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The Total Eccentricity Sum of Non-adjacent Vertex Pairs in Graphs

Hongbo Hua[1] · **Zhengke Miao**[2]

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Abstract Classical topological indices, such as Zagreb indices (M_1 and M_2) and the well-studied eccentric connectivity index (ξ^c) directly or indirectly consider the total contribution of all edges in a graph. By considering the total degree sum of all non-adjacent vertex pairs in a graph, Ashrafi et al. (Discrete Appl Math 158:1571–1578, 2010) proposed two new Zagreb-type indices, namely the first Zagreb coindex ($\overline{M}_1$) and second Zagreb coindex ($\overline{M}_2$), respectively. Motivated by Ashrafi et al., we consider the total eccentricity sum of all non-adjacent vertex pairs, which we call the eccentric connectivity coindex ($\overline{\xi}^c$), of a connected graph. In this paper, we study the extremal problems of $\overline{\xi}^c$ for connected graphs of given order, connected graphs of given order and size, and the trees, unicyclic graphs, bipartite graphs containing cycles and triangle-free graphs of given order, respectively. Additionally, we establish various lower bounds for $\overline{\xi}^c$ in terms of several other graph parameters.

Keywords Degree · Distance · Eccentricity · Bounds · Extremal graphs

Mathematics Subject Classification 05C35 · 05C12 · 05C90

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✉ Hongbo Hua
hongbo_hua@163.com; hbhua@hyit.edu.cn

Zhengke Miao
miaozkmath@sina.com; zkmiao@jsnu.edu.cn

1 Faculty of Mathematics and Physics, Huaiyin Institute of Technology, Huai'an 223003, Jiangsu, People's Republic of China

2 School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou 221116, Jiangsu, People's Republic of China

1 Introduction

Throughout this paper, we consider only simple connected graphs. For a graph $G = (V, E)$ with vertex set $V = V(G)$ and edge set $E = E(G)$, the *degree* of a vertex v in G, denoted by $d_G(v)$, is the number of edges incident with v. Let $d_G(u, v)$ be the distance between vertices u and v in G. The *eccentricity* of a vertex v in a graph G is defined to be $\varepsilon_G(v) = \max\{d_G(u, v)|u \in V(G)\}$. The *diameter* of a connected graph is the greatest distance between any pair of vertices in this graph. A vertex subset S of a graph G is said to be an *independent set* of G, if the subgraph induced by S is an empty graph. An edge subset T of a graph G is said to be a *matching* of G, if any two edges in T do not share a common end vertex. Then, $\alpha = \max\{|S| : S$ is an independent set of $G\}$ and $\beta = \max\{|T| : T$ is a matching of $G\}$ are said to be the *independence number* and *matching number* of G, respectively. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colors needed to guarantee that G can be colored with these colors so that no two adjacent vertices have the same color. The *vertex-connectivity* $\kappa(G)$ (where G is not a complete graph) is the size of a minimal vertex-cut, and the *edge-connectivity* is $\kappa'(G)$ is the size of a smallest edge cut.

A topological index is a function defined on a (molecular) graph regardless of the labeling of its vertices. Till now, hundreds of different topological indices have been employed in QSAR/QSPR studies, some of which have been proved to be successful [25]. Among those successful topological indices, there are two degree-based topological indices, called the *first Zagreb index* and the *second Zagreb index*, which are defined to be

$$M_1(G) = \sum_{u \in V(G)} \Big(d_G(u)\Big)^2 \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v),$$

respectively. During the past decades, a large amount of papers dealt with the properties of these two indices. For more details on Zagreb indices, see the recent papers [4,5,8–12,15,18,21,23,24,28,29]. Recall that the first Zagreb index can be rewritten as

$$M_1(G) = \sum_{uv \in E(G)} \Big(d_G(u) + d_G(v)\Big).$$

According to the above equality, Ashrafi et al. [1,2] considered the total contribution of all non-adjacent vertex pairs in a graph, and they proposed two new Zagreb-type indices, namely the *first Zagreb coindex* and *second Zagreb coindex*, which are defined to be

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} \Big(d_G(u) + d_G(v)\Big) \quad \text{and} \quad \overline{M}_2(G) = \sum_{uv \notin E(G)} d_G(u) d_G(v),$$

respectively. For recent results on Zagreb coindices, see [1,2,16,17].

The *eccentric connectivity index* of a connected graph G, denoted by $\xi^c(G)$, is defined as

$$\xi^c(G) = \sum_{v \in V(G)} d_G(v) \varepsilon_G(v),$$

where $\varepsilon_G(v)$ is the eccentricity of the vertex v.

The eccentric connectivity index is a graph invariant which can be used to predict biological and physical properties and has a vast potential in structure activity/property relationships see [13,14,19]. For the mathematical properties of this index, see [3,20,22] and the references cited therein. The eccentric connectivity index of a connected graph G can be rewritten as

$$\xi^c(G) = \sum_{uv \in E(G)} \Big(\varepsilon_G(u) + \varepsilon_G(v)\Big).$$

Motivated by Ashrafi et al.'s definition for Zagreb coindices, we consider the total eccentricity sum of all non-adjacent vertex pairs, which is defined for a connected graph G as

$$\overline{\xi}^c(G) = \sum_{uv \notin E(G)} \Big(\varepsilon_G(u) + \varepsilon_G(v)\Big). \tag{1}$$

Similar to Ashrafi et al.'s definition for Zagreb coindices, we call this new eccentricity-based graph invariant the *eccentric connectivity coindex* ($\overline{\xi}^c$).

By (1), we can rewrite $\overline{\xi}^c$ of a connected graph G as

$$\overline{\xi}^c(G) = \sum_{u \in V(G)} \varepsilon_G(u)\Big(n - 1 - d_G(u)\Big). \tag{2}$$

In this paper, we mainly study extremal properties of $\overline{\xi}^c$. We organize this paper as follows. In Sect. 2, we characterize all extremal graphs with the maximum and minimum $\overline{\xi}^c$, respectively, among all connected graphs of given order. In Sect. 3, we characterize the connected graph with given order, size and the minimum $\overline{\xi}^c$ as well as the tree, unicyclic graph, bipartite graph containing cycles and triangle-free graph with the minimum $\overline{\xi}^c$, respectively. In Sect. 4, we establish various lower bounds for $\overline{\xi}^c$ in terms of several other graph parameters including the number of pendent vertices, independence number, matching number, chromatic number, vertex-connectivity, and edge-connectivity.

Before proceeding, we introduce some further notation and terminology. A vertex in a graph is said to be a *branch vertex* if it is of degree no less than three. If the length of an internal path in a connected graph is equal to diameter, then it is said to be a *diametrical path*. Let G and H be two vertex-disjoint graphs. The *join* of graphs G and H, denoted by $G \vee H$, is defined as a graph whose vertex set is $V(G) \cup V(H)$ and edge set is $E(G) \cup E(H) \cup \{xy | x \in V(G), y \in V(H)\}$. Let tK_1 be the union of t copies of K_1. Denote by $T_{n,t}$ the *Turán graph*, a complete t-partite graph of order n with $|n_i - n_j| \leq 1$, where $n_i, i = 1, \ldots, t$, is the number of vertices in the ith partite set of $T_{n,t}$. When $t = 2$, $T_{n,2}$ is just the balanced bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. Denoted by P_n, S_n and K_n the path, star and complete graph on n vertices, respectively. Let K_n^p denote the graph obtained by attaching p pendent edges to a vertex of K_{n-p}. Other notation and terminology not defined here will conform to those in [7].

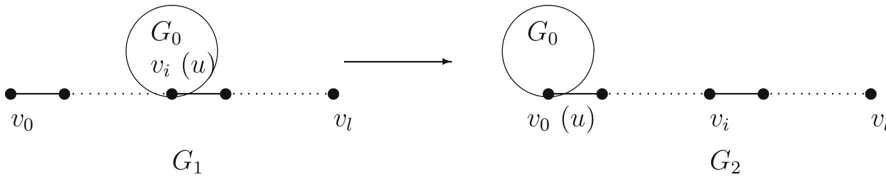

Fig. 1 Operation I: $G_1 \longrightarrow G_2$

2 General Connected Graphs

In this section, we characterize all extremal graphs with the maximum and minimum $\overline{\xi}^c$, respectively, among all connected graphs of given order.

We first give two lemmas which will be used in the proof of our main results.

Lemma 2.1 *Let G be a connected graph with at least three vertices.*

(i) *If G is not isomorphic to K_n, then $\overline{\xi}^c(G) > \overline{\xi}^c(G+e)$, where $e \in E(\overline{G})$;*
(ii) *If G has an edge e not being a cut edge, then $\overline{\xi}^c(G) < \overline{\xi}^c(G-e)$.*

Proof We first prove (i) holds. Suppose that G is not a complete graph. Then, there exists a pair of vertices u and v in G such that $uv \in E(\overline{G})$. It is obvious that $\varepsilon_G(x) \geq \varepsilon_{G+uv}(x)$ and $d_G(x) \leq d_{G+uv}(x)$ for any vertex x in G. Also, we have $d_{G+uv}(u) > d_G(u)$. By (2), we have $\overline{\xi}^c(G) > \overline{\xi}^c(G+e)$, as claimed.

Now, we consider (ii). Suppose that e is not a cut edge in G. Since $G-e$ is connected and not the complete graph, by (i), we have $\overline{\xi}^c(G-e) > \overline{\xi}^c\Big((G-e)+e\Big) = \overline{\xi}^c(G)$, as desired (Fig. 1). □

Lemma 2.2 *Suppose that G_0 is a nontrivial connected graph and u is a vertex in G_0. Let G_1 (resp., G_2) be a graph obtained by identifying the vertex u of G_0 with a non-pendent vertex v_i (resp., a pendent vertex, say v_0,) of the path $P_{l+1} : v_0v_1 \ldots, v_l$ $(l \geq 2)$, where $1 \leq i \leq l-1$. If $\varepsilon_{G_0}(u) \geq \max\{l-i,\ i\}$, then $\overline{\xi}^c(G_1) < \overline{\xi}^c(G_2)$.*

Proof Suppose without loss of generality that $l-i \geq i$. Then, $\varepsilon_{G_0}(u) \geq l-i \geq i$. For each $x \in V(G_0)\backslash\{u\}$, we have $\varepsilon_{G_2}(x) = \max\{\varepsilon_{G_0}(x),\ d_{G_0}(x,\ u)+l\}$, $\varepsilon_{G_1}(x) = \max\{\varepsilon_{G_0}(x),\ d_{G_0}(x,\ u)+l-i\}$ (as $l-i \geq i$). So, $\varepsilon_{G_2}(x) \geq \varepsilon_{G_1}(x)$ for each $x \in V(G_0)\backslash\{u\}$. Also, $d_{G_2}(x) = d_{G_1}(x)$ for each $x \in V(G_0)\backslash\{u\}$. Thus,

$$\sum_{x\in V(G_0)\backslash\{u\}} \varepsilon_{G_2}(x)\Big(n-1-d_{G_2}(x)\Big) - \sum_{x\in V(G_0)\backslash\{u\}} \varepsilon_{G_1}(x)\Big(n-1-d_{G_1}(x)\Big) \geq 0. \quad (3)$$

We first assume that $i \geq 2$. Thus, $l \geq 2i \geq 4$.

For each $k = 1,\ \ldots,\ i-1$, we have $\varepsilon_{G_2}(v_k) = \max\{k+\varepsilon_{G_0}(u),\ l-k\} \geq k+\varepsilon_{G_0}(u)$, $\varepsilon_{G_1}(v_k) = \max\{i-k+\varepsilon_{G_0}(u),\ l-k\} = i-k+\varepsilon_{G_0}(u)$ (as $\varepsilon_{G_0}(u) \geq l-i$). Also, $d_{G_2}(v_k) = d_{G_1}(v_k) = 2$.

Since $\varepsilon_{G_2}(v_k) \geq k+\varepsilon_{G_0}(u)$, we have

$$\varepsilon_{G_2}(v_k) - \varepsilon_{G_1}(v_k) \geq \Big(k+\varepsilon_{G_0}(u)\Big) - \Big(i-k+\varepsilon_{G_0}(u)\Big) = 2k-i.$$

So,

$$\begin{aligned}
&\sum_{k=1}^{i-1} \varepsilon_{G_2}(v_k)\Big(n-1-d_{G_2}(v_k)\Big) - \sum_{k=1}^{i-1} \varepsilon_{G_1}(v_k)\Big(n-1-d_{G_1}(v_k)\Big) \\
&= (n-3)\sum_{k=1}^{i-1}\Big(\varepsilon_{G_2}(v_k) - \varepsilon_{G_1}(v_k)\Big) \\
&\geq (n-3)\sum_{k=1}^{\lfloor \frac{i}{2} \rfloor}(2k-i) \quad (\text{as } 2k-i \geq 0 \text{ for } k \geq \tfrac{i}{2}) \\
&\geq (n-3)\lfloor \frac{i}{2} \rfloor (2-i).
\end{aligned} \tag{4}$$

For each $k = i+1, \ldots, l-1$, we have $\varepsilon_{G_2}(v_k) = \max\{k + \varepsilon_{G_0}(u),\ l-k\}$, $\varepsilon_{G_1}(v_k) = \max\{k-i+\varepsilon_{G_0}(u),\ l-k,\ k\}$. Since $k+\varepsilon_{G_0}(u) \geq k+(l-i) \geq l+1 > l-k$, then $\varepsilon_{G_2}(v_k) = k + \varepsilon_{G_0}(u)$. Also, because $2k - i + \varepsilon_{G_0}(u) \geq k + 1 + \varepsilon_{G_0}(u) > i + \varepsilon_{G_0}(u) \geq l$, we get $k-i+\varepsilon_{G_0}(u) > l-k$. Moreover, $k-i+\varepsilon_{G_0}(u) \geq k-i+(l-i) \geq k$. Therefore, $\varepsilon_{G_1}(v_k) = k - i + \varepsilon_{G_0}(u)$. Then,

$$\varepsilon_{G_2}(v_k) = \varepsilon_{G_1}(v_k) + i$$

for each $k = i+1, \ldots, l-1$. Also, for each $k = i+1, \ldots, l-1$, $d_{G_2}(v_k) = d_{G_1}(v_k) = 2$. So,

$$\begin{aligned}
&\sum_{k=i+1}^{l-1} \varepsilon_{G_2}(v_k)\Big(n-1-d_{G_2}(v_k)\Big) - \sum_{k=i+1}^{l-1} \varepsilon_{G_1}(v_k)\Big(n-1-d_{G_1}(v_k)\Big) \\
&= (n-3)\sum_{k=i+1}^{l-1}\Big(\varepsilon_{G_2}(v_k) - \varepsilon_{G_1}(v_k)\Big) \\
&= (n-3)(l-i-1)i.
\end{aligned} \tag{5}$$

As $l \geq 2i$, we obtain $(l-i-1)i + \lfloor \frac{i}{2} \rfloor (2-i) \geq (l-i-1)i + \frac{i-1}{2}(2-i) \geq (i-1)i + \frac{i-1}{2}(2-i) = \frac{i(i+1)}{2} - 1 \geq 0$. This, in conjunction with (4) and (5), gives

$$\sum_{k=1,\, k\neq i}^{l-1} \varepsilon_{G_2}(v_k)\Big(n-1-d_{G_2}(v_k)\Big) - \sum_{k=1,\, k\neq i}^{l-1} \varepsilon_{G_1}(v_k)\Big(n-1-d_{G_1}(v_k)\Big) \geq 0. \tag{6}$$

By (3) and (6), we have

$$\sum_{y\in V(G_2)\backslash\{v_0,\, v_i,\, v_l\}} \varepsilon_{G_2}(y)\Big(n-1-d_{G_2}(y)\Big) - \sum_{y\in V(G_1)\backslash\{v_0,\, v_i,\, v_l\}} \varepsilon_{G_1}(y)\Big(n-1-d_{G_1}(y)\Big) \geq 0. \tag{7}$$

Now, by (7), it suffices to prove that

$$\sum_{y\in\{v_0,\, v_i,\, v_l\}} \varepsilon_{G_2}(y)\Big(n-1-d_{G_2}(y)\Big) - \sum_{y\in\{v_0,\, v_i,\, v_l\}} \varepsilon_{G_1}(y)\Big(n-1-d_{G_1}(y)\Big) \geq 0.$$

For vertex v_0, $\varepsilon_{G_2}(v_0) = \max\{\varepsilon_{G_0}(u),\, l\}$, $\varepsilon_{G_1}(v_0) = \max\{i+\varepsilon_{G_0}(u),\, l\} = i+\varepsilon_{G_0}(u)$ (as $\varepsilon_{G_0}(u) \geq l-i$), $d_{G_2}(v_0) = 1+d_{G_0}(u)$, $d_{G_1}(v_0) = 1$.

For vertex v_i, $\varepsilon_{G_2}(v_i) = \max\{i+\varepsilon_{G_0}(u),\, l-i\} = i+\varepsilon_{G_0}(u)$, $\varepsilon_{G_1}(v_i) = \max\{\varepsilon_{G_0}(u),\, l-i\} = \varepsilon_{G_0}(u)$, $d_{G_2}(v_i) = 2$, $d_{G_1}(v_i) = d_{G_0}(u)+2$.

For vertex v_l, $\varepsilon_{G_2}(v_l) = l+\varepsilon_{G_0}(u)$, $\varepsilon_{G_1}(v_l) = \max\{l-i+\varepsilon_{G_0}(u),\, l\} = l-i+\varepsilon_{G_0}(u)$ (since $\varepsilon_{G_0}(u) \geq l-i \geq i$, we have $l-i+\varepsilon_{G_0}(u) \geq (l-i)+i = l$). Also, $d_{G_2}(v_l) = d_{G_1}(v_l) = 1$.

By (2) and (7), we have

$$\begin{aligned}
\overline{\xi}^c(G_2) - \overline{\xi}^c(G_1) &\geq \sum_{y\in\{v_0,\, v_i,\, v_l\}} \varepsilon_{G_2}(y)(n-1-d_{G_2}(y)) \\
&\quad - \sum_{y\in\{v_0,\, v_i,\, v_l\}} \varepsilon_{G_1}(y)(n-1-d_{G_1}(y)) \\
&= \Big[(l+\varepsilon_{G_0}(u))(n-2) - (l-i+\varepsilon_{G_0}(u))(n-2)\Big] \\
&\quad + \Big[(i+\varepsilon_{G_0}(u))(n-3) - \varepsilon_{G_0}(u)(n-d_{G_0}(u)-3)\Big] \\
&\quad + \Big[\max\{l,\ \varepsilon_{G_0}(u)\}(n-d_{G_0}(u)-2) - (i+\varepsilon_{G_0}(u))(n-2)\Big] \\
&= (n-2)i+(n-3)(i+\varepsilon_{G_0}(u))-(n-3)\varepsilon_{G_0}(u) + d_{G_0}(u)\varepsilon_{G_0}(u) \\
&\quad + \max\{l,\ \varepsilon_{G_0}(u)\}(n-d_{G_0}(u)-2) - (n-2)(i+\varepsilon_{G_0}(u)) \\
&= (n-2)i - (i+\varepsilon_{G_0}(u)) - (n-3)\varepsilon_{G_0}(u) + d_{G_0}(u)\varepsilon_{G_0}(u) \\
&\quad + \max\{l,\ \varepsilon_{G_0}(u)\}(n-d_{G_0}(u)-2).
\end{aligned} \tag{8}$$

We distinguish the following two cases.

Case 1 $\varepsilon_{G_0}(u) \geq l$.

Then, by (8), we have

$$\begin{aligned}
\overline{\xi}^c(G_2) - \overline{\xi}^c(G_1) &\geq (n-2)i - (i+\varepsilon_{G_0}(u)) - (n-3)\varepsilon_{G_0}(u) + d_{G_0}(u)\varepsilon_{G_0}(u) \\
&\quad + \varepsilon_{G_0}(u)(n-d_{G_0}(u)-2) \\
&= (n-3)i > 0.
\end{aligned}$$

Case 2 $\varepsilon_{G_0}(u) < l$.
Then, by (8), we have

$$\begin{aligned}\overline{\xi}^c(G_2) - \overline{\xi}^c(G_1) &\geq (n-2)i - (i+\varepsilon_{G_0}(u)) - (n-3)\varepsilon_{G_0}(u) + d_{G_0}(u)\varepsilon_{G_0}(u)\\ &\quad + l(n - d_{G_0}(u) - 2)\\ &= (n-3)i + (n-2)l - (n-2)\varepsilon_{G_0}(u) + d_{G_0}(u)\varepsilon_{G_0}(u) - ld_{G_0}(u)\\ &> (n-3)i + d_{G_0}(u)\varepsilon_{G_0}(u) - ld_{G_0}(u). \end{aligned} \tag{9}$$

By our assumption $\varepsilon_{G_0}(u) \geq l - i$, we have $i \geq l - \varepsilon_{G_0}(u)$. By (9), we have

$$\begin{aligned}\overline{\xi}^c(G_2) - \overline{\xi}^c(G_1) &> (n-3)(l - \varepsilon_{G_0}(u)) + d_{G_0}(u)\varepsilon_{G_0}(u) - ld_{G_0}(u)\\ &= [d_{G_0}(u) - (n-3)](\varepsilon_{G_0}(u) - l). \end{aligned} \tag{10}$$

Clearly, $d_{G_0}(u) \leq n-3$. By our assumption, $\varepsilon_{G_0}(u) < l$, we have $[d_{G_0}(u) - (n-3)](\varepsilon_{G_0}(u) - l) \geq 0$. So, $\overline{\xi}^c(G_2) > \overline{\xi}^c(G_1)$.

Summarizing above, when $i \geq 2$, $\overline{\xi}^c(G_2) > \overline{\xi}^c(G_1)$.

Now, we consider the case of $i = 1$. If $l \geq 3$, by the same approach as above, we can prove that (5) holds. So, no matter whether $l = 2$ or $l \geq 3$, by (3), we can prove that (7) holds. What remains to do is exactly the same as that used in the case of $i \geq 2$.

This completes the proof. □

For graphs G_1 and G_2 as introduced in Lemma 2.2, we call the graph operation: $G_1 \Longrightarrow G_2$ the *Operation I* on G_1.

By means of Lemmas 2.1 and 2.2, we are in a position to characterize connected graphs with the maximum and minimum $\overline{\xi}^c$, respectively. Our result is as follows.

Theorem 2.3 *Among all connected graphs of order n, the graphs with the minimum and maximum $\overline{\xi}^c$ are K_n and P_n, respectively.*

Proof The case of $n = 2$ is trivial. So we suppose that $n \geq 3$.

We first prove that K_n is minimal with respect to $\overline{\xi}^c$. If G is not a complete graph, then we can repeatedly add edges into G until we obtain $G \cong K_n$. By Lemma 2.1 (i), $\overline{\xi}^c(G) \geq \overline{\xi}^c(K_n)$, with equality if and only if $G \cong K_n$.

Now, let us assume that G is maximal with respect to $\overline{\xi}^c$. We shall prove that $G \cong P_n$. Suppose first that G is not isomorphic to a tree. Let $Span(G)$ be one spanning tree of G. It then follows from Lemma 2.1 (ii) that $\overline{\xi}^c(G) < \overline{\xi}^c(Span(G))$, a contradiction to our choice of G. So, G must be a tree. We further claim that $G \cong P_n$. Suppose, to the contrary, that $G \ncong P_n$. Then, G has at least a branched vertex.

We choose a diametrical path, say $P_{d+1} : v_0 v_1 \ldots v_d$, in G. We claim that there exists no branched vertices outside the path P_{d+1}. If it is not so, we choose a branched vertex, say u, among all branched vertices outside the path P_{d+1} such that $\max\{d_G(u, v_0), d_G(u, v_d)\} = \max\left\{\max\{d_G(x, v_0), d_G(x, v_d)\}\right\}$, where x is any one branched vertex in $V(G)\backslash V(P_{d+1})$. Assume without loss of generality that $d_G(u, v_d) = \max\{d_G(u, v_0), d_G(u, v_d)\}$. Let $G - u = G_1 \cup G_2 \cup G_3 \cdots \cup G_k$, where G_1 is assumed to be the component containing v_0 and v_d. Since u is a branched vertex, $k \geq 3$. By our choice of u, each G_i $(i \geq 2)$ can not have branched vertices, that is, each induced subgraph $G[G_i \cup \{u\}] \cong P_{n_i}$ for $i = 2, \ldots, k$. Let u_i be another pendent vertex, different from u, of P_{n_i} for $i = 2, \ldots, k$. Then, $d_G(u, v_d) \geq d_G(u, u_i)$ for each $i = 2, \ldots, k$, for otherwise, there exists some i such that $d_G(u_i, v_d) = d_G(u_i, u) + d_G(u, v_d) > 2d_G(u, v_d) \geq d_G(u, v_0) + d_G(u, v_d) > d$, a contradiction.

Now, let $G_0 = G\Big[\{u\}\cup\Big(V(G)\backslash(V(G_2)\cup V(G_3))\Big)\Big]$. Then, $\varepsilon_{G_0}(u) \geq d_G(u, u_d) \geq d_G(u, u_2)$ and $\varepsilon_{G_0}(u) \geq d_G(u, u_d) \geq d_G(u, u_3)$. So, we can employ Operation I, introduced as in Lemma 2.1, on G, and we get a new graph $G^{'}$. By Lemma 2.2, we have $\overline{\xi}^c(G) < \overline{\xi}^c(G^{'})$, a contradiction to our choice of G.

Similarly, the diametrical path P_{d+1} cannot have branched vertices. If it is not so, we may choose a branched vertex, say u, among all branched vertices along the path $P_{d+1} : v_0v_1 \ldots v_d$ such that $\max\{d_G(u, v_0), d_G(u, v_d)\} = \max\Big\{\max\{d_G(x, v_0), d_G(x, v_d)\}\Big\}$, where x is a branched vertex in $V(P_{d+1})$. Similar to above, we can employ Operation I on G to obtain a contradiction.

Therefore, $G \cong P_n$, as desired. □

Remark 2.4 In fact, we may give a more direct proof than above for the first part of Theorem 2.3. According to (2), if a vertex is of degree $n-1$, then the contribution of this vertex to $\overline{\xi}^c$ is equal to 0. Since K_n is the unique graph having the maximum number of vertices of degree $n-1$, K_n is the unique graph minimal with respect to $\overline{\xi}^c$. But, Lemma 2.1 will be frequently used in the subsequent part of this paper. So, we use the current approach to prove the first part of Theorem 2.3.

3 Trees, Unicyclic Graphs, Bipartite Graphs Containing Cycles and Triangle-Free Graphs

In this section, we shall determine the tree, unicyclic graph, bipartite graph containing cycles and triangle-free graph with the minimum $\overline{\xi}^c$, respectively. First, we deduce a lower bound for $\overline{\xi}^c$ of a connected graph in terms of its order and size.

Theorem 3.1 *Let G be a connected graph of order n, size m and diameter d. Then*

$$\overline{\xi}^c(G) \geq 2n(n-1) - 4m$$

with equality if and only if $d \leq 2$.

Proof Suppose that N is the set of vertices of degree $n-1$, and n_0 is the number of elements in N. For any u in $V(G)\backslash N$, we have $\varepsilon_G(u) \geq 2$. By (2), we have

$$\begin{aligned}\overline{\xi}^c(G) &= \sum_{u\in V(G)} \varepsilon_G(u)(n-1-d_G(u)) \\ &= \sum_{u\in N} \varepsilon_G(u)(n-1-d_G(u)) + \sum_{u\in V(G)\backslash N} \varepsilon_G(u)(n-1-d_G(u))\end{aligned}$$

$$
\begin{aligned}
&\geq 2 \sum_{u\in V(G)\setminus N} (n-1-d_G(u)) \\
&= 2(n-1)(n-n_0) - 2\left(\sum_{u\in V(G)} d_G(u) - n_0(n-1)\right) \\
&= 2n(n-1) - 4m,
\end{aligned}
$$

where the equality is attained if and only if $\varepsilon_G(x) = 2$ for each $x \in V(G)\setminus N$, i.e., $\varepsilon_G(x) \leq 2$ for each $x \in V(G)$, i.e., $d \leq 2$.

This completes the proof. □

According to Theorem 3.1, we get the following two results on $\overline{\xi}^c$ for trees and unicyclic graphs, respectively.

Corollary 3.2 *Let T be a tree of order n. Then*

$$\overline{\xi}^c(T) \geq 2n^2 - 6n + 4$$

with equality if and only if $T \cong S_n$.

Proof Suppose that n_0 is the number of vertices of degree $n-1$ in G. Then, $n_0 = 0$ or 1. If $n_0 = 1$, then $G \cong S_n$, and $\overline{\xi}^c(G) = 2n^2 - 6n + 4$. Now, we assume that $n_0 = 0$. Let d be the diameter of G. Then, $d \geq 3$. By Theorem 3.1, $\overline{\xi}^c(G) > 2n(n-1) - 4m = 2n(n-1) - 4(n-1) = 2n^2 - 6n + 4$. This completes the proof. □

Similarly, for a unicyclic graph, we have

Corollary 3.3 *Let G be a unicyclic graph of order n. Then,*

$$\overline{\xi}^c(G) \geq 2n^2 - 6n$$

with equality if and only if $G \cong S_n^3$, where S_n^3 is the graph obtained by introducing an edge between two pendent vertices of the star S_n.

Now, we consider bipartite graphs containing cycles. We first prove a more general result which deals with the graphs with given chromatic number.

Theorem 3.4 *Let G be a connected graph of order n with chromatic number χ such that $n = q\chi + p$, $0 \leq p \leq \chi - 1$. Then,*

$$\overline{\xi}^c(G) \geq 4nq - 2q(q+1)\chi$$

with equality if and only if $G \cong T_{n,\chi}$.

Proof Let $G_{\min}$ be a graph chosen among all connected graphs of order n with chromatic number χ such that $G_{\min}$ has the smallest $\overline{\xi}^c$. By Lemma 2.1(i), the addition of edges into a graph decreases its $\overline{\xi}^c$. Thus, we have $G_{\min} \cong \overline{K_{n_1}} \vee \overline{K_{n_2}} \vee \cdots \vee \overline{K_{n_\chi}}$, where n_i is the number of vertices in the ith partite set.

By (2), we obtain

$$\begin{aligned}\overline{\xi}^c(G_{\min}) &= \sum_{i=1}^{\chi} n_i \cdot 2\Big[n-1-(n-n_i)\Big]\\ &= \sum_{i=1}^{\chi} 2n_i(n_i-1)\\ &= 2\sum_{i=1}^{\chi} n_i^2 - 2n.\end{aligned}$$

Suppose that $G_{\min} \ncong T_{n,\chi}$. Then, there exists $n_j \geq n_i + 2$ for some $1 \leq i,\ j \leq \chi$. We construct a new graph $G^{'} = \overline{K_{n_1}} \vee \cdots \overline{K_{n_i+1}} \vee \cdots \vee \overline{K_{n_j-1}} \vee \cdots \overline{K_{n_\chi}}$.

Then,

$$\begin{aligned}\overline{\xi}^c(G^{'}) - \overline{\xi}^c(G_{\min}) &= 2[(n_j-1)^2 - n_j^2 + (n_i+1)^2 - n_i^2]\\ &= 4(n_i + 1 - n_j)\\ &< 0,\end{aligned}$$

a contradiction.

So, $G_{\min} \cong T_{n,\chi}$. Moreover, we have

$$\begin{aligned}\overline{\xi}^c(T_{n,\chi}) &= p(q+1)\cdot 2\Big[n-1-(n-q-1)\Big] + (\chi-p)q\cdot 2\Big[n-1-(n-q)\Big]\\ &= 2pq(q+1) + 2q(q-1)(\chi-p)\\ &= 4pq + 2q(q-1)\chi\\ &= 4q(n-q\chi) + 2q(q-1)\chi\\ &= 4nq - 2q(q+1)\chi.\end{aligned}$$

This completes the proof. □

Since a bipartite graph is a graph with chromatic number $\chi = 2$, by Theorem 3.4, we have

Corollary 3.5 *Let G be a cycle-containing bipartite graph of order n. Then,*

$$\overline{\xi}^c(G) \geq \begin{cases} n^2 - 2n + 1 & \text{if } n \text{ is odd,}\\ n^2 - 2n & \text{if } n \text{ is even.}\end{cases}$$

Each of above equalities holds if and only if $G \cong K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

Now, we consider the triangle-free graph. First, we recall Turán's Theorem, which is stated as follows.

Theorem 3.6 ([26]) *Let G be a connected K_{q+1}-free graph of order n and size m. Then,*

$$m \leq \left\lfloor \left(1-\frac{1}{q}\right)\cdot\frac{n^2}{2} \right\rfloor$$

with equality if and only if G is a complete q-partite graph in which all classes are of almost equal cardinality.

Theorem 3.7 *Let G be a connected triangle-free graph of order n. Then,*

$$\overline{\xi}^c(G) \geq \begin{cases} n^2-2n+1 & \text{if } n \text{ is odd,} \\ n^2-2n & \text{if } n \text{ is even.} \end{cases} \tag{11}$$

Each of above equalities holds if and only if $G \cong K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

Proof Suppose that G is a connected triangle-free graph of order n and size m. Since G is triangle-free, by Theorem 3.6, we have

$$m \leq \left\lfloor \left(1-\frac{1}{2}\right)\cdot\frac{n^2}{2} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor \tag{12}$$

with equality holds if and only if $G \cong K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

By Theorem 3.1 and (12),

$$\begin{aligned} \overline{\xi}^c(G) &\geq 2n(n-1)-4m & (13)\\ &\geq 2n(n-1)-4\left\lfloor \frac{n^2}{4} \right\rfloor \\ &= \begin{cases} n^2-2n+1 & \text{if } n \text{ is odd,} \\ n^2-2n & \text{if } n \text{ is even.} \end{cases} & (14) \end{aligned}$$

By Theorem 3.1, the equality in (13) holds if and only if $d \leq 2$. From (12), we know that the equality in (14) holds if and only if $G \cong K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. So, the equality in (11) holds if and only if $G \cong K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

This completes the proof. □

4 Connected Graphs with Given Parameters

In this section, we will establish bounds for $\overline{\xi}^c$ of connected graphs with given parameters such as the number of pendent vertices, independence number, matching number, vertex-connectivity and edge-connectivity, respectively.

Theorem 4.1 *Let G be a connected graph of order n with p pendent vertices. Then,*

$$\overline{\xi}^c(G) \geq 4np-6p-2p^2$$

with equality if and only if $G \cong K_n^p$.

Proof Let $G_{\min}$ be a graph chosen among all connected graphs of order n with p pendent vertices such that $G_{\min}$ has the smallest $\overline{\xi}^c$. Let $v_1, \ldots, v_p$ be pendent vertices in $G_{\min}$. By Lemma 2.1(i), the addition of edges into a graph decreases its $\overline{\xi}^c$. So, the subgraph induced by vertices in $V(G_{\min})\backslash\{v_1, \ldots, v_p\}$ must be a complete subgraph in $G_{\min}$.

It is obvious that $G_{\min}$ has $p + \binom{n-p}{2} = p + \frac{(n-p)(n-p-1)}{2}$ edges. By Theorem 3.1, we have

$$\begin{aligned}\overline{\xi}^c(G_{\min}) &\geq 2n(n-1) - 4m\\ &= 2n(n-1) - 4\Big[p + \frac{(n-p)(n-p-1)}{2}\Big]\\ &= 2n(n-1) - 4p - 2(n-p)(n-p-1)\\ &= 4np - 6p - 2p^2. \end{aligned} \tag{15}$$

By the equality condition in Theorem 3.1, we know that the diameter of $G_{\min}$ must be equal to two. So, all pendent edges in $G_{\min}$ must be attached to the same vertex in K_{n-p}. Thus, the equality in (15) holds if and only if $G_{\min} \cong K_n^p$.

This completes the proof. □

Theorem 4.2 *Let G be a connected graph of order n with independence number α. Then,*

$$\overline{\xi}^c(G) \geq 2\alpha^2 - 2\alpha$$

with equality if and only if $G \cong \alpha K_1 \vee K_{n-\alpha}$.

Proof Let $G_{\min}$ be a graph chosen among all connected graphs of order n with independence number α such that $G_{\min}$ has the smallest $\overline{\xi}^c$. Let S be a maximal independent set in $G_{\min}$ with $|S| = \alpha$. Since adding edges into a graph will decrease its $\overline{\xi}^c$ by Lemma 2.1, each vertex u in S is adjacent to every vertex v in $G_{\min} - S$. Moreover, the subgraph induced by vertices in $G_{\min} - S$ is a complete subgraph of $G_{\min}$. So $G_{\min} \cong \alpha K_1 \vee K_{n-\alpha}$. An elementary calculation gives $\overline{\xi}^c(\alpha K_1 \vee K_{n-\alpha}) = \alpha \cdot 2\Big[n - 1 - (n-\alpha)\Big] = 2\alpha^2 - 2\alpha$, as claimed. □

The following result on matching number is the well-known Tutte–Berge formula due to Tutte and Berge [6,27].

Lemma 4.3 *Suppose that G is a graph of order n with matching number β. Then,*

$$n - 2\beta = \max\{o(G-S) - |S| : S \subseteq V(G)\},$$

where $o(G)$ denotes the number of odd components in G.

Theorem 4.4 *Let G be a connected graph of order n with matching number $\beta \geq 1$.*

(i) *If* $\beta = \lfloor \frac{n}{2} \rfloor$, *then*

$$\overline{\xi}^c(G) \geq 0$$

with equality if and only if $G \cong K_n$.

(ii) *If* $1 \leq \beta < \lfloor \frac{n}{2} \rfloor$, *then*

$$\overline{\xi}^c(G) \geq 2n^2 - 4n\beta + 2\beta^2 - 2n + 2\beta$$

with equality if and only if $G \cong K_\beta \vee (n - \beta)K_1$.

Proof When $\beta = 1$, we must have $G \cong K_3$ or $G \cong S_n$. If $n = 2$, then $\beta = \lfloor \frac{2}{2} \rfloor$, and the result is obvious, as $G \cong S_2 \cong K_2$. If $n = 3$, then $\beta = \lfloor \frac{3}{2} \rfloor$. By Lemma 2.1, it is easy to check that $\overline{\xi}^c(S_3) > \overline{\xi}^c(K_3)$, and the result follows readily. If $n \geq 4$, then $\beta = 1 < \lfloor \frac{4}{2} \rfloor$. Since $S_n \cong K_\beta \vee (n - \beta)K_1$ for $\beta = 1$, the result holds.

Now, we assume that $\beta \geq 2$, and then $n \geq 4$.

We choose $G_{\min}$ to be a graph such that $G_{\min}$ has the smallest $\overline{\xi}^c$ among all connected graphs of order n with matching number β. According to Lemma 4.3, there exists a vertex subset S, satisfying $|S| = s$, in $V(G_{\min})$ such that $G_{\min} - S$ has $t = n - 2\beta + s$ odd components, say $G_1, \ldots, G_t$. For each $i = 1, \ldots, t$, let n_i be the order of G_i. Then, each n_i is a positive odd number for $i = 1, \ldots, t$ and $\sum_{i=1}^{t} n_i = n - s$. □

We have the following claim.

Claim 1 $G_{\min} \cong K_s \vee \left(\bigcup_{i=1}^{t} K_{n_i} \right)$.

Proof Assume without loss of generality that $n_1 \leq n_2 \leq \ldots \leq n_t$. We first show that $G_{\min} - S$ contains no even components. If it is not so, we may let U be the union of all even components of $G_{\min} - S$. Now, one can add all possible edges between vertices in U and those in G_t, until the resulting subgraph induced by vertices both in U and in G_t is a complete subgraph. The resulting graph obtained from $G_{\min}$ by adding edges in such a way as above is denoted by G^*. By Lemma 4.3, on one hand, we have

$$n - 2\beta(G^*) \geq o(G^* - S) - |S| = o(G - S) - |S| = n - 2\beta(G),$$

implying that $\beta(G^*) \leq \beta(G)$. On the other hand, we have $\beta(G^*) \geq \beta(G)$. Thus, $\beta(G^*) = \beta(G)$. But then, by Lemma 2.1, we have $\overline{\xi}^c(G_{\min}) > \overline{\xi}^c(G^*)$, a contradiction to our choice of $G_{\min}$. So, all components of $G_{\min} - S$ are odd and thus, $G_{\min} - S = G_1 \cup \cdots \cup G_t$. It is not difficult to see that each G_i is a complete subgraph, for otherwise, we can add edges into any one non-complete subgraph, say G_j, and we obtain a new graph G^{**} of order n. Similar to above, we have $\beta(G^{**}) = \beta(G)$. Again, by Lemma 2.1, we have $\overline{\xi}^c(G_{\min}) > \overline{\xi}^c(G^{**})$, a contradiction. Similarly, we can prove that, for each $i = 1, \ldots, t$, $G[V(G_i) \cup S]$ is a complete subgraph of $G_{\min}$. So, $G_{\min} \cong K_s \vee \left(\bigcup_{i=1}^{t} K_{n_i} \right)$. □

By Claim 1, there exists a vertex subset S having s vertices in $G_{\min}$ such that $G_{\min} \cong K_s \vee \left(\bigcup_{i=1}^{t} K_{n_i}\right)$, where each n_i is a positive odd number for $i = 1, \ldots, t$.

If $s = 0$, then $n - 2\beta = t$. Since $G_{\min}$ is connected, we have $t \leq 1$. If $t = 0$, then $n = 2\beta$; If $t = 1$, then $n = 2\beta+1$. So, when $s = 0$, we have $\beta = \lfloor \frac{n}{2} \rfloor$. When $\beta = \lfloor \frac{n}{2} \rfloor$, we conclude that $G_{\min} \cong K_n$, for otherwise, we can add edges into $G_{\min}$ so that we can obtain a new graph with strictly smaller $\overline{\xi}^c$ that that of $G_{\min}$, a contradiction.

Now, we assume that $s \geq 1$ and $\beta < \lfloor \frac{n}{2} \rfloor$.

By (2), we have $\overline{\xi}^c\left(K_s \vee \left(\bigcup_{i=1}^{t} K_{n_i}\right)\right) = \sum_{i=1}^{t} 2n_i[n - (n_i - 1 + s) - 1] = 2(n-s)^2 - 2\sum_{i=1}^{t} n_i^2$. We claim that $n_1 = \ldots = n_{t-1} = 1$ and $n_t = n-s-t+1$, that is, $G_{\min} \cong K_s \vee \left(K_{n-s-t+1} \bigcup (t-1)K_1\right)$. Suppose to the contrary that $n_j \geq 3$ for some $j \in \{1, 2, \ldots, t-1\}$. Let $G^{***} = K_s \vee \left(K_{n_t+2} \bigcup K_{n_j-2} \bigcup \left(\bigcup_{i=1,\, i\neq j}^{t-1} K_{n_i}\right)\right)$. Clearly, $o(G^{***} - S) = o(G_{\min} - S)$, and thus, $\beta(G^{***}) = \beta(G_{\min})$.

But then, we have

$$\begin{aligned}\overline{\xi}^c(G_{\min}) - \overline{\xi}^c(G^{***}) &= -2(n_j^2 + n_t^2) + 2[(n_j - 2)^2 + (n_t + 2)^2] \\ &= 8(-n_j + n_t + 2) > 0,\end{aligned}$$

a contradiction to our choice of $G_{\min}$.

Thus, $n_1 = \ldots = n_{t-1} = 1$ and $n_t = n - s - t + 1$, that is, $G_{\min} \cong K_s \vee \left(K_{n-s-t+1} \bigcup (t-1)K_1\right)$. By (2), $\overline{\xi}^c\left(K_s \vee \left(K_{n-s-t+1} \bigcup (t-1)K_1\right)\right) = 2(t-1)(2n - 2s - t)$. Let $f(x) = 2(t-1)(2n - 2x - t)$. Clearly $t \geq 2$. Then, $f(x)$ is a strictly decreasing function on the interval $[1, \beta]$.

Since each $n_i \geq 1$, we have $n \geq s + t$. So, $n \geq n + 2s - 2\beta$, resulting in $s \leq \beta$. When $s = \beta$, we have $n-s-t+1 = n-\beta-t+1$. Recall that $n-2\beta = t-s = t-\beta$, implying that $t = n - \beta$. Thus, $n - s - t + 1 = 1$. If $s < \beta$, then $f(s) > f(\beta)$, that is, $\overline{\xi}^c(G_{\min}) = \overline{\xi}^c\left(K_s \vee \left(K_{n-s-t+1} \bigcup (t-1)K_1\right)\right) > \overline{\xi}^c\left(K_\beta \vee (n-\beta)K_1\right)$, a contradiction to our choice of $G_{\min}$.

So, $s = \beta$, and $G_{\min} \cong K_\beta \vee (n-\beta)K_1$.

This completes the proof. □

In the following two theorems, we shall determine graphs with the minimum $\overline{\xi}^c$ among graphs with given vertex-connectivity and edge-connectivity, respectively.

Theorem 4.5 *Let G be a graph of order n with vertex-connectivity κ. Then,*

$$\overline{\xi}^c(G) \geq 4(n - \kappa - 1)$$

with equality if and only if $G \cong K_\kappa \vee (K_1 + K_{n-\kappa-1})$.

Proof We choose $G_{\min}$ to be a graph such that $G_{\min}$ has the smallest $\overline{\xi}^c$ within all connected graphs of order n with vertex-connectivity κ. Let C be a vertex-cut in $G_{\min}$ such that $|C| = \kappa$ and let $G_{\min} - C = G_1 \cup G_2 \cup \cdots G_t$ $(t \geq 2)$. By Lemma 2.1, we must have $t = 2$, for otherwise, we can add edges between any two components, resulting in a new graph G^* with vertex-connectivity κ and a strictly smaller $\overline{\xi}^c$ than that of $G_{\min}$, a contradiction to our choice of $G_{\min}$.

By the same reason, we can deduce that both G_1 and G_2 are cliques of $G_{\min}$, that the subgraph of $G_{\min}$ induced by C is a clique, and that any vertex in $G_1 \cup G_2$ is adjacent to each vertex in C. Let n_i denote the order of G_i. Thus, we have $G_{\min} \cong K_\kappa \vee (K_{n_1} + K_{n_2})$.

Without loss of generality, we may assume that $n_2 \geq n_1$. If $n_1 = 1$, then the theorem follows. Suppose now that $n_1 \geq 2$. By (2), we obtain

$$\begin{aligned}\overline{\xi}^c(G_{\min}) &= \sum_{u \in V(G_1)} \varepsilon_{G_{\min}}(u)\Big(n-1-d_{G_{\min}}(u)\Big) \\ &\quad + \sum_{u \in V(G_2)} \varepsilon_{G_{\min}}(u)\Big(n-1-d_{G_{\min}}(u)\Big) \\ &\quad + \sum_{u \in C} \varepsilon_{G_{\min}}(u)\Big(n-1-d_{G_{\min}}(u)\Big) \\ &= 4n_1n_2.\end{aligned}$$

Let $G^* = K_\kappa \vee (K_{n_1-1} + K_{n_2+1})$. Then

$$\begin{aligned}\overline{\xi}^c(G^*) - \overline{\xi}^c(G_{\min}) &= 4\Big[(n_1-1)(n_2+1) - n_1n_2\Big] \\ &= 4(n_1 - n_2 - 1) < 0,\end{aligned}$$

a contradiction to our choice of $G_{\min}$.

So, $n_1 = 1$ and $n_2 = n - \kappa - 1$. Thus, $G_{\min} \cong K_\kappa \vee (K_1 + K_{n-\kappa-1})$.

An elementary calculation gives $\overline{\xi}^c(K_\kappa \vee (K_1 + K_{n-\kappa-1})) = 4(n-\kappa-1)$, completing the proof. □

In our last theorem, we determine the graph with the minimum $\overline{\xi}^c$ among all graphs of order n with edge-connectivity κ'.

Theorem 4.6 *Let G be a graph of order n with edge-connectivity κ'. Then*

$$\overline{\xi}^c(G) \geq 4(n - \kappa' - 1)$$

with equality if and only if $G \cong K_{\kappa'} \vee (K_1 + K_{n-\kappa'-1})$.

Proof Let $g(x) = 4(n-x-1)$. It is easily seen that $g(x)$ is a strictly decreasing function. Suppose that G is a graph of order n with vertex-connectivity κ and edge-connectivity κ'. Then, $\kappa \leq \kappa'$. It follows from Theorem 4.5 that $\overline{\xi}^c(G) \geq g(\kappa)$. Since $g(\kappa) \geq g(\kappa')$, we get $\overline{\xi}^c(G) \geq g(\kappa') = 4(n-\kappa'-1)$. It is easy to check that the equality holds if and only if $G \cong K_{\kappa'} \vee (K_1 \cup K_{n-1-\kappa'})$.

This completes the proof. □

5 Concluding Remarks

In this paper, we considered a new eccentricity-based graph invariant, named the eccentric connectivity coindex. We mainly investigated extremal properties of this graph invariant. More specifically, we characterized extremal graphs with the maximum and minimum $\overline{\xi}^c$, respectively, among all connected graphs of given order. Also, we characterized the connected graph with given order, size and the minimum $\overline{\xi}^c$ as well as the tree, unicyclic graph, bipartite graph containing cycles and triangle-free graph with the minimum $\overline{\xi}^c$, respectively. Moreover, we established various lower bounds for $\overline{\xi}^c$ in terms of several other graph parameters including the number of pendent vertices, independence number, matching number, chromatic number, vertex-connectivity, and edge-connectivity.

Our research on this new graph invariant is just a beginning. Similar to other distance-based invariants, there are many interesting problems about this graph invariant left for us to discover and solve in the future.

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