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A General Representation Theorem of a Kind of Super B-Quasi-Ehresmann Semigroups

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Abstract We first study the structure of a special generalized regular semigroup, namely the B-semiabundant semigroup which can be expressed as the join of the pseudo-varieties of finite groups and finite aperiodic groups. In the literature, the weakly B-semiabundant semigroups have recently been thoughtfully investigated and considered by Wang. One easily observes that the class of good B-semiabundant semigroups is a special class of semigroups embraces all abundant (and hence regular) semigroups. In particular, a super B-quasi-Ehresmann semigroup is an analogy of an orthodox semigroup within the class of B-semiabundant semigroups. Thus, the class of super B-quasi-Ehresmann semigroups is obviously a subclass of the class of good B-quasi-Ehresmann semigroups which contains all orthodox semigroups. Thus, the super B-quasi-Ehresmann semigroup behaves similarly as the Clifford subsemigroups within the class of regular semigroups. Consequently, a super B-quasi-Ehresmann semigroup is now recognized as an important generalized regular semigroup. Our aim in this paper is to describe the properties and intrinsic structure of a super B-quasi-Ehresmann semigroup whose band of projections is right regular, right normal, left semiregular, left seminormal, regular, left quasinormal or normal, respectively. Hence, our representation theorem of the super B-quasi-Ehresmann semigroups improves, strengthens and generalizes the well-known "standard representation theorem of an

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orthodox semigroup" established by He et al. (Commun. Algebra 33:745–761, 2005). Finally, a general representation theorem in the category of Ehresmann semigroups is given.

Keywords Quasi-Ehresmann semigroup · Super B-quasi-Ehresmann semigroups · Right regular triple · Left semiregular band · Spined products

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1 Introduction

Throughout this paper, we follow the convention and notation of [10,13]. In the theory of semigroups, the commutativity *bands* of a semigroup S are the idempotent subsemigroups and are also called *semilattices*. It was stated in [13] that the varieties of bands listed in the following table are those bands defined by identities in at most three variables:
The standard representation of bands is a powerful tool for giving the structure theorems of bands in the above varieties.

It is well known that the regular semigroups are the core semigroups to be studied in the theory of semigroups. Recall that the regular semigroups whose idempotents form a subband (a semilattice) are called *orthodox semigroups* (*inverse semigroups*). In the literature, the inverse semigroups have attracted a wide attention among authors in semigroup theory. In particular, Yamada [17], [18] described the structure of the orthodox semigroups whose band of idempotents is normal, left regular and regular, respectively. As a generalization of bands, He et al. [8] established a standard representation theorem of the orthodox semigroups and offered a systematic description for the structures of orthodox semigroups whose band of idempotents lies in the varieties of bands listed in the below Table 1.

On the other hand, the abundant semigroups introduced by Fountain in [4] are the semigroups more general than the regular semigroups. We call the abundant semigroups whose idempotents form a subband the *quasi-adequate semigroups*. Investigations concerning the structure of quasi-adequate semigroups were initiated by El-Quallali and Fountain [2]. The main results in [17] and [18] have been extended and amplified into the corresponding classes of quasi-adequate semigroups(in general,

Table 1 The varieties of bands defined by identities in at most three variables

Left zero band	Trivial semigroup	Right zero band
Left normal band	Semilattice	Right normal band
Left regular band	Rectangular band	Right regular band
Left quasinormal band	Normal band	Right quasinormal band
Left seminormal band	Regular band	Right seminormal band
Left semiregular band	Band	Right semiregular band

the so-called IC condition, that is , the idempotent-connected condition, is mentioned in the results). For more detail information, the reader is referred to [3,5,6,14,15].

As a further generalization of abundant semigroups, Lawson [11] defined the B-semiabundant semigroups. The notions of B-Ehresmann semigroup and B-quasi-Ehresmann semigroup are analogies of the inverse semigroups and the orthodox semigroups , respectively, in the class of B-semiabundant semigroups. In this aspect, Lawson [12] proved that every B-Ehresmann semigroup is equivalent to an Ehresmann category. On the other hand, Wang [16] has generalized and extended the well-known Hall–Yamada theorem for orthodox semigroups to B-quasi-Ehresmann semigroups (under the name of *weakly B-orthodox semigroups*).

We now call a special class of B-quasi-Ehresmann semigroups " a good B-quasi-Ehresmann semigroup" which has been investigated in [9]. Such a general class of semigroups contains all orthodox semigroups, B-Ehresmann semigroups, type W-semigroups considered in [2], orthodox supper rpp semigroups considered in [7] and $\mathscr{P}$-orthomonoids considered in [7]. In view of the properties of the above semigroups, we are able to establish the representation theorems of semigroups. Among all the above representation theorems, it is worth to mention the structure theorem of good B-quasi-Ehresmann semigroups, namely "The global presentation theorem" which was first deduced and given in [9]. The aim of this paper is to further describe the properties and to investigate the various intrinsic properties concerning the structure of a super B-quasi-Ehresmann semigroup whose band of projections is right regular, right normal, left semiregular, left seminormal, regular, left quasinormal or normal, respectively. Because our new representation theorem given in this paper, in compare with the so-called "global presentation theorem of good B-quasi-Ehresmann semigroups", involves at least seven different band varieties and hence we call our more general representation theorem established in this paper "a general representation theorem of a super B-quasi-Ehresmann semigroup with seven different kind of bands of projections".

2 Preliminaries

In this section, we first state the properties and the basic results of a good B-quasi-Ehresmann semigroup.

Let A be a non-empty set. Then, we write $A = \biguplus_{e\in Y} A_e$ if $\{A_e\}_{e\in Y}$ is a partition of A. The full transformation semigroup of A is denoted by $\mathscr{T}(A)$ whose elements are assumed to act on the set A from the right-hand side. Two members ϕ, ψ of $\mathscr{T}(A)$ are composed with $a(\phi \circ \psi) = (a\phi)\psi$ for all $a \in A$. The dual semigroup of $\mathscr{T}(A)$ is denoted by $\mathscr{T}^*(A)$, whose elements are assumed to act on the set A from the left-hand side. Two members ϕ, ψ of $\mathscr{T}^*(A)$ are composed by $(\phi * \psi)(a) = \phi(\psi(a))$ for all $a \in A$.

Let B be a band. Then, for each $x \in B$, we denote the $\mathscr{D}$-class of B containing x by $B(x)$. Clearly, $B(x)$ is a rectangular subband of B. It is known that B is a semilattice of rectangular bands since the Green's equivalence $\mathscr{D}$ on B is the least semilattice congruence. We write $B = (E; B_e)$ if B is a semilattice E of rectangular bands B_e's, and $B=(E;B_e;\vartheta_{e,f})$ if B is a strong semilattice E of the rectangular bands B_e's with respect to the transitive system $\{\vartheta_{e,f}|e\geq f \text{ in } E\}$ of homomorphisms.

Let S be a semigroup and B a non-empty set of idempotents of S. Define two equivalences $\tilde{\mathscr{L}}^B$ and $\tilde{\mathscr{R}}^B$ on S as below:

$$\tilde{\mathscr{L}}^B=\{(a,b)\in S\times S\,|\,(\forall e\in B)\; ae=a\Leftrightarrow be=b\},$$

$$\tilde{\mathscr{R}}^B=\{(a,b)\in S\times S\,|\,(\forall e\in B)\; ea=a\Leftrightarrow eb=b\}.$$

For each $a\in S$, denote the $\tilde{\mathscr{L}}^B$ and $\tilde{\mathscr{R}}^B$-classes of S containing a by $\tilde{L}_a^B$ and $\tilde{R}_a^B$, respectively. If each $\tilde{\mathscr{L}}^B$ and $\tilde{\mathscr{R}}^B$-class of S meets with B, then we call S a *B-semiabundant semigroup*.

In the following lemmas, we state some of the properties of a good quasi-B-Ehresmann semigroup.

Lemma 2.1 [11] *If S is a B-semiabundant semigroup then $\mathscr{L}\subseteq\tilde{\mathscr{L}}^B$ on S, and the restrictions $\tilde{\mathscr{L}}^B$ and $\mathscr{L}$ on B coincide.* □

Now, we call a B-semiabundant semigroup S a *B-quasi-Ehresmann semigroup* (with B as the *band of projections*) if B is a subband of S and the equivalences $\tilde{\mathscr{L}}^B$ and $\tilde{\mathscr{R}}^B$ on S are a right and a left congruence, respectively. In this case, we call $S(B)$ a *quasi-Ehresmann semigroup*. Two quasi-Ehresmann semigroups $S_1(B_1)$, $S_2(B_2)$ are said to be *good isomorphic* if there exists an isomorphism $\xi:S_1\to S_2$ of semigroups such that $B_1\xi=B_2$.

For each element a of a quasi-Ehresmann semigroup $S(B)$, we take a typical element a^+ from $B\cap\tilde{R}_a^B$ and a typical element a^* from $B\cap\tilde{L}_a^B$. In particular, we fix the element $a^*=a^+=a$ whenever $a\in B$. Then it is obvious that

$$a^+a=a=aa^*.$$

Define an equivalence γ on $S(B)$ by

$$(\forall x,y\in S)\quad (x,y)\in\gamma\Longleftrightarrow B(x^+)xB(x^*)=B(y^+)yB(y^*).$$

Now, we call the quasi-Ehresmann semigroup $S(B)$ *good* if it satisfies the following condition

$$(\forall x,y\in S)\qquad xB(x^*)B(y^+)y\subseteq B((xy)^+)xyB((xy)^*).$$

In particular, a good quasi-Ehresmann semigroup $S(B)$ is a *right regular quasi-Ehresmann semigroup*, if B is a right regular band. Then, similar notions such as *right normal quasi-Ehresmann semigroup* will be adopted through out this paper without explanation. Recall from [12] , a quasi-Ehresmann semigroup is said to be an *Ehresmann semigroup* if its band of projections is a semilattice. If $T(E)$ is an Ehresmann semigroup, then it follows from Lemma 2.1 and its left–right dual result concerning the relation $\tilde{\mathscr{R}}^B$ that each $\tilde{\mathscr{L}}^E$ and $\tilde{\mathscr{R}}^E$-class of T contains a unique element of E. Consequently, for each $s\in T$, the pair (s^+,s^*) of elements of E is uniquely determined. Moreover, we see that $T(E)$ is a good quasi-Ehresmann semigroup on which the equivalence γ is the identical relation. Now, we call a congruence α on a quasi-Ehresmann semigroup $S(B)$ a *good* congruence when $S\alpha^{\natural}(B\alpha^{\natural})$ is a quasi-Ehresmann semigroup. In this case, $S\alpha^{\natural}(B\alpha^{\natural})$ is written as $S(B)/\alpha$. We call a good congruence α on $S(B)$ an *Ehresmann congruence* if $S(B)/\alpha$ is an Ehresmann semigroup. It is also called a *right regular quasi-Ehresmann congruence* if $S(B)/\alpha$ is a right regular quasi-Ehresmann semigroup. The notions of *right normal quasi-Ehresmann congruence* and so on can be defined similarly.

Lemma 2.2 [9] *A quasi-Ehresmann semigroup $S(B)$ is said to be "good" if and only if the relation γ on S is a congruence. Moreover, in this case, γ is the least Ehresmann congruence on $S(B)$ such that $(a\gamma)^+ = a^+\gamma$ and $(a\gamma)^* = a^*\gamma$ for all $a \in S$.* □

Let $T(E)$ be an Ehresmann semigroup, while $I = \biguplus_{e\in E} I_e$ and $\Lambda = \biguplus_{e\in E} \Lambda_e$ are two non-empty sets. Then, we consider the following two sets :

$$\mathbb{S} = \bigcup_{s\in T}(I_{s^+} \times \{s\} \times \Lambda_{s^*}), \qquad \mathbb{B} = \bigcup_{e\in E}(I_e \times \{e\} \times \Lambda_e), \tag{1}$$

$$\bar{\mathbb{S}} = \bigcup_{s\in T}(\{s\} \times \Lambda_{s^*}), \qquad \bar{\mathbb{B}} = \bigcup_{e\in E}(\{e\} \times \Lambda_e), \tag{2}$$

$$\bar{\bar{\mathbb{S}}} = \bigcup_{s\in T}(I_{s^+} \times \{s\}), \qquad \bar{\bar{\mathbb{B}}} = \bigcup_{e\in E}(I_e \times \{e\}). \tag{3}$$

By a *global tuple* we mean a sextuple $\mathfrak{G} = (E; T; I, \Phi; \Lambda, \Psi)$, where Φ and Ψ are two mappings of the form

$$\Phi : \mathbb{S} \longrightarrow \mathscr{T}^*(I),\ x \longmapsto \phi_x, \quad \Psi : \mathbb{S} \longrightarrow \mathscr{T}(\Lambda),\ x \longmapsto \psi_x,$$

such that the following statements hold for all $e \in E$ and $(i, s, \lambda), (j, t, \mu) \in \mathbb{S}$:

$$\phi_{(i,s,\lambda)}(I_{s^*}) = \{i\}, \qquad \Lambda_{s^+}\psi_{(i,s,\lambda)} = \{\lambda\}; \tag{G1}$$

$$\phi_{(i,s,\lambda)}(I_e) \subseteq I_{(se)^+}, \qquad \Lambda_e\psi_{(i,s,\lambda)} \subseteq \Lambda_{(es)^*}; \tag{G2}$$

$$\left.\begin{aligned}\phi_{(i,s,\lambda)} * \phi_{(j,t,\mu)} &= \phi_{(\phi_{(i,s,\lambda)}(j), st, \lambda\psi_{(j,t,\mu)})},\\ \psi_{(i,s,\lambda)} \circ \psi_{(j,t,\mu)} &= \psi_{(\phi_{(i,s,\lambda)}(j), st, \lambda\psi_{(j,t,\mu)})}.\end{aligned}\right\} \tag{G3}$$

We state below a structure theorem of a good quasi-Ehresmann semigroup with a global tuple .

Theorem 2.3 [9] *Let $\mathfrak{G} = (E; T; I, \Phi; \Lambda, \Psi)$ be a global tuple, and $\mathbb{S}, \mathbb{B}$ the sets defined as in (1). Then $\mathbb{S}(\mathbb{B})$ forms a good quasi-Ehresmann semigroup under the following operation*

$$(i, s, \lambda)(j, t, \mu) = (\phi_{(i,s,\lambda)}(j), st, \lambda\psi_{(j,t,\mu)}). \tag{4}$$

Moreover, the following statements hold in S:
(i) $\mathbb{B} = (E; I_e \times \{e\} \times \Lambda_e)$;
(ii) $(i, s, \lambda)\ \tilde{\mathscr{L}}^{\mathbb{B}}(j, t, \mu)$ *if and only if* $\lambda = \mu$;
(iii) $(i, s, \lambda)\ \tilde{\mathscr{R}}^{\mathbb{B}}(j, t, \mu)$ *if and only if* $i = j$;
(iv) $(i, s, \lambda)\ \gamma\ (j, t, \mu)$ *if and only if* $s = t$.
Conversely, up to good isomorphism, every good quasi-Ehresmann semigroup can be constructed as above. □.

The good quasi-Ehresmann semigroup $\mathbb{S}(\mathbb{B})$ constructed as in the above theorem is called the *semigroup defined by a global tuple* $\mathfrak{G}$. If $S(B)$ is a good quasi-Ehresmann semigroup which is good isomorphic to $\mathbb{S}(\mathbb{B})$. We hence call $\mathbb{S}(\mathbb{B})$ a *global representation* of $S(B)$.

Remark 2.4 Let $S = I \times M \times \Lambda$ be the direct product of a left zero band I, a monoid M and a right zero band Λ, and let $B = I \times \{1_M\} \times \Lambda$, where 1_M is the identity of the monoid M. Then $S(B)$ is a rectangular quasi-Ehresmann semigroup which is called a $\mathscr{P}$*-plank* in [1]. It follows from Theorem 2.3 that each rectangular quasi-Ehresmann semigroup is good isomorphic to a $\mathscr{P}$-plank.

3 Right Regular and Right Normal Quasi-Ehresmann Semigroups

Recall that a band is *right regular* if it is a semilattice of right zero bands, and *right normal* if it is a strong semilattice of right zero bands. Note that a band is right normal if and only if it satisfies the identity $xyz = yxz$. Of course, right normal bands are right regular. The aim of this section is to describe the structures of right regular and right normal quasi-Ehresmann semigroups.

By a *right regular triple* we mean a quadruple $\mathfrak{R} = (E; T; \Lambda, \bar{\Psi})$ consisting of a semilattice E, an Ehresmann semigroup $T(E)$, a non-empty set $\Lambda = \biguplus_{e \in E} \Lambda_e$ as well as a mapping

$$\bar{\Psi} : \bar{\mathbb{S}} \longrightarrow \mathscr{T}(\Lambda),\ x \longmapsto \bar{\psi}_x,$$

such that the following statements hold for all $e \in E$ and $(s, \lambda), (t, \mu) \in \bar{\mathbb{S}}$:

$$\Lambda_{s^+}\psi_{(s,\lambda)} = \{\lambda\}, \quad \Lambda_e\psi_{(s,\lambda)} \subseteq \Lambda_{(es)^*}, \quad \psi_{(s,\lambda)} \circ \psi_{(t,\mu)} = \psi_{(st,\lambda\psi_{(t,\mu)})}.$$

Clearly, the semigroup $\mathbb{S}(\mathbb{B})$ defined by a global triple $\mathfrak{G} = (E; T; I, \Phi; \Lambda, \Psi)$ is a right regular quasi-Ehresmann semigroup if and only if each I_e $(e \in E)$ contains a single element say i.e. If this is the case, then the mapping Φ is uniquely defined by

$$(\forall e \in E)\,(\forall (i, s, \lambda) \in S) \qquad \phi_{(i,s,\lambda)}(i_e) = i_{(se)^+}.$$

Therefore, we have no need to mention I and Φ. In this case, the global tuple $\mathfrak{G}$ is “reduced” to a right regular triple $(E; T; \Lambda, \Psi)$. Consequently, by Theorem 2.3, we deduce the following corollary on right regular quasi-Ehresmann semigroups.

Corollary 3.1 *Let $\mathfrak{R} = (E; T; \Lambda, \bar{\Psi})$ be a right regular triple, and $\bar{\mathbb{S}}, \bar{\mathbb{B}}$ the sets defined as in (2). Then $\bar{\mathbb{S}}(\bar{\mathbb{B}})$ forms a right regular quasi-Ehresmann semigroup under the following operation*

$$(s, \lambda)(t, \mu) = (st, \lambda\bar{\psi}_{(t,\mu)}). \tag{5}$$

Moreover, the following statements hold in $\bar{\mathbb{S}}$:
(i) $\bar{\mathbb{B}} = (E; \{e\} \times \Lambda_e)$;
(ii) $(s, \lambda)\ \tilde{\mathscr{L}}^{\bar{\mathbb{B}}}(t, \mu)$ *if and only if* $\lambda = \mu$;
(iii) $(s, \lambda)\ \gamma\ (t, \mu)$ *if and only if* $s = t$.
Conversely, up to good isomorphism, every right regular quasi-Ehresmann semigroup can be constructed as in the above manner. □

We now consider the left regular quasi-Ehresmann semigroups. The right regular quasi-Ehresmann semigroup $\bar{\mathbb{S}}(\bar{\mathbb{B}})$ constructed in the above corollary is called the *semigroup defined by the right regular triple* $\mathfrak{R} = (E; T; \Lambda, \bar{\Psi})$. Recall that a band is *left regular* if it is a semilattice of left zero bands. As the dual to right regular triple, the concept *left regular triple* of the form $\mathfrak{L} = (E; T; I, \bar{\bar{\Phi}})$ may be defined for constructing a left regular quasi-Ehresmann semigroup $\bar{\bar{\mathbb{S}}}(\bar{\bar{\mathbb{B}}})$.

We now have the following corollary on right normal quasi-Ehresmann semigroups.

Corollary 3.2 *Let $T(E)$ be an Ehresmann semigroup, $\Lambda = (E; \Lambda_e, \vartheta_{e,f})$ a right normal band, and let $\bar{\mathbb{S}}, \bar{\mathbb{B}}$ be the sets defined as in (2). Then $\bar{\mathbb{S}}(\bar{\mathbb{B}})$ forms a right normal quasi-Ehresmann semigroup under the operation*

$$(s, \lambda)(t, \mu) = (st, \mu\vartheta_{t^*,(st)^*}). \tag{6}$$

Conversely, up to good isomorphism, every right normal quasi-Ehresmann semigroup can be constructed as in the above manner.

Proof To establish the first part of the corollary, we first observe that

$$(\forall e \in E)(\forall s, t \in T) \qquad (es)^*s^* = (es)^*, \quad ((es)^*t)^* = (est)^*.$$

In fact, since $(es)^* \in E \cap \tilde{L}^E_{es}$ and $ess^* = es$, we have $(es)^*s^* = (es)^*$. Moreover, since the relation $\tilde{\mathscr{L}}^E$ on T is a right congruence, we conclude that $(es)^*t$ and est are $\tilde{\mathscr{L}}^E$-related in T, and hence, we have $((es)^*t)^* = (est)^*$. For each $(s, \lambda) \in \bar{\mathbb{S}}$, define a transformation $\bar{\psi}_{(s,\lambda)}$ on Λ by

$$(\forall e \in E) \qquad \Lambda_e\bar{\psi}_{(s,\lambda)} = \{\lambda\vartheta_{s^*,(es)^*}\}.$$

Let $\mathfrak{R} = (E; T; \Lambda, \bar{\Psi})$, where $\bar{\Psi}$ is a mapping given below:

$$\bar{\Psi} : \bar{\mathbb{S}} \longrightarrow \mathscr{T}(\Lambda),\ x \longmapsto \bar{\psi}_x.$$

For any $e \in E$ and $(s, \lambda), (t, \mu) \in \bar{\mathbb{S}}$, it is evident that $\Lambda_{s^+}\bar{\psi}_{(s,\lambda)} = \{\lambda\}$ and that $\Lambda_e\bar{\psi}_{(s,\lambda)} \subseteq \Lambda_{(es)^*}$. Furthermore, the equality $\bar{\psi}_{(s,\lambda)} \circ \bar{\psi}_{(t,\mu)} = \bar{\psi}_{(st,\lambda\bar{\psi}_{(t,\mu)})}$ holds since, for any $\nu \in \Lambda_e$,

$$(\nu\bar{\psi}_{(s,\lambda)})\bar{\psi}_{(t,\mu)} = (\nu\vartheta_{s^*,(es)^*})\bar{\psi}_{(t,\mu)} = \mu\vartheta_{t^*,((es)^*t)^*} = \mu\vartheta_{t^*,(est)^*},$$
$$\nu\bar{\psi}_{(st,\lambda\bar{\psi}_{(t,\mu)})} = \nu\bar{\psi}_{(st,\mu\vartheta_{t^*,(st)^*})} = (\mu\vartheta_{t^*,(st)^*})\vartheta_{(st)^*,(est)^*} = \mu\vartheta_{t^*,(est)^*}.$$

Therefore, $\mathfrak{R}$ is a right regular triple, and hence $\bar{\mathbb{S}}(\bar{\mathbb{B}})$ forms a right regular quasi-Ehresmann semigroup under the operation (5). For three arbitrary elements $(e, \lambda), (f, \nu), (g, \pi)$ of $\bar{\mathbb{B}}$, it is routine to check that

$$(e, \lambda)(f, \nu)(g, \pi) = (efg, \pi\vartheta_{g,efg}) = (f, \nu)(e, \lambda)(g, \pi).$$

Thus, $\bar{\mathbb{B}}$ satisfies the identity $xyz = yxz$, and hence it is a right normal band. Observe that the operations (5) and (6) on $\bar{\mathbb{S}}$ coincide, we hence conclude that $\bar{\mathbb{S}}(\bar{\mathbb{B}})$ forms a right normal quasi-Ehresmann semigroup under the operation (6).

For the converse part, we assume that $S(B)$ is an arbitrary right normal quasi-Ehresmann semigroup. Then $S(B)$ is certainly a right regular quasi-Ehresmann semigroup, so that $S(B)$ is good isomorphic to the semigroup $\bar{\mathbb{S}}(\bar{\mathbb{B}})$ defined by some right regular triple $\mathfrak{R} = (E; T; \Lambda, \bar{\Psi})$. Of course, the band $\bar{\mathbb{B}}$ is right normal since it is isomorphic to the band B. Suppose if possible that $\bar{\mathbb{B}} = (E; \{e\} \times \Lambda_e, \theta_{e,f})$. Then, by identifying each element λ of Λ to the corresponding one (e, λ) of $\bar{\mathbb{B}}$, the set Λ may be constructed as a right normal band of the form $\Lambda = (E; \Lambda_e, \vartheta_{e,f})$. Consequently $\bar{\mathbb{S}}(\bar{\mathbb{B}})$ also forms a right normal quasi-Ehresmann semigroup under the operation (6).

For any $e \in E$, $\lambda \in \Lambda_e$ and $(t, \mu) \in \bar{\mathbb{S}}$, since $(et)^*t^* = (et)^*$ in E, we have

$$(e, \lambda)(t^*, \mu) = ((et)^*, \lambda)\left((t^*, \mu)\theta_{t^*,(et)^*}\right) = (t^*, \mu)\theta_{t^*,(et)^*} = (t^*, \mu\vartheta_{t^*,(et)^*}),$$

so that we deduce that $\Lambda_{(et)^*}\bar{\psi}_{(t^*,\mu)} = \{\mu\vartheta_{t^*,(et)^*}\}$. This yields $\Lambda_e\bar{\psi}_{(t,\mu)} = \{\mu\vartheta_{t^*,(et)^*}\}$ since

$$\Lambda_e\bar{\psi}_{(t,\mu)} = \Lambda_e\bar{\psi}_{(t,\mu)(t^*,\mu)} = (\Lambda_e\psi_{(t,\mu)})\bar{\psi}_{(t^*,\mu)} \subseteq \Lambda_{(et)^*}\bar{\psi}_{(t^*,\mu)} = \{\mu\vartheta_{t^*,(et)^*}\}.$$

Now, for any $(s, \lambda), (t, \mu) \in \bar{\mathbb{S}}$, we have

$$(s, \lambda)(t, \mu) = (st, \lambda\bar{\psi}_{(t,\mu)}) = (st, \mu\vartheta_{t^*,(st)^*}).$$

Hence the operations (5) and (6) on $\bar{\mathbb{S}}(\bar{\mathbb{B}})$ coincide. This completes the proof. □

4 Left Semiregular and Left Seminormal Quasi-Ehresmann Semigroups

Recall that a band is said to be *left semiregular* if the relation $\mathscr{L}$ is a right regular band congruence, and *left seminormal* if the relation $\mathscr{L}$ is a right normal band congruence. Of course, all left seminormal bands are left semiregular. Our aim in this section is to describe the structures of left semiregular quasi-Ehresmann semigroups and also the left seminormal quasi-Ehresmann semigroups.

Let $\mathbb{S}(\mathbb{B})$ be the semigroup defined by a global triple $(E; T; I, \Phi; \Lambda, \Psi)$. Then it follows from Theorem 2.3 (ii) and (iv) that

$$(\forall (i, s, \lambda), (j, t, \mu) \in \mathbb{S}) \qquad (i, s, \lambda)\ (\gamma \cap \tilde{\mathscr{L}}^{\mathbb{B}})\ (j, t, \mu) \Longleftrightarrow (s, \lambda) = (t, \mu).$$

Moreover, by Lemma 2.1 and Theorem 2.3 (ii), we have

$$(\forall (i, e, \lambda), (j, f, \mu) \in \mathbb{B}) \qquad (i, e, \lambda)\ \mathscr{L}\ (j, f, \mu) \Longleftrightarrow (e, \lambda) = (f, \mu).$$

Hence the restriction of $\gamma \cap \tilde{\mathscr{L}}^{\mathbb{B}}$ on $\mathbb{B}$ is precisely the relation $\mathscr{L}$ on $\mathbb{B}$. The following lemma is a crucial lemma of left semiregular bands.

Lemma 4.1 *The band $\mathbb{B}$ is left semiregular if and only if $\psi_{(j,t,\mu)} = \psi_{(k,t,\mu)}$ whenever $(j, t, \mu), (k, t, \mu) \in \mathbb{S}$.*

Proof If $\psi_{(j,t,\mu)} = \psi_{(k,t,\mu)}$ whenever $(j, t, \mu), (k, t, \mu) \in \mathbb{S}$, then the left regularity of $\mathbb{B}$ follows from the next equalities in $\mathbb{B}$:

$$(k, f, \nu)(i, e, \lambda) = (\phi_{(k,f,\nu)}(i), e, \nu\psi_{(i,e,\lambda)}),$$
$$(k, f, \nu)(j, e, \lambda) = (\phi_{(k,f,\nu)}(j), e, \nu\psi_{(j,e,\lambda)}).$$

Conversely, suppose that $\mathbb{B}$ is a left semiregular band and that $(j, t, \mu), (k, t, \mu) \in \mathbb{S}$. Then, for any $\nu \in \Lambda_{t^+}$ and $(m, e, \lambda) \in \mathbb{B}$, since $(j, t^+, \nu)\ \mathscr{L}\ (k, t^+, \nu)$ in $\mathbb{B}$, we have

$$(m, e, \lambda)(j, t^+, \nu)\ \mathscr{L}\ (m, e, \lambda)(k, t^+, \nu),$$

so that we have $\lambda\psi_{(j,t^+,\nu)} = \lambda\psi_{(k,t^+,\nu)}$. This implies by (G3) that

$$\lambda\psi_{(j,t,\mu)} = (\lambda\psi_{(j,t^+,\nu)})\psi_{(j,t,\mu)} = (\lambda\psi_{(k,t^+,\nu)})\psi_{(j,t,\mu)} = \lambda\psi_{(k,t,\mu)}. \qquad \square$$

Since the restriction of $\gamma \cap \tilde{\mathscr{L}}^{\mathbb{B}}$ to $\mathbb{B}$ is exactly the relation $\mathscr{L}$ on $\mathbb{B}$, we claim that $\mathbb{S}(\mathbb{B})$ is a left semiregular Ehresmann semigroup when $\gamma \cap \tilde{\mathscr{L}}^{\mathbb{B}}$ is a right regular quasi-Ehresmann congruence. To prove that the converse also holds, we suppose that $\mathbb{S}(\mathbb{B})$ is a left semiregular quasi-Ehresmann semigroup, and let $\bar{\mathbb{S}}, \bar{\mathbb{B}}$ be the sets defined as in (2). For each $(t, \mu) \in \bar{\mathbb{S}}$, we put $\bar{\psi}_{(t,\mu)} = \psi_{(j,t,\mu)}$ for some $j \in I_{t^+}$. In fact, by Lemma 4.1, the definition of $\bar{\psi}_{(t,\mu)}$ is independent of the choice of j. This leads to a mapping

$$\bar{\Psi} : \bar{\mathbb{S}} \longrightarrow \mathscr{T}(\Lambda),\ x \longmapsto \bar{\psi}_x.$$

Since $\mathfrak{G}$ satisfies (G1), (G2) and (G3), We see immediately that the quadruple $\mathfrak{R} = (E; T; \Lambda, \bar{\Psi})$ is a right regular triple so that $\bar{\mathbb{S}}(\bar{\mathbb{B}})$ forms a right regular quasi-Ehresmann semigroup under the operation (5). In what follows, we denote the right regular triple $\mathfrak{R}$ and the right regular quasi-Ehresmann semigroup $\bar{\mathbb{S}}(\bar{\mathbb{B}})$ constructed just by $R_{\mathfrak{G}}$ and $R_{\mathbb{S}(\mathbb{B})}$, respectively. Clearly, the mapping

$$\xi : \mathbb{S} \longrightarrow \bar{\mathbb{S}},\ (i, s, \lambda) \longmapsto (s, \lambda)$$

is an epimorphism of semigroups with kernel $\gamma \cap \tilde{\mathscr{L}}^{\mathbb{B}}$ such that $\mathbb{B}\xi = \bar{\mathbb{B}}$. Hence the equivalence $\gamma \cap \tilde{\mathscr{L}}^{\mathbb{B}}$ on $\mathbb{S}(\mathbb{B})$ is a right regular quasi-Ehresmann congruence. Now, by Theorem 2.3, we state the following two important results on Ehresmann semigroups.

Theorem 4.2 *A good quasi-Ehresmann semigroup $S(B)$ is a left semiregular Ehresmann semigroup if and only if the equivalence $\gamma \cap \tilde{\mathscr{L}}^B$ on it is a right regular quasi-Ehresmann congruence.* □

Theorem 4.3 *A good quasi-Ehresmann semigroup $S(B)$ is a left seminormal quasi-Ehresmann semigroup if and only if the equivalence $\gamma \cap \tilde{\mathscr{L}}^B$ on it is a right normal quasi-Ehresmann congruence.* □

The following theorem describes the construction of the left semiregular and left seminormal quasi-Ehresmann semigroups.

Theorem 4.4 *Let $\bar{S}(\bar{B})$ be a right regular quasi-Ehresmann semigroup where $\bar{B} = (E; \bar{B}_e)$, and let $I = \biguplus_{e \in E} I_e$ be a non-empty set. For any $u \in \bar{B}$, if $e \in E$ such that $u \in \bar{B}_e$, Then we put $\overline{u} = e$. Build two sets*

$$S = \bigcup_{a \in \bar{S}} \left(I_{\overline{a^+}} \times \{a\}\right), \qquad B = \bigcup_{u \in \bar{B}} \left(I_{\overline{u}} \times \{u\}\right), \tag{7}$$

and assume that there is a mapping

$$\Phi : \ S \longrightarrow \mathscr{T}^*(I), \ x \longmapsto \phi_x$$

such that the following statements hold for all $u \in \bar{B}$ and $(i, a), (j, b) \in S$:

$$\phi_{(i,a)}(I_{\overline{a^*}}) = \{i\}, \quad \phi_{(i,a)}(I_{\overline{u^+}}) \subseteq I_{\overline{(au)^+}}, \quad \phi_{(\phi_{(i,a)}(j),ab)} = \phi_{(i,a)} * \phi_{(j,b)}. \tag{GL}$$

This shows that $S(B)$ forms a left semiregular quasi-Ehresmann semigroup under the operation

$$(i, a)(j, b) = (\phi_{(i,a)}(j), ab). \tag{8}$$

In particular, $S(B)$ is a left seminormal quasi-Ehresmann semigroup when $\bar{S}(\bar{B})$ is a right normal quasi-Ehresmann semigroup.

Conversely, up to good isomorphism, each left semiregular and left seminormal quasi-Ehresmann semigroup can be constructed in the above manner. □

Proof We only need consider the left semiregular case. To establish the first part, without loss of generality, we may assume that $\bar{S}(\bar{B}) = \mathbb{S}(\mathbb{B})$ is the semigroup defined by a right regular triple $\mathfrak{R} = (E; T; \Lambda, \bar{\Psi})$. Then it follows by Lemma 2.2 and Corollary 3.1 that

$$(\forall a = (s, \lambda) \in \bar{S}) \qquad \overline{a^+} = s^+. \tag{9}$$

In particular, we have

$$(\forall u = (e, \lambda) \in \bar{B}) \qquad \overline{u^+} = \overline{u} = e. \tag{10}$$

Now, we denote an arbitrary element $(i, (s, \lambda))$ of S as (i, s, λ). In this case, we have $S = \mathbb{S}$ and $B = \mathbb{B}$ for the sets $\mathbb{S}, \mathbb{B}$ defined as in (1). Consider a sextuple $\mathfrak{G} = (E; T; I, \Phi; \Lambda, \Psi)$ by defining a mapping

$$\Psi : \ S \longrightarrow \mathscr{T}(\Lambda), \ (i, s, \lambda) \longmapsto \bar{\psi}_{(s,\lambda)}.$$

By the statements in (GL), we can easily verify that $\mathfrak{G}$ is a global tuple, and thereby $S(B)$ forms a good quasi-Ehresmann semigroup under the operation (4). In fact, it follows from Lemma 4.1 that $S(B)$ is a left semiregular quasi-Ehresmann semigroup. Moreover, the operation (4) on S coincides with the operation (8) since, for any two elements $(i, a) = (i, s, \lambda)$ and $(j, b) = (j, t, \mu)$ of S, we have

$$(i, a)(j, b) = (\phi_{(i,s,\lambda)}(j), st, \lambda\psi_{(j,t,\mu)}) = (\phi_{(i,a)}(j), st, \lambda\bar{\psi}_{(t,\mu)}) = (\phi_{(i,a)}(j), ab).$$

Thus, $S(B)$ forms a left semiregular quasi-Ehresmann semigroup under the operation (8).

To prove the converse part of this theorem, it suffices to show that, when $S(B)$ is a left semiregular quasi-Ehresmann semigroup, the semigroup $\mathbb{S}(\mathbb{B})$ defined by a global tuple, $\mathfrak{G} = (E; T; I, \Phi; \Lambda, \Psi)$ can be reconstructed as in the first part. Suppose that $\mathbb{S}(\mathbb{B})$ is a left semiregular quasi-Ehresmann semigroup, and let $\bar{S}(\bar{B}) = R_{\mathbb{S}(\mathbb{B})}$. Then the equalities in (9) and (10) also hold. If we write an arbitrary element (i, s, λ) of $\mathbb{S}$ as $(i, (s, \lambda))$, then $\mathbb{S} = S$ and $\mathbb{B} = B$ for the sets S, B defined as in (7). Of course, the statements in (GL) hold for all $u \in \bar{B}$ and $(i, a), (j, b) \in \mathbb{S}$. Consequently, by the first part, we see immediately that $\mathbb{S}(\mathbb{B})$ forms a left semiregular quasi-Ehresmann semigroup under the operation (8). In fact, the operation (8) on $\mathbb{S}$ coincides with that (4). The proof is hence completed. □

5 Regular and Normal Quasi-Ehresmann Semigroups

Recall that a band is said to be *regular* if it is left semiregular and right semiregular, to be *left quasinormal* if it is left seminormal and right semiregular, and to be *normal* if it is left seminormal and right seminormal. Of course, normal bands are left quasinormal, while left quasinormal bands are regular. As a consequence of Theorem 4.2, Corollary 4.3 and their left–right dual results concerning right semiregular and right seminormal quasi-Ehresmann semigroups, we have the following corollary.

Corollary 5.1 *Let $S(B)$ be a good quasi-Ehresmann semigroup. Then the following statements hold :*

(i) *$S(B)$ is a regular quasi-Ehresmann semigroup if and only if the equivalence $\gamma \cap \mathscr{L}^B$ is a right regular quasi-Ehresmann congruence and the equivalence $\gamma \cap \mathscr{R}^B$ is a left regular quasi-Ehresmann congruence;*

(ii) *$S(B)$ is a left quasinormal quasi-Ehresmann semigroup if and only if the equivalence $\gamma \cap \mathscr{L}^B$ is a right normal quasi-Ehresmann congruence and the equivalence $\gamma \cap \mathscr{R}^B$ is a left regular quasi-Ehresmann congruence;*

(iii) *$S(B)$ is a normal quasi-Ehresmann semigroup if and only if the equivalence $\gamma \cap \mathscr{L}^B$ is a right normal quasi-Ehresmann congruence and the equivalence $\gamma \cap \mathscr{R}^B$ is a left normal quasi-Ehresmann congruence.* □

Recall from Howie [10] that if both of $\xi : S \to T$ and $\zeta : S' \to T$ are homomorphisms of semigroups, then the set

$$S \times_T S' = \{(x, y) \in S \times S' | x\xi = y\zeta\}$$

forms a subsemigroup of the direct product $S \times S'$ of S and S', which is called a *spined product* of S and S' (with respect to T, ξ and ζ if necessary).

The following result can be found in [13].

Lemma 5.2 *A band B is regular if and only if there are a left regular band $I = (E; I_e)$ and a right regular band $\Lambda = (E; \Lambda_e)$ such that $B \cong I \times_E \Lambda$. In particular, we have the following characterizations for the band B.*

(i) *B is left quasinormal if and only if Λ is right normal;*

(ii) *B is normal if and only if I is left normal and Λ is right normal.* □

Now, we state the following corollary.

Corollary 5.3 *Let $\bar{\bar{S}}(\bar{\bar{B}})$ and $\bar{S}(\bar{B})$ be a left regular and a right regular quasi-Ehresmann semigroup, respectively, such that*

$$T(E) = \bar{\bar{S}}(\bar{\bar{B}})/\gamma = \bar{S}(\bar{B})/\gamma.$$

Let $S = \bar{\bar{S}} \times_T \bar{S}$ and $B = \bar{\bar{B}} \times_E \bar{B}$. Then $S(B)$ is a regular quasi-Ehresmann semigroup. In particular, we have the following statements.

(i) *if $\bar{S}(\bar{B})$ is a right normal quasi-Ehresmann semigroup, then $S(B)$ is a left quasi-normal quasi-Ehresmann semigroup;*

(ii) *if $\bar{\bar{S}}(\bar{\bar{B}})$ and $\bar{S}(\bar{B})$ are a left normal and a right normal quasi-Ehresmann semigroup, respectively, then $S(B)$ is a normal quasi-Ehresmann semigroup.*

Conversely, up to good isomorphism, each regular, left quasinormal and normal quasi-Ehresmann semigroup can be constructed as above.

Proof Only need consider the regular case. To establish the first part, without loss of generality, we assume that $\bar{S}(\bar{B}) = \bar{\mathbb{S}}(\bar{\mathbb{B}})$ is the semigroup defined by a right regular triple $\mathfrak{R} = (E; T; \Lambda, \bar{\Psi})$, and that $\bar{\bar{S}}(\bar{\bar{B}}) = \bar{\bar{\mathbb{S}}}(\bar{\bar{\mathbb{B}}})$ is the semigroup defined by a left regular triple $\mathfrak{L} = (E; T; I, \bar{\bar{\Phi}})$. Let $\mathbb{S}, \mathbb{B}$ be the sets defined as in (1), and hence we form a global tuple (that is, a sextuple) $\mathfrak{G} = (E; T; I, \Phi; \Lambda, \Psi)$ by defining two mappings

$$\Phi: \ S \longrightarrow \mathscr{T}^*(I), \ (i, s, \lambda) \longmapsto \bar{\bar{\psi}}_{(i,s)},$$

$$\Psi: \ S \longrightarrow \mathscr{T}(\Lambda), \ (i, s, \lambda) \longmapsto \bar{\psi}_{(s,\lambda)}.$$

By routine checking, we can easily verify that $\mathfrak{G}$ is a global tuple so that $\mathbb{S}(\mathbb{B})$ forms a good quasi-Ehresmann semigroup under the operation (4). Moreover, it follows from Lemma 4.1 and its left–right dual result concerning the right semiregular quasi-Ehresmann semigroups that $\mathbb{S}(\mathbb{B})$ is indeed a regular quasi-Ehresmann semigroup.

By Corollary 3.1 (iii) and its left–right dual result concerning the left regular quasi-Ehresmann semigroups, we deduce the following equations:

$$S = \{((i, s), (s, \lambda)) | s \in T, i \in I_{s^+}, \lambda \in \Lambda_{s^*}\},$$
$$B = \{((i, e), (e, \lambda)) | e \in E, i \in I_e, \lambda \in \Lambda_e\}.$$

We now proceed to identify an arbitrary element $((i, s), (s, \lambda))$ of S to the element (i, s, λ) of $\mathbb{S}$ such that $S = \mathbb{S}$ and $B = \mathbb{B}$ as sets. Then, for any $(i, s, \lambda), (j, t, \mu) \in \mathbb{S}$, we have

$$\begin{aligned}(i, s, \lambda)(j, t, \mu) &= (\phi_{(i,s,\lambda)}(j), st, \lambda\psi_{(j,t,\mu)}) \\ &= (\bar{\bar{\phi}}_{(i,s)}(j), st, \lambda\bar{\psi}_{(t,\mu)}) \\ &= ((\bar{\bar{\phi}}_{(i,s)}(j), st), (st, \lambda\bar{\psi}_{(t,\mu)})) \\ &= ((i, s)(j, t), (s, \lambda)(t, \mu)) \\ &= ((i, s), (s, \lambda))((j, t), (t, \mu)).\end{aligned}$$

Thus $S = \mathbb{S}$ and $B = \mathbb{B}$ as semigroups so that we have shown that $S(B)$ is a regular quasi-Ehresmann semigroup.

The proof of the converse part of the corollary is similar to Theorem 4.4 and hence we omit the details. □

By using our Corollarys 3.2, 5.3, as well as its left–right dual result concerning the left normal quasi-Ehresmann semigroups, we obtain another structure theorem of normal quasi-Ehresmann semigroups as given below.

Corollary 5.4 *Let $T(E)$ be an Ehresmann semigroup. We also let $I = (E; I_e, \theta_{e,f})$ be a left normal band and $\Lambda = (E; \Lambda_e; \vartheta_{e,f})$ a right normal band. If $\mathbb{S}, \mathbb{B}$ is the sets defined as in (1), then $\mathbb{S}(\mathbb{B})$ forms a normal quasi-Ehresmann semigroup under the operation*

$$(i, s, \lambda)(j, t, \mu) = (j\theta_{s^+,(st)^+}, st, \mu\vartheta_{t^*,(st)^*}).$$

Conversely, up to good isomorphism, every normal quasi-Ehresmann semigroup can be similarly constructed in the above manner. □

It is noteworthy that a super B-quasi-Ehresmann semigroup is an analogy of the orthodox semigroup among the class of good B-quasi-Ehresmann semigroups.

For the sake of convenience, we call a super B-quasi-Ehresmann semigroups with the following seven different projective bands, namely (1^*) right regular bands, (2^*) right normal bands, (3^*) left semiregular bands, (4^*) left seminormal bands, (5^*) regular bands, (6^*) left quasinormal bands and (7^*) normal bands the 7^*-super B-quasi-Ehresmann semigroup.

In closing this paper, summarizing all our results, we establish the following "general representation theorem of a 7^*-super B-quasi-Ehresmann semigroup" :

Theorem 5.5 (The general representation theorem of a generalized regular semigroup (the 7^*-super B-quasi-Ehresmann semigroup))

Let S be a super B-quasi-Ehresmann semigroup with seven different kind bands of projections. Then we give the following representation theorem for the generalized regular semigroup S (the 7^*-super B-quasi-Ehresmann semigroup).

(i) *The projective bands of S are (1^*) if and only if S is good isomorphic to some semigroup $\bar{\mathbb{S}}(\bar{\mathbb{B}})$ defined by the right regular triple $\mathfrak{R} = (E; T; \Lambda, \bar{\Psi})$, with the following operation*

$$(s, \lambda)(t, \mu) = \left(st, \lambda\bar{\psi}_{(t,\mu)}\right).$$

(ii) *The projective bands of S are* (2*) *if and only if S is good isomorphic to some semigroup* $\bar{\mathbb{S}}(\bar{\mathbb{B}})$ *with the following operation*

$$(s, \lambda)(t, \mu) = (st, \mu\vartheta_{t^*,(st)^*}).$$

where $T(E)$ *is an Ehresmann semigroup,* $\Lambda = (E; \Lambda_e, \vartheta_{e,f})$ *a right normal band, and* $\bar{\mathbb{S}}, \bar{\mathbb{B}}$ *the sets defined as in* (2).

(iii) *The projective bands of S are* (3*) *if and only if there exists a right regular quasi-Ehresmann semigroup* $\bar{S}(\bar{B})$ *where* $\bar{B} = (E; \bar{B}_e)$*, and a non-empty set* $I = \biguplus_{e\in E} I_e$*, S is good isomorphic to some semigroup* $S(B)$*, with the following operation*

$$(i, a)(j, b) = (\phi_{(i,a)}(j), ab).$$

where

$$S = \bigcup_{a\in\bar{S}} \left(I_{\overline{a^+}} \times \{a\}\right), \qquad B = \bigcup_{u\in\bar{B}} \left(I_{\overline{u}} \times \{u\}\right),$$

and

$$\Phi : S \longrightarrow \mathscr{T}^*(I),\ x \longmapsto \phi_x$$

is a mapping satisfying the condition (GL).

(iv) *The projective bands of S are* (4*) *if and only if there exists a right normal quasi-Ehresmann semigroup* $\bar{S}(\bar{B})$*, where* $\bar{B} = (E; \bar{B}_e)$*, and a non-empty set* $I = \biguplus_{e\in E} I_e$ *so that S is good isomorphic to some semigroup* $S(B)$*, with the operation*

$$(i, a)(j, b) = (\phi_{(i,a)}(j), ab),$$

where

$$S = \bigcup_{a\in\bar{S}} (I_{\overline{a^+}} \times \{a\}), \qquad B = \bigcup_{u\in\bar{B}} (I_{\overline{u}} \times \{u\}),$$

and

$$\Phi : S \longrightarrow \mathscr{T}^*(I),\ x \longmapsto \phi_x$$

is a mapping satisfying the condition (GL).

(v) *The projective bands of S are* (5*) *if and only if there exists a left regular quasi-Ehresmann semigroup* $\bar{\bar{S}}(\bar{\bar{B}})$ *and a right regular quasi-Ehresmann semigroup* $\bar{S}(\bar{B})$ *such that* $T(E) = \bar{\bar{S}}(\bar{\bar{B}})/\gamma = \bar{S}(\bar{B})/\gamma$*, so that S is good isomorphic to some semigroup* $S(B)$*, where* $S = \bar{\bar{S}} \times_T \bar{S}$ *and* $B = \bar{\bar{B}} \times_E \bar{B}$*.*

(vi) *The projective bands of S are* (6*) *if and only if there exists a left regular quasi-Ehresmann semigroup* $\bar{\bar{S}}(\bar{\bar{B}})$ *and a right normal quasi-Ehresmann semigroup*

$\bar{S}(\bar{B})$ *such that* $T(E) = \bar{\bar{S}}(\bar{\bar{B}})/\gamma = \bar{S}(\bar{B})/\gamma$, *so that S is good isomorphic to some semigroup* $S(B)$, *where* $S = \bar{\bar{S}} \times_T \bar{S}$ *and* $B = \bar{\bar{B}} \times_E \bar{B}$.

(vii) *The projective bands of S are* (7^*) *if and only if there exists a left normal quasi-Ehresmann semigroup* $\bar{\bar{S}}(\bar{\bar{B}})$ *and a right normal quasi-Ehresmann semigroup* $\bar{S}(\bar{B})$ *such that* $T(E) = \bar{\bar{S}}(\bar{\bar{B}})/\gamma = \bar{S}(\bar{B})/\gamma$ *so that S is good isomorphic to some semigroup* $S(B)$, *where* $S = \bar{\bar{S}} \times_T \bar{S}$ *and* $B = \bar{\bar{B}} \times_E \bar{B}$.

(viii) *The projective bands of S are also* (7^*) *if and only if there exists a left normal band* $I = (E; I_e, \theta_{e,f})$ *and a right normal band* $\Lambda = (E; \Lambda_e; \vartheta_{e,f})$ *so that S is good isomorphic to some semigroup* $\mathbb{S}(\mathbb{B})$, *with the operation*

$$(i, s, \lambda)(j, t, \mu) = (j\theta_{s^+,(st)^+}, st, \mu\vartheta_{t^*,(st)^*}),$$

where $\mathbb{S}, \mathbb{B}$ *is the sets defined as in* (1).

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