leq C$. Let $r(t)^{2\alpha} = \frac{1}{(t+e)\ln(t+e)}$ in (3.4), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\ln(t+e)^2\|\hat{u}\|_{L^2}^2(t)\right]\\ &\quad\leq C\frac{\ln(t+e)}{t+e}\left\{(t+e)^{-\theta} + (\ln(t+e)(t+e))^{-\frac{n+2}{2\alpha}}\left(\int_0^t \|u\|_{L^2}^2 \mathrm{d}\tau\right)^2\right\}.\end{aligned}$$

Note that $\|u\|_{L^2}^2$ is bounded, we have $\ln(t+e)^2\|u\|_{L^2}^2 \leq C$. It follows that $\int_0^s \|u(\tau)\|_{L^2}^2 \mathrm{d}\tau \leq C(s+1)\ln(s+e)^{-2}$. Then, by the same argument as that in [28], we complete the claim.

Suppose that $\|u(s)\|_{L^2}^2 \leq C(1+s)^\beta$ with $\beta > 0$ for $\alpha = \frac{n+2}{4}$ and $\beta \geq 0$ for $\alpha \in (0, \frac{n+2}{4})$. Hence, from (3.5), we obtain

$$\ln(t+e)\|\hat{u}\|_{L^2}^2(t) \leq C(t+e)^{-\theta} + C(t+e)^{-\frac{n+2}{2\alpha}+2-2\beta},$$

which implies that

$$\|u\|_{L^2}^2 \leq (1+t)^{-\tilde{\beta}},$$

with $\tilde{\beta} = \min\{\theta, \frac{n+2}{2\alpha} - 2 + 2\beta\}$. When $\alpha = \frac{n+2}{4}$, if we start with $\beta = 0$, we would get $\tilde{\beta} = 0$. This is why we need the claim above.

Now, starting with the new exponent, and after finitely many iterations, we get if $\theta \leq 1$, then $\|u\|_{L^2}^2 \leq C(1+t)^{-\theta}$. If $\theta > 1$, then we have $\tilde{\beta} = 1 + \varepsilon$ with $\varepsilon > 0$. It follows $\int_0^s \|u(\tau)\|_{L^2}^2 \mathrm{d}\tau \leq C$; here C is without respect to the time s. By (3.5), we have

$$\ln(t+e)\|\hat{u}\|_{L^2}^2 \leq C(t+e)^{-\theta} + C(t+e)^{-\frac{n+2}{2\alpha}}.$$

It follows that

$$\|u\|_{L^2}^2 \leq (1+t)^{-\theta_0} \text{ for } \theta_0 = \min\left\{\theta, \frac{n+2}{2\alpha}\right\}.$$

This completes the proof of Theorem 3.1. □

Then, we optimize the upper bound of decay for the strong solutions and obtain their algebraic lower bound.

Theorem 3.2 *Assume v is the solution to the generalized heat equation $v_t + \Lambda^{2\alpha} v = 0$ with the same initial data $u_0 \in H^1(\mathbb{R}^n) \cap R_\mu^\epsilon$ for some $\mu, \epsilon > 0$, and*

$$\|v(t)\|_{L^2}^2 \leq M(1+t)^{-\frac{n}{2\alpha}}.$$

Then, v satisfies

$$m(1+t)^{-\frac{n}{2\alpha}} \leq \|v(t)\|_{L^2}^2 \leq M(1+t)^{-\frac{n}{2\alpha}},$$

here m, M are positive constants. Then for $n \geq 2$ and $\alpha \in (0, \frac{n+2}{4}]$, we have

$$\|u(t)\|_{L^2} \geq C(1+t)^{-\frac{n}{4\alpha}}.$$

Here, $R_\mu^\epsilon = \{u : |\hat{u}(\xi)| \geq \mu \ \text{ for } \ |\xi| \leq \epsilon\}$ as that in [28].

In order to prove Theorem 3.2, we need the following lemma.

Lemma 3.3 *Choose T_1 large enough and fixed (will be chosen later). Let h be the solution to the generalized heat equation*

$$\begin{cases} h_t + \Lambda^{2\alpha} h = 0, & t \geq 0, x \in \mathbb{R}^n; \\ h(x,0) = u(x, T_1), \ x \in \mathbb{R}^n. \end{cases}$$

For $t > T_1$, we have

$$C(\delta)(1+t)^{-\frac{n}{2\alpha}} \leq \|h(t)\|_{L^2}^2 \leq C_1(1+t)^{-\frac{n}{2\alpha}}.$$

Proof For $|\xi| \leq T_1^{-\frac{1}{2\alpha}}$, $T_1 \geq \max\{\epsilon^{-2\alpha}, 1\}$, by direct calculation we have

$$\begin{aligned} |\hat{u}(\xi, T_1)| &= \left| e^{-|\xi|^{2\alpha} T_1} \hat{u}_0 - \int_0^{T_1} e^{-|\xi|^{2\alpha}(T_1 - s)} (\widehat{u \cdot \nabla u} + \widehat{\nabla p})(\xi, s) \mathrm{d}s \right| \\ &\geq \left| e^{-|\xi|^{2\alpha} T_1} \hat{u}_0 \right| \\ &\quad - \left| \int_0^{T_1} e^{-|\xi|^{2\alpha}(T_1 - s)} \left(\sum_{j=1}^n i\xi_j \widehat{u_j u} + i\xi \sum_{i,j=1}^n \frac{\xi_i \xi_j}{|\xi|^2} \widehat{u_i u_j} \right) (\xi, s) \mathrm{d}s \right| \\ &\geq e^{-1}\mu - C(|\xi|) \end{aligned}$$

For $|\xi| \leq T_1^{-\frac{1}{2\alpha}}$, we can obtain

$$|\hat{u}(\xi, T_1)| \geq \delta.$$

Then

$$\|h\|_{L^2}^2 \geq \int_{|\xi|\leq T_1^{-\frac{1}{2\alpha}}} e^{-2|\xi|^{2\alpha}t}|\hat{u}(\xi, T_1)|^2 \mathrm{d}\xi \geq \delta^2 t^{-\frac{n}{2\alpha}} \int_{|y|\leq \frac{\sqrt{t}}{\sqrt{T_1}}} e^{-2|y|^{2\alpha}} \mathrm{d}y.$$

For $t > T_1$, we have

$$\|h\|_{L^2}^2 \geq \delta^2 t^{-\frac{n}{2\alpha}} \int_{|y|\leq 1} e^{-2|y|^{2\alpha}} \mathrm{d}y \geq C(\delta)(1+t)^{-\frac{n}{2\alpha}},$$

here $C(\delta) = \frac{\delta^2 \pi^{\frac{n}{2}}}{e^2 \Gamma(\frac{n}{2}+1)}$.

Now we give the upper bound for $\|h\|_{L^2}$; we obtain

$$\begin{aligned}
|\hat{h}(\xi, t)| &= |e^{-|\xi|^{2\alpha}t}\hat{u}(\xi, T_1)| \\
&\leq |e^{-|\xi|^{2\alpha}(t+T_1)}\hat{u_0}| + \left| e^{-|\xi|^{2\alpha}t} \int_0^{T_1} e^{-|\xi|^{2\alpha}(T_1-s)} (\widehat{u\cdot\nabla u} + \widehat{\nabla p})(\xi, s)\mathrm{d}s \right| \\
&\leq |e^{-|\xi|^{2\alpha}(t+T_1)}\hat{u_0}| + |Ce^{-|\xi|^{2\alpha}t}\xi|.
\end{aligned}$$

Then we have

$$\begin{aligned}
\|h(\cdot, t)\|_{L^2} &= |\hat{h}(\xi, t)| \\
&\leq C\|\hat{h}(t+T_1)\|_{L^2} + C\||\xi|e^{-|\xi|^{2\alpha}t}\|_{L^2} \\
&\leq C_1(1+t)^{-\frac{n}{4\alpha}}.
\end{aligned}$$

□

Now, we give the proof of Theorem 3.2.

Set $U(x,t) = u(x, t+T_1)$ and $V(x,t) = U(x,t) - h(x,t)$. Multiplying both sides of the equation of V by V, and integrating over $\mathbb{R}^n$, after suitable integration by parts, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|V(t)\|_{L^2}^2 + 2\|\nabla V\|_{L^2}^2 &= 2\int_{\mathbb{R}^n} (U\cdot\nabla)U\cdot V\mathrm{d}x \\
&\leq 2\|\nabla h\|_{L^\infty}\|U\|_{L^2}^2.
\end{aligned}$$

Using the parseval's equality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{V}(t)\|_{L^2}^2 + 2\|\widehat{\nabla V}\|_{L^2}^2 \leq 2\|\nabla h\|_{L^\infty}\|U\|_{L^2}^2.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{V}(t)\|_{L^2}^2 + \frac{k}{1+t}\|\widehat{\nabla V}\|_{L^2}^2 \leq \frac{k}{1+t}\int_{|\xi|\leq r(t)} |\hat{V}(t)|^2 \mathrm{d}\xi + 2\|\nabla h\|_{L^\infty}\|U\|_{L^2}^2. \tag{3.6}$$

On the other hand, we have

$$\hat{V}(\xi,t)=\int_0^t e^{-|\xi|^{2\alpha}(t-s)}\hat{H}(\xi,s)\mathrm{d}s,$$

here

$$\hat{H}(\xi,t)=-\widehat{U\cdot\nabla U}-\widehat{\nabla P},$$

which follows

$$|\hat{H}(\xi,t)|\leq C|\xi|\|U\|_{L^2}^2.$$

Thanks to the above inequality, we have

$$\begin{aligned}\hat{V}(\xi,t)&\leq C\int_0^t(|\xi|\|U\|_{L^2}^2)\mathrm{d}s\\&\leq C|\xi|\int_{T_1}^{t+T_1}\|u\|_{L^2}^2\mathrm{d}s\\&\leq C|\xi|(1+T_1)^{-\frac{n}{2\alpha}+1}.\end{aligned}$$

Inserting above inequality into the right-hand side of (3.6), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(1+t)^k\|\hat{V}(t)\right]\|_{L^2}^2\leq C(1+t)^{k-1-\frac{n+2}{2\alpha}}(1+T_1)^{-\frac{n}{\alpha}+2}+2\|\nabla h\|_{L^\infty}\|U\|_{L^2}^2. \tag{3.7}$$

Before completing the proof, we need to show

$$\begin{aligned}\|\nabla h\|_{L^\infty}\leq\|\widehat{\nabla h}\|_{L^1}&\leq\int_{\mathbb{R}^n}|\xi||\hat{h}\left(\frac{t-1}{2}\right)|e^{-|\xi|^{2\alpha}\frac{t+1}{2}}\mathrm{d}\xi\\&\leq C\|\hat{h}\left(\frac{t-1}{2}\right)\|_{L^2}(1+t)^{-\frac{n+2}{4\alpha}}\leq C(1+t)^{-\frac{n+1}{2\alpha}}.\end{aligned} \tag{3.8}$$

Combining (3.8) into (3.7) and choosing T_1 large enough, we have

$$\|V(\cdot,t)\|_{L^2}^2\leq\frac{C(\delta)}{4}(1+t)^{-\frac{n}{2\alpha}}\ \ as\ \ t\to\infty.$$

Then, we can deduce

$$\|V(\cdot,t)\|_{L^2}^2\geq\frac{\sqrt{C(\delta)}}{2}(1+t)^{-\frac{n}{2\alpha}}\ \ as\ \ t\to\infty.$$

Now, we complete the proof of Theorem 3.2. □

4 Regularity Criterion

In [11], they proved that the generalized Navier–Stokes system is local well posed for any given initial datum $\|u_0\| \in H^s$ with $s > \max\{\frac{5}{2} - \alpha, 1\}$. However, whether this unique local solution (for the case $\alpha < \frac{5}{4}$) can exist globally is still an open problem. In this section, a new regularity criterion will be established to (1.1)–(1.2) in dimension three. Our main result is as follows

Theorem 4.1 *Suppose* $\alpha \in (0, 1]$, $u_0 \in H^m(\mathbb{R}^3)$ *with* $m \geq \frac{5}{2} - \alpha$ *and* $\mathrm{div} u_0 = 0$. *If on* $[0, T]$, $u(x, t)$ *satisfies*

$$\Lambda^\theta u(x,t) \in L^{t,s}, \quad with \ \frac{2\alpha}{t} + \frac{3}{s} \leq 2\alpha - 1 + \theta, \quad \theta \in [1-\alpha, 1], \quad \frac{3}{2\alpha - 1 + \theta} < s < \infty. \tag{4.1}$$

Then the solution remains smooth on $[0, T]$ *and satisfies*

$$u \in L^\infty(0, T; H^m(\mathbb{R}^3)) \cap L^2(0, T; H^{m+\alpha}(\mathbb{R}^3)).$$

Remark 4.1 If $\alpha = 1$, system (1.1)–(1.2) is the classical Navier–Stokes equations. Our regularity criterion reduces to

$$\Lambda^\theta u(x,t) \in L^{t,s}, \quad \text{with} \ \frac{2}{t} + \frac{3}{s} \leq 1 + \theta, \quad \theta \in [0, 1], \quad \frac{3}{1+\theta} < s < \infty. \tag{4.2}$$

When $\theta = 0$, (4.2) is almost the famous result established by Prodi [20] and Serrin [24]. When $\theta = 1$, (4.2) reduces to H. Beirão da Veiga's work in [2].

Remark 4.2 It is worth to point out that $\|\Lambda^\theta u\|_{L^{t,s}}$ is scaling invariant for $\frac{2\alpha}{t} + \frac{3}{s} = 2\alpha - 1 + \theta$. It is interesting and difficult to get similar results for $\theta \in [0, 1-\alpha)$.

Remark 4.3 Our result can also be established to any dimension by the same argument. We only present the theorem and omit the detail proof.

Theorem 4.2 *Suppose* $\alpha \in (0, 1]$, $u_0 \in H^m(\mathbb{R}^N)$ *with* $m \geq \frac{n}{2} + 1 - \alpha$, $N \geq 2$ *and* $\mathrm{div} u_0 = 0$. *If on* $[0, T]$, $u(x, t)$ *satisfies*

$$\Lambda^\theta u(x,t) \in L^{t,s}, \quad with \ \frac{2\alpha}{t} + \frac{N}{s} \leq 2\alpha - 1 + \theta, \quad \theta \in [1-\alpha, 1], \quad \frac{N}{2\alpha - 1 + \theta} < s < \infty.$$

Then the solution remains smooth on $(0, T)$ *and satisfies*

$$u \in L^\infty(0, T; H^m(\mathbb{R}^N)) \cap L^2(0, T; H^{m+\alpha}(\mathbb{R}^N)).$$

Proof Multiplying (1.1) by u, after integration by parts and taking the divergence-free property into account, we have the following energy estimate

$$\|u\|_{L^2}^2 + 2\int_0^T \|\Lambda^{\alpha} u\|_{L^2}^2 \mathrm{d}\tau = \|u_0\|_{L^2}^2.$$

H^1*-estimation*. Multiplying (1.1) by Δu , after integration by parts and taking the divergence-free property into account, we have the following energy estimate

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2 + \|\Lambda^{1+\alpha} u\|_{L^2}^2 &= \int_{\mathbb{R}^3} u \cdot \nabla u \Delta u \mathrm{d}x \\
&= \int_{\mathbb{R}^3} -\partial_k u_i \partial_i u_j \partial_k u_j \mathrm{d}x \\
&\le \int_{\mathbb{R}^3} -\widehat{\partial_k u_i} \mathcal{F}^{-1}(\partial_i u_j \partial_k u_j)\mathrm{d}x \\
&\le \int_{\mathbb{R}^3} -|\xi|^{\theta-1}\widehat{\partial_k u_i}|\xi|^{1-\theta}\mathcal{F}^{-1}(\partial_i u_j \partial_k u_j)\mathrm{d}x \\
&\le \int_{\mathbb{R}^3} -\Lambda^{\theta-1}(\partial_k u_i)\Lambda^{1-\theta}(\partial_i u_j \partial_k u_j)\mathrm{d}x \\
&\le \|\Lambda^{\theta-1}(\partial_k u_i)\|_{L^s}\|\Lambda^{1-\theta}(\partial_i u_j \partial_k u_j)\|_{L^{\frac{s}{s-1}}}
\end{aligned}$$

Note that

$$\Lambda^{\theta-1}\partial_k u = -\Lambda^{\theta-1}R_k(\Lambda u),$$

where R_i is the Riesz transform, $\widehat{R_i g}(\xi) = -i(\xi_i/|\xi|)\widehat{g}(\xi)$, and the boundedness of the operator $R_i : L^p \to L^p$, $1 < p < \infty$, we have

$$\|\Lambda^{\theta-1}(\partial_k u_i)\|_{L^p} \le \|\Lambda^{\theta} u\|_{L^p}, \quad 1 < p < \infty.$$

By the Kato–Ponce inequality, Gagliardo–Nirenberg inequality and the above estimate, we have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2 + \|\Lambda^{1+\alpha} u\|_{L^2}^2 &\le C\|\Lambda^{\theta} u\|_{L^s}\|\nabla u\|_{L^A}\|\Lambda^{2-\theta} u\|_{L^B} \\
&\le C\|\Lambda^{\theta} u\|_{L^s}\|\nabla u\|_{L^2}^{2-\mu-\nu}\|\Lambda^{1+\alpha} u\|_{L^2}^{\mu+\nu},
\end{aligned}$$

where

$$\begin{aligned}
&\frac{1}{s} + \frac{1}{A} + \frac{1}{B} = 1; \\
&\frac{1}{A} = \left(\frac{1}{2} - \frac{\alpha}{3}\right)\mu + \frac{1-\mu}{2}; \\
&\frac{1}{B} = \frac{1-\theta}{3} + \left(\frac{1}{2} - \frac{\alpha}{3}\right)\nu + \frac{1-\nu}{2}.
\end{aligned}$$

By direct calculation, we have

$$\mu = \frac{3(A-2)}{2\alpha A}, \quad \nu = \frac{5}{2\alpha} - \frac{\theta}{\alpha} - \frac{3}{\alpha B}.$$

Then,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2 + \|\Lambda^{1+\alpha}u\|_{L^2}^2 \\
&\le C\|\Lambda^\theta u\|_{L^s}\|\nabla u\|_{L^2}^{2-\frac{4-\theta}{\alpha}+\frac{3}{\alpha A}+\frac{3}{\alpha B}}\|\Lambda^{1+\alpha}u\|_{L^2}^{\frac{4-\theta}{\alpha}-\frac{3}{\alpha A}-\frac{3}{\alpha B}} \\
&\le C\|\Lambda^\theta u\|_{L^s}\|\nabla u\|_{L^2}^{2-\frac{1-\theta}{\alpha}-\frac{3}{\alpha s}}\|\Lambda^{1+\alpha}u\|_{L^2}^{\frac{1-\theta}{\alpha}+\frac{3}{\alpha s}} \\
&\le C\|\Lambda^\theta u\|_{L^s}^{\frac{2\alpha s}{2\alpha s-s+\theta s-3}}\|\nabla u\|_{L^2}^2 + \frac{1}{2}\|\Lambda^{1+\alpha}u\|_{L^2}^2.
\end{aligned}$$

Actually, the choice of A and B depends on s. Here we can choose $A = \frac{2(\alpha+1)s}{(\alpha+1)s-2}$ and $B = \frac{2(\alpha+1)s}{2(\alpha+1)s-2\alpha}$. By the Gronwall's inequality and assumption (4.1), we obtain

$$u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^{1+\alpha}).$$

H^2-estimation. Taking Δ to (1.1) and multiplying (1.1) by Δu, after integration by parts and taking the divergence-free property into account, we have the following energy estimate

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|_{L^2}^2 + \|\Lambda^{2+\alpha}u\|_{L^2}^2 &= \left|\int_{\mathbb{R}^3}\Delta(u\cdot\nabla u)\Delta u\,\mathrm{d}x\right| \\
&\le \left|\int_{\mathbb{R}^3}\partial_{hh}(u_i\partial_i u_j)\partial_{kk}u_j\,\mathrm{d}x\right| \\
&\le \left|\int_{\mathbb{R}^3}\partial_{hh}u_i\partial_i u_j\partial_{kk}u_j\,\mathrm{d}x + \int_{\mathbb{R}^3}\partial_h u_i\partial_{ih}u_j\partial_{kk}u_j\,\mathrm{d}x\right|
\end{aligned}$$

By the same argument as above.

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|_{L^2}^2 + \|\Lambda^{2+\alpha}u\|_{L^2}^2 &\le C\|\Lambda^\theta u\|_{L^s}\|\Delta u\|_{L^A}\|\Lambda^{3-\theta}u\|_{L^B} \\
&\le C\|\Lambda^\theta u\|_{L^s}\|\Delta u\|_{L^2}^{2-\mu-\nu}\|\Lambda^{2+\alpha}u\|_{L^2}^{\mu+\nu} \\
&\le C\|\Lambda^\theta u\|_{L^s}\|\Delta u\|_{L^2}^{2-\frac{4-\theta}{\alpha}+\frac{3}{\alpha A}+\frac{3}{\alpha B}}\|\Lambda^{2+\alpha}u\|_{L^2}^{\frac{4-\theta}{\alpha}-\frac{3}{\alpha A}-\frac{3}{\alpha B}} \\
&\le C\|\Lambda^\theta u\|_{L^s}\|\Delta u\|_{L^2}^{2-\frac{1-\theta}{\alpha}-\frac{3}{\alpha s}}\|\Lambda^{2+\alpha}u\|_{L^2}^{\frac{1-\theta}{\alpha}+\frac{3}{\alpha s}} \\
&\le C\|\Lambda^\theta u\|_{L^s}^{\frac{2\alpha s}{2\alpha s-s+\theta s-3}}\|\Delta u\|_{L^2}^2 + \frac{1}{2}\|\Lambda^{2+\alpha}u\|_{L^2}^2.
\end{aligned}$$

By the Gronwall's inequality and the assumption, we have

$$u \in L^\infty(0,T;H^2(\mathbb{R}^3)) \cap L^2(0,T;H^{2+\alpha}(\mathbb{R}^3)).$$

Due to Sobolev's embedding $H^s \hookrightarrow L^\infty$ with $s > \frac{3}{2}$, we have $\nabla u \in L^2(0,T;L^\infty)$ for $\alpha > \frac{1}{2}$. This completes the proof for $\alpha > \frac{1}{2}$ by the BKM criterion [1].

H^3-estimation. Then, for $\alpha \in (0, \frac{1}{2}]$, we need to show the H^3-estimation. Taking $\nabla\Delta$ to (1.1) and multiplying (1.1) by $\nabla\Delta u$, after integration by parts and taking the divergence-free property into account, we have the following energy estimate

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^3 u\|_{L^2}^2 + \|\Lambda^{3+\alpha}u\|_{L^2}^2 = -\int_{\mathbb{R}^3}\nabla\Delta(u\cdot\nabla u)\nabla\Delta u\mathrm{d}x\\
&\quad = -\int_{\mathbb{R}^3}\partial_{ikk}u\cdot\nabla u\partial_{ikk}u + 2\partial_{ik}u\cdot\partial_k\nabla u\partial_{ikk}u\\
&\qquad + \partial_{kk}u\cdot\partial_i\nabla u\partial_{ikk}u + \partial_i u\cdot\partial_{kk}\nabla u\partial_{ikk}u + 2\partial_k u\cdot\partial_{ik}\nabla u\partial_{ikk}u\mathrm{d}x\\
&\quad = K_1 + K_2 + K_3 + K_4 + K_5.
\end{aligned} \tag{4.3}$$

One can use the methods above to estimate K_1, K_4, K_5.

The section term can be estimated as above

$$\begin{aligned}
|K_1| + |K_4| + |K_5| &\le C\|\Lambda^\theta u\|_{L^s}\|\Lambda^3 u\|_{L^A}\|\Lambda^{4-\theta}u\|_{L^B}\\
&\le C\|\Lambda^\theta u\|_{L^s}\|\Lambda^3 u\|_{L^2}^{2-\mu-\nu}\|\Lambda^{3+\alpha}u\|_{L^2}^{\mu+\nu}\\
&\le C\|\Lambda^\theta u\|_{L^s}\|\Lambda^3 u\|_{L^2}^{2-\frac{4-\theta}{\alpha}+\frac{3}{\alpha A}+\frac{3}{\alpha B}}\|\Lambda^{3+\alpha}u\|_{L^2}^{\frac{4-\theta}{\alpha}-\frac{3}{\alpha A}-\frac{3}{\alpha B}}\\
&\le C\|\Lambda^\theta u\|_{L^s}\|\Lambda^3 u\|_{L^2}^{2-\frac{1-\theta}{\alpha}-\frac{3}{\alpha s}}\|\Lambda^{3+\alpha}u\|_{L^2}^{\frac{1-\theta}{\alpha}+\frac{3}{\alpha s}}\\
&\le C\|\Lambda^\theta u\|_{L^s}^{\frac{2\alpha s}{2\alpha s-s+\theta s-3}}\|\Lambda^3 u\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{3+\alpha}u\|_{L^2}^2.
\end{aligned}$$

By the Hölder inequality, Gagliardo–Nirenberg inequality and Young's inequality, we can estimate K_2 and K_3 in the right-hand side of (4.3)

$$\begin{aligned}
|K_2| + |K_3| &\le C\|\Lambda^2 u\|^2_{L^{\frac{12s}{6-2\theta s+5s}}}\|\Lambda^3 u\|_{L^{\frac{6s}{s+2\theta s-6}}}\\
&\le C(\|\Lambda^\theta u\|_{L^s}\|\Lambda^3 u\|_{L^2})\|\Lambda^3 u\|_{L^2}^{1-\frac{1}{\alpha}(1-\theta+\frac{3}{s})}\|\Lambda^{3+\alpha}u\|_{L^2}^{\frac{1}{\alpha}(1-\theta+\frac{3}{s})}\\
&\le C\|\Lambda^\theta u\|_{L^s}\|\Lambda^3 u\|_{L^2}^{2-\frac{1-\theta}{\alpha}-\frac{3}{\alpha s}}\|\Lambda^{3+\alpha}u\|_{L^2}^{\frac{1-\theta}{\alpha}+\frac{3}{\alpha s}}\\
&\le C\|\Lambda^\theta u\|_{L^s}^{\frac{2\alpha s}{2\alpha s-s+\theta s-3}}\|\Lambda^3 u\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{3+\alpha}u\|_{L^2}^2.
\end{aligned}$$

Combining the above estimates to (4.3), by the Gronwall's inequality, we have

$$u \in L^\infty(0,T;H^3(\mathbb{R}^3)) \cap L^2(0,T;H^{3+\alpha}(\mathbb{R}^3)).$$

Due to Sobolev's embedding $H^s \hookrightarrow L^\infty$ with $s > \frac{3}{2}$, we have $\nabla u \in L^2(0, T; L^\infty)$. This complete the proof of Theorem 3.1 by the regularity criterion in [1]. □

5 Global Existence

In this section, we will show that the local solution can exist globally with the small initial data.

Theorem 5.1 *Suppose $\alpha \in (0, \frac{5}{4})$, $u_0 \in H^m(\mathbb{R}^3)$ with $m \geq \frac{5}{2} - \alpha$ and $\mathrm{div} u_0 = 0$. There exists a constant $K > 0$ such that if $\|u_0\|_{\dot{H}^{\frac{5}{2}-2\alpha}} \leq K$, then there exists a unique global solution.*

Remark 5.1 Our result is partially motivated by Chae and Lee's work for Hall-MHD system in [4]. As we know, Leary [16] proved that if $\|u_0\|_{H^1}$ is small enough, then the classical Navier–Stokes equations exist globally. In 1964, Fujita and Kato [8] improved Leary's result as the initial data $\|u_0\|_{\dot{H}^{\frac{1}{2}}}$ are small enough. It seems that our result is a generalization.

Proof The proof of the global existence is based on the energy method by combining the local existence and the closure of the a priori estimate. We can use the similar method as that in [11] to obtain the local existence. Here we only need to close the priori estimate. That is, under the priori assumption that $\|u\|_{H^m(\mathbb{R}^3)}(t)$, $m \geq \frac{5}{2} - 2\alpha$ is very small, say, $\|u\|_{H^m(\mathbb{R}^3)}(t) < \delta$ where δ is a sufficiently small positive constant, we want to prove the following energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{H^m(\mathbb{R}^3)} + \|u\|^2_{H^{m+\alpha}(\mathbb{R}^3)} \leq \|u\|_{H^m(\mathbb{R}^3)}(t)\|u\|^2_{H^{m+\alpha}(\mathbb{R}^3)}.$$

In fact, this inequality is to say that if $\|u\|_{H^m(\mathbb{R}^3)}$ is priori small uniformly in time, then it will be smaller than what is expected. Now, as long as it is initially small, it must be uniformly bounded in all time due to the continuity argument.

Assume $\|u\|_{H^m(\mathbb{R}^3)}(t) < \delta$ where δ is a sufficiently small positive constant.

Taking $\Lambda^{\frac{5}{2}-2\alpha}$ to (1.1) and multiplying (1.1) by $\Lambda^{\frac{5}{2}-2\alpha}u$, we have the following energy estimate

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^{\frac{5}{2}-2\alpha}u\|^2_{L^2} + \|\Lambda^{\frac{5}{2}-\alpha}u\|^2_{L^2}\\
&= \left|\int_0^T \Lambda^{\frac{5}{2}-2\alpha}(u\cdot\nabla u)\Lambda^{\frac{5}{2}-2\alpha}u\mathrm{d}x\right|\\
&\leq C\|\Lambda^{\frac{5}{2}-2\alpha}(u\cdot\nabla u) - u\cdot\nabla\Lambda^{\frac{5}{2}-2\alpha}u\|_{L^2}\|\Lambda^{\frac{5}{2}-2\alpha}u\|_{L^2}\\
&\leq C\|\nabla u\|_{L^{\frac{3}{\alpha}}}\|\Lambda^{\frac{5}{2}-2\alpha}u\|_{L^{\frac{6}{3-2\alpha}}}\|\Lambda^{\frac{5}{2}-2\alpha}u\|_{L^2}\\
&\leq C\|\Lambda^{\frac{5}{2}-2\alpha}u\|_{L^2}\|\Lambda^{\frac{5}{2}-\alpha}u\|^2_{L^2}.
\end{aligned}$$

Here, we have used the Kato–Ponce inequality.

Choosing K so small that

$$C\|\Lambda^{\frac{5}{2}-2\alpha}u_0\|_{L^2} \le \frac{1}{2},$$

then, for any $T > 0$, we have

$$\|\Lambda^{\frac{5}{2}-2\alpha}u\|_{L^2}^2 + \int_0^T \|\Lambda^{\frac{5}{2}-\alpha}u\|_{L^2}^2 d\tau \le \|\Lambda^{\frac{5}{2}-2\alpha}u_0\|_{L^2}^2.$$

Then, we will show the H^m-estimate for the generalized Navier–Stokes equations.

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Lambda^m u\|_{L^2}^2 + \|\Lambda^{m+\alpha}u\|_{L^2}^2 &= \left|\int_{\mathbb{R}^3} \Lambda^m(u\cdot\nabla u)\Lambda^m u dx\right| \\ &\le C\|\nabla u\|_{L^{\frac{3}{\alpha}}}\|\Lambda^m u\|_{L^2}\|\Lambda^m u\|_{L^{\frac{6}{3-2\alpha}}} \\ &\le C\|\Lambda^{\frac{5}{2}-\alpha}u\|_{L^2}\|\Lambda^m u\|_{L^2}\|\Lambda^{m+\alpha}u\|_{L^2} \\ &\le C\|\Lambda^{\frac{5}{2}-\alpha}u\|_{L^2}^2\|\Lambda^m u\|_{L^2}^2 + \frac{1}{2}\|\Lambda^{m+\alpha}u\|_{L^2}^2.\end{aligned}$$

Choosing K so small that

$$C\|\Lambda^{\frac{5}{2}-2\alpha}u_0\|_{L^2} \le \frac{1}{2},$$

then, for any $T > 0$, we have

$$\|\Lambda^m u\|_{L^2}^2 + \int_0^T \|\Lambda^{m+\alpha}u\|_{L^2}^2 d\tau \le C\|\Lambda^m u_0\|_{L^2}^2.$$

This completes the proof of Theorem 4.1. □

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