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The Closed Subset Theorem for Inverse Limits with Upper Semicontinuous Bonding Functions

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Abstract We give several characterizations of inverse limits of compact metric spaces with upper semicontinuous set-valued bonding functions having the property that any closed subset of the inverse limit is the inverse limit of its projections. This solves a problem stated by Ingram.

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1 Introduction

The following theorem is a well-known result in the theory of inverse limits of inverse sequences of compact metric spaces with continuous single-valued bonding functions.

Theorem 1 *Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of compact metric spaces X_n with continuous single-valued bonding functions $f_n : X_{n+1} \to X_n$ and let A be a closed subset of $\varprojlim\{X_n, f_n\}_{n=1}^{\infty}$. For each positive integer i, let $\pi_i : \varprojlim\{X_n, f_n\}_{n=1}^{\infty} \to X_i$ denote the i-th projection map. Then $\{\pi_n(A), f_n|_{\pi_{n+1}(A)}\}_{n=1}^{\infty}$ is an inverse sequence with onto bonding functions and*

$$A = \varprojlim\{\pi_n(A), f_n|_{\pi_{n+1}(A)}\}_{n=1}^{\infty}.$$

Proof [1, Lemma 2.6, p. 20].

The theorem is known as "the closed subset theorem" (for inverse limits of inverse sequences of compact metric spaces with continuous single-valued bonding functions). It is well known that an analogue of the theorem does not hold for inverse limits of inverse sequences of compact metric spaces with upper semicontinuous set-valued bonding functions, see [2, Example 3.4, p. 51]. Based on this example, W. T. Ingram stated the following open problems concerning "the closed subset theorem" (for inverse limits of inverse sequences of compact metric spaces with upper semicontinuous set-valued bonding functions).

Problem 1 [2, Problem 6.14, p. 78] Find sufficient conditions on the bonding functions so that closed subsets of the inverse limits are the inverse limits of their projections.

Problem 2 [2, Problem 6.15, p. 78] Solve Problem 6.14 on [0, 1].

Problem 3 [2, Problem 6.16, p. 78] Solve Problem 6.14 for closed subsets of the inverse limits that are connected.

Problem 4 [2, Problem 6.17, p. 78] Solve Problem 6.16 on [0, 1].

In this paper, we solve Problems 1, 2, 3 and 4. Explicitly, we give sufficient and necessary conditions on the bonding functions $f_n : X_{n+1} \multimap X_n$ for the inverse limit $\underset{\circ\!-\!\!-}{\lim}\{X_n, f_n\}_{n=1}^{\infty}$ to have the property that for any closed subset A of $\underset{\circ\!-\!\!-}{\lim}\{X_n, f_n\}_{n=1}^{\infty}$, A is the inverse limit of its projections. It turns out that there are two possible interpretations of these problems and we precisely formulate and solve both of them.

We proceed as follows. In Sect. 2 we introduce basic definitions and notation. In Sect. 3 we present and prove our main results concerning the first interpretation of Ingram's problems, while in Sect. 4 we present and prove our main results concerning the second interpretation of Ingram's problems. In Sect. 5 we state some open problems.

2 Definitions and Notation

Our definitions and notation mostly follow [3], [2] and [1].

If (X, d) is a compact metric space, then 2^X denotes the set of all nonempty closed subsets of X, and $C(X)$ the set of all connected elements of 2^X.

Let X and Y be compact metric spaces. A function $f : X \to 2^Y$ is called a *set-valued function* from X to Y. We denote set-valued functions $f : X \to 2^Y$ by $f : X \multimap Y$.

A function $f : X \multimap Y$ is an *upper semicontinuous* set-valued function if for each open set $V \subseteq Y$ the set $\{x \in X \mid f(x) \subseteq V\}$ is an open set in X.

The *graph* $\Gamma(f)$ of a set-valued function $f : X \multimap Y$ is the set of all points $(x, y) \in X \times Y$ such that $y \in f(x)$.

There is a simple characterization of upper semicontinuous set-valued functions ([2, Theorem 1.2, p. 3]).

Theorem 2 *Let X and Y be compact metric spaces and $f : X \multimap Y$ a set-valued function. Then f is upper semicontinuous if and only if its graph $\Gamma(f)$ is closed in $X \times Y$.*

If $F : X \multimap Y$ is a set-valued function, where for each $x \in X$, the image $F(x)$ is a singleton in Y, then we can interpret it as a single-valued function, identifying it with the function $f : X \to Y$, where $F(x) = \{f(x)\}$ for any $x \in X$. Conversely, any single-valued function $f : X \to Y$ can be identified with the set-valued function $F : X \multimap Y$, defined by $F(x) = \{f(x)\}$. By an abuse of notation, we will use the same letter for both functions, i.e., for such functions we will use both $f : X \to Y$ and $f : X \multimap Y$ interchangeably, making our choice depending on the aspect of the function we want to emphasize.

Let $f : X \multimap Y$ be a set-valued function. Then we define the preimage $f^{-1}(y) = \{x \in X \mid y \in f(x)\}$ for any $y \in Y$.

The set-valued function mapping $y \in Y$ to $f^{-1}(y) \in 2^X$ is defined only on $f(X) = \bigcup_{x \in X} f(x)$. Therefore, whenever we treat f^{-1} as a function, we treat it as a function $f^{-1} : f(X) \to 2^X$, i.e., as $f^{-1} : f(X) \multimap X$. We call it the *inverse function* of the function f. Note that f^{-1} is upper semicontinuous if f is upper semicontinuous.

We say that the graph of a set-valued function $f : X \multimap Y$ is *surjective* if for each $y \in Y$, $|f^{-1}(y)| \geq 1$, i.e., if $f(X) = Y$.

In this paper we deal with *inverse sequences* $\{X_n, f_n\}_{n=1}^{\infty}$, where X_n are compact metric spaces and $f_n : X_{n+1} \multimap X_n$ are upper semicontinuous set-valued functions.

The *inverse limit* of an inverse sequence $\{X_n, f_n\}_{n=1}^{\infty}$ is defined to be the subspace of the product space $\prod_{n=1}^{\infty} X_n$ of all $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \prod_{n=1}^{\infty} X_n$, such that $x_n \in f_n(x_{n+1})$ for each n. The inverse limit is denoted by $\underset{\circ\!\!-}{\lim}\{X_n, f_n\}_{n=1}^{\infty}$.

Such inverse limits are a recent generalization (by Ingram and Mahavier [4,5]) of inverse limits $\varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ of inverse sequences $\{X_n, f_n\}_{n=1}^{\infty}$ of compact metric spaces with continuous single-valued bonding functions $f_n : X_{n+1} \to X_n$.

Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of compact metric spaces X_n and upper semicontinuous functions $f_n : X_{n+1} \multimap X_n$. For all positive integers $i, j, i < j$, we define the function $f_{i,j} : X_j \multimap X_i$ by

$$f_{i,j} = f_i \circ f_{i+1} \circ f_{i+2} \circ \ldots \circ f_{j-2} \circ f_{j-1}.$$

As an important tool in this paper, we use different types of projections; hence, we introduce notation for them:

Definition 1 Let X_n be a compact metric space for each positive integer n. For each positive integer m, we define the function $\pi_m : \prod_{n=1}^{\infty} X_n \to X_m$ to be the projection to the m-th factor, $\pi_m(\mathbf{x}) = x_m$, and the function $\pi_{m+1,m} : \prod_{n=1}^{\infty} X_n \to X_{m+1} \times X_m$ by $\pi_{m+1,m}(\mathbf{x}) = (x_{m+1}, x_m)$.

In [6], Nall introduced the following important notion:

Definition 2 Let X_n be a compact metric space for each positive integer n. If $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \prod_{n=1}^{\infty} X_n$ and $\mathbf{y} = (y_1, y_2, y_3, \ldots) \in \prod_{n=1}^{\infty} X_n$ and $x_i = y_i$ for some $i > 1$, then we denote by $Cr_i(\mathbf{x}, \mathbf{y})$ the point

$$Cr_i(\mathbf{x}, \mathbf{y}) = (x_1, x_2, x_3, \ldots, x_i, y_{i+1}, y_{i+2}, y_{i+3}, \ldots) \in \prod_{n=1}^{\infty} X_n.$$

$Cr_i(\mathbf{x}, \mathbf{y})$ is called the i-th *crossover* of $\mathbf{x}$ and $\mathbf{y}$.

For a subset A of $\prod_{n=1}^{\infty} X_n$ we let $Cr(A)$ to be the set of all $\mathbf{z} \in \prod_{n=1}^{\infty} X_n$ such that there is an i and elements $\mathbf{x}$ and $\mathbf{y}$ in A such that $\mathbf{z} = Cr_i(\mathbf{x}, \mathbf{y})$.

Among other results proved in [6], Nall proved the following theorem ([6, Theorem 3.2.]).

Theorem 3 *Let X_n be a compact metric space for each positive integer n and $A \neq \emptyset$ a closed subset of $\prod_{n=1}^{\infty} X_n$. The following statements are equivalent:*

1. *There exist upper semicontinuous set-valued functions $f_n : \pi_{n+1}(A) \multimap \pi_n(A)$ such that $A = \varprojlim\{\pi_n(A), f_n\}_{n=1}^{\infty}$.*
2. $Cr(A) = A$.

To prove the implication from 2 to 1 in Theorem 3, Nall defined the upper semicontinuous set-valued functions $f_n : \pi_{n+1}(A) \to 2^{\pi_n(A)}$ to be the functions with the graphs $\Gamma(f_n) = \pi_{n+1,n}(A)$. We use the following notation introduced in [3] for such subspaces $\pi_n(A)$ and functions f_n:

Definition 3 Let X_n be a compact metric space for each positive integer n and $A \neq \emptyset$ a closed subset of $\prod_{n=1}^{\infty} X_n$. Then

1. $\Theta_n = \pi_n(A)$,
2. $\psi_n : \Theta_{n+1} \multimap \Theta_n$ is defined by $\Gamma(\psi_n) = \pi_{n+1,n}(A)$,
3. $L(A) = \varprojlim\{\Theta_n, \psi_n\}_{n=1}^{\infty}$.

Note that $\{\Theta_n, \psi_n\}_{n=1}^{\infty}$ is an inverse sequence with bonding functions whose graphs are surjective.

Lemma 1 *Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of compact metric spaces with single-valued bonding functions, and let A be a closed subset of $\varprojlim\{X_n, f_n\}_{n=1}^{\infty}$. Then*

$$\psi_n(x) = f_n|_{\pi_{n+1}(A)}(x)$$

for each positive integer n and for each $x \in X_{n+1}$.

Proof The claim is equivalent to $\Gamma(\psi_n) = \Gamma(f_n|_{\pi_{n+1}(A)})$. It follows from [3, Lemma 2.10, p. 159], that $\Gamma(\psi_n) \subseteq \Gamma(f_n|_{\pi_{n+1}(A)})$. To show that $\Gamma(\psi_n) \supseteq \Gamma(f_n|_{\pi_{n+1}(A)})$, let $(x, y) \in \Gamma(f_n|_{\pi_{n+1}(A)})$. It follows that $y = f_n|_{\pi_{n+1}(A)}(x)$. On the other hand, since $x \in \pi_{n+1}(A)$, it follows that there is $\mathbf{x} \in A$ such that $x = \pi_{n+1}(\mathbf{x})$. Since f_n is single-valued, it follows that $y = \pi_n(\mathbf{x})$. Therefore $(x, y) = \pi_{n+1,n}(\mathbf{x}) \in \pi_{n+1,n}(A) = \Gamma(\psi_n)$. □

This enables us to interpret Problem 1 in two possible ways, since ψ_n and $f_n|_{\pi_{n+1}(A)}$ may differ for set-valued functions f_n. The first interpretation gives:

Problem 5 Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of compact metric spaces X_n with upper semicontinuous set-valued bonding functions $f_n : X_{n+1} \multimap X_n$. Find sufficient conditions on f_n under which

$$A = \varprojlim\{\pi_n(A), f_n|_{\pi_{n+1}(A)}\}_{n=1}^{\infty}$$

for each closed subset $A \subseteq \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$.

Note that the question asked in Problem 5 has two parts. The first part asks if $f_n|_{\pi_{n+1}(A)}(x)$ is a subset of $\pi_n(A)$ for each $x \in \pi_{n+1}(A)$ (since this is not necessarily so in general), in order for $\{\pi_n(A), f_n|_{\pi_{n+1}(A)}\}_{n=1}^{\infty}$ to be an inverse sequence. The second part then asks: if $\{\pi_n(A), f_n|_{\pi_{n+1}(A)}\}_{n=1}^{\infty}$ is an inverse sequence, does A equal the inverse limit of the inverse sequence $\{\pi_n(A), f_n|_{\pi_{n+1}(A)}\}_{n=1}^{\infty}$?

The second interpretation gives:

Problem 6 Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of compact metric spaces X_n with upper semicontinuous set-valued bonding functions $f_n : X_{n+1} \multimap X_n$. Find sufficient conditions on f_n under which

$$A = L(A) \quad \text{(i.e., } A = \varprojlim\{\pi_n(A), \psi_n\}_{n=1}^{\infty}\text{)}$$

for each closed subset $A \subseteq \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$.

We will show that the first interpretation is much more restrictive than the second one. We will treat both of them independently.

3 The First Interpretation

In [2, Example 3.4, p. 51] an example of an inverse sequence $\{X_n, f_n\}_{n=1}^{\infty}$ and a closed subset A in $\varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ is given such that

$$A = \varprojlim\{\pi_n(A), f_n|_{\pi_{n+1}(A)}\}_{n=1}^{\infty}$$

does not hold (where $f_n|_{\pi_{n+1}(A)} : \pi_{n+1}(A) \multimap X_n$ is defined by $f_n|_{\pi_{n+1}(A)}(x) = f_n(x)$ for each positive integer n and for each $x \in \pi_{n+1}(A)$; in this example all $X_n = [0, 1]$ and $f_n(x) = \{0, x\}$ for all $x \in [0, 1]$; $A = \{(t, t, t, \ldots) \mid t \in [0, 1]\}$). Implicitly, this example shows that Ingram interpreted Problem 1 as a question about the inverse sequence $\{\pi_n(A), f_n|_{\pi_{n+1}(A)}\}_{n=1}^{\infty}$, showing that he understood his problem as in our first interpretation.

The following obvious result is used in the proof of Theorem 4, the main result in this section.

Lemma 2 *Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of compact metric spaces X_n with upper semicontinuous set-valued bonding functions $f_n : X_{n+1} \multimap X_n$ and let $X = \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$. Then $\{\pi_n(X), f_n|_{\pi_{n+1}(X)}\}_{n=1}^{\infty}$ is an inverse sequence and*

$$X = \varprojlim\{\pi_n(X), f_n|_{\pi_{n+1}(X)}\}_{n=1}^{\infty}.$$

The following theorem (which we call the closed subset theorem for inverse limits with upper semicontinuous bonding functions [for the first interpretation of Ingram's problem]) answers Problem 5. At the same time it gives answers also to Problems 1–4 under the first interpretation.

Theorem 4 *Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of compact metric spaces X_n with upper semicontinuous set-valued bonding functions $f_n : X_{n+1} \multimap X_n$ and let $X = \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$. The following statements are equivalent.*

1. *For each closed subset A of $\varprojlim\{X_n, f_n\}_{n=1}^{\infty}$, $\{\pi_n(A), f_n|_{\pi_{n+1}(A)}\}_{n=1}^{\infty}$ is an inverse sequence and $A = \varprojlim\{\pi_n(A), f_n|_{\pi_{n+1}(A)}\}_{n=1}^{\infty}$.*
2. *For each $\mathbf{x} \in \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ and for each positive integer n, $|f_n(x_{n+1})| = 1$.*
3. *For each positive integer n, $f_n|_{\pi_{n+1}(X)} : \pi_{n+1}(X) \multimap \pi_n(X)$ is single-valued.*

Proof Suppose that there is a point $\mathbf{x} \in \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ such that $f_n(x_{n+1})$ contains at least two different points, say x_n and y_n, then for $A = \{\mathbf{x}\}$, $\{\pi_n(A), f_n|_{\pi_{n+1}(A)}\}_{n=1}^{\infty}$ is not an inverse sequence since $f_n|_{\pi_{n+1}(A)}$ does not map $\pi_{n+1}(A)$ into $\pi_n(A)$. Therefore, 2 follows from 1.

Obviously, 2 implies 3, and 3 implies 1 follows by Lemma 2 and Theorem 1. □

The Ingram's example, mentioned above, obviously does not satisfy the property 2 and 3 from Theorem 4, and hence it does not satisfy the property 1.

In Theorem 4, the graphs of the bonding functions are not necessarily surjective. We conclude this section by proving the following corollary which deals with inverse sequences of compact metric spaces with upper semicontinuous set-valued bonding functions whose graphs are surjective.

Corollary 1 *Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of compact metric spaces X_n with upper semicontinuous set-valued bonding functions $f_n : X_{n+1} \multimap X_n$ whose graphs are surjective. The following statements are equivalent.*

1. *For each closed subset A of $\varprojlim\{X_n, f_n\}_{n=1}^{\infty}$, $\{\pi_n(A), f_n|_{\pi_{n+1}(A)}\}_{n=1}^{\infty}$ is an inverse sequence and $A = \varprojlim\{\pi_n(A), f_n|_{\pi_{n+1}(A)}\}_{n=1}^{\infty}$.*
2. *For each positive integer n, f_n is single-valued.*

Proof Let $X = \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$. Since for bonding functions whose graphs are surjective, $\pi_n(X) = X_n$ holds for each n, 3 of Theorem 4 holds if and only if all the functions $f_n : X_{n+1} \multimap X_n$ are single-valued. □

The following example shows that 3 of Theorem 4 may be satisfied even if the functions $f_n : X_{n+1} \multimap X_n$ are not all single-valued.

Example 1 For each positive integer n, let $X_n = [0, 1]$ and let $f_n : X_{n+1} \multimap X_n$ be defined by $f_n(x) = \{0\}$ for all $x < 1$ and $f_n(1) = \{0, \frac{1}{2}\}$. Then $\varprojlim\{X_n, f_n\}_{n=1}^{\infty} = \{(0, 0, 0, \ldots)\}$ and 1, 2 and 3 of Theorem 4 obviously hold.

4 The Second Interpretation

In this section, we study the second interpretation (Problem 6).

We introduce several notions and abbreviations in order to enable a more precise and concise formulations of results in this section as well as of open problems in the last section.

Definition 4 Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of compact metric spaces with upper semicontinuous bonding functions $f_n : X_{n+1} \multimap X_n$ and let $\mathcal{A}$ be any set of nonempty closed subsets of $\varprojlim\{X_n, f_n\}_{n=1}^{\infty}$. We say that the inverse sequence $\{X_n, f_n\}_{n=1}^{\infty}$ has *the inverse limit property for the class* $\mathcal{A}$ (for short ILP($\mathcal{A}$)), if $A = L(A)$ for any $A \in \mathcal{A}$.

If $\mathcal{A} = 2^{\varprojlim\{X_n, f_n\}_{n=1}^{\infty}}$ we shorten ILP($\mathcal{A}$) to ILP.

Problem 6 can now be reformulated as follows.

Problem 7 Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of compact metric spaces X_n with upper semicontinuous set-valued bonding functions $f_n : X_{n+1} \multimap X_n$. Find sufficient conditions on f_n under which the inverse sequence $\{X_n, f_n\}_{n=1}^{\infty}$ has ILP.

The following definitions introduce the notions needed for a characterization that solves Problem 7.

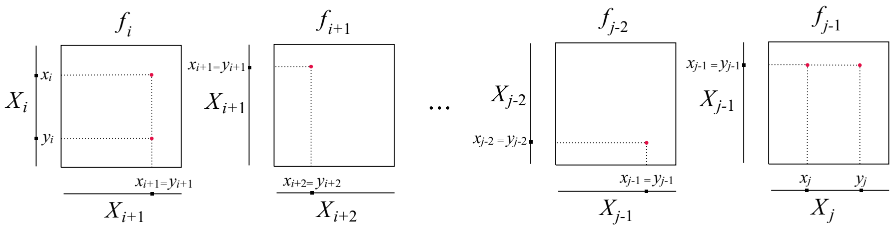

Fig. 1 Producing a non-trivial crossover

Definition 5 Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of compact metric spaces with upper semicontinuous bonding functions $f_n : X_{n+1} \multimap X_n$. We say that points

$$\mathbf{x} = (x_1, x_2, x_3, \ldots), \mathbf{y} = (y_1, y_2, y_3, \ldots) \in \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$$

form a non-trivial crossover if there are positive integers i and j such that

1. $i < j - 1$,
2. $x_i \neq y_i, x_j \neq y_j$, and
3. $x_k = y_k$ for each $k = i+1, i+2, i+3, \ldots, j-1$.

See Fig. 1.

The name has been chosen because under conditions given in Definition 5, the crossover $Cr_{i+1}(\mathbf{x}, \mathbf{y})$ differs from both $\mathbf{x}$ and $\mathbf{y}$.

Definition 6 Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of compact metric spaces with upper semicontinuous bonding functions $f_n : X_{n+1} \multimap X_n$. We say that the sequence $\{f_n\}_{n=1}^{\infty}$ *produces a non-trivial crossover* if there are points

$$\mathbf{x}, \mathbf{y} \in \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$$

that form a non-trivial crossover.

Next we state and prove one of our main results (which we call the closed subset theorem for inverse limits with upper semicontinuous bonding functions [for the second interpretation of Ingram's problem]). Note that the last two statements are formulated in terms of bonding functions.

Theorem 5 *Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of compact metric spaces with upper semicontinuous set-valued bonding functions $f_n : X_{n+1} \multimap X_n$. The following statements are equivalent.*

1. *The inverse sequence $\{X_n, f_n\}_{n=1}^{\infty}$ has ILP.*
2. *$A = Cr(A)$ for any closed subset $A \subseteq \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$.*
3. *$Cr(\{\mathbf{x}, \mathbf{y}\}) = \{\mathbf{x}, \mathbf{y}\}$ for any $\mathbf{x}, \mathbf{y} \in \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$.*
4. *The sequence $\{f_n\}_{n=1}^{\infty}$ does not produce a non-trivial crossover.*

Moreover, each of the statements listed above is implied by each of

5. *For each positive integer* n *and for each* $x_n \in X_n$, $|f_n^{-1}(x_n)| > 1$ *implies* $|f_{i,n}(x_n)| = 1$ *for each* $i \leq n-1$.

and

6. *For each positive integer* n *and for each* $x_{n+1} \in X_{n+1}$, $|f_n(x_{n+1})| > 1$ *implies* $|f_{n+1,i}^{-1}(x_{n+1})| = 1$ *for each* $i > n+1$.

If the bonding functions f_n *have surjective graphs, the converse of each of these implications also holds true.*

Proof It follows from [3, Theorem 5.1, p. 165] that 2 is equivalent to 1. Next we show that 2 is equivalent to 3. Obviously, 3 follows from 2. Let A be a closed subset of $\varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ such that $Cr(A) \neq A$. Then there are $\mathbf{x}, \mathbf{y} \in A$ and a positive integer n such that $Cr_n(\mathbf{x}, \mathbf{y}) \notin A$. Therefore, $Cr_n(\mathbf{x}, \mathbf{y}) \notin \{\mathbf{x}, \mathbf{y}\}$. It follows that $Cr(\{\mathbf{x}, \mathbf{y}\}) \neq \{\mathbf{x}, \mathbf{y}\}$ and hence 2 follows from 3. It is easy to see that 3 is equivalent to 4.

Suppose that 4 does not hold. Then there are points

$$\mathbf{x} = (x_1, x_2, x_3, \ldots), \mathbf{y} = (y_1, y_2, y_3, \ldots) \in \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$$

such that there are positive integers i and j such that

- $i < j-1$,
- $x_i \neq y_i$, $x_j \neq y_j$, and
- $x_k = y_k$ for each $k = i+1, i+2, i+3, \ldots, j-1$.

Therefore, $|f_{j-1}^{-1}(x_{j-1})| > 1$ and $|f_{i,j-1}(x_{j-1})| > 1$ and 5 does not hold. Hence, 4 follows from 5.

Under the same assumption that 4 does not hold for the same points $\mathbf{x}, \mathbf{y} \in \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ we see that $|f_i(x_{i+1})| > 1$ and $|f_{i+1,j}^{-1}(x_{i+1})| > 1$. It follows that 6 does not hold. Hence, 4 follows from 6.

Suppose that 5 does not hold and that the bonding functions f_n have surjective graphs. This means that there are positive integers n and $i \leq n-1$ and $x_n \in X_n$ such that $|f_n^{-1}(x_n)| > 1$ and $|f_{i,n}(x_n)| > 1$. Using surjectivity of the graphs of the bonding functions, one can easily construct points $\mathbf{x}$ and $\mathbf{y}$ in $\varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ that form a non-trivial crossover and 4 does not hold. Hence 4 implies 5.

Finally, assuming that the bonding functions f_n have surjective graphs, suppose that 6 does not hold. This means that there are positive integers n and $i \geq n+1$ and $x_{n+1} \in X_{n+1}$ such that $|f_n(x_{n+1})| > 1$ and $|f_{n+1,i}^{-1}(x_{n+1})| > 1$ (since the bonding functions have surjective graphs). Therefore, 4 does not hold, hence 4 implies 6. □

Corollary 2 *Let* $\{X_n, f_n\}_{n=1}^{\infty}$ *be an inverse sequence of compact metric spaces with upper semicontinuous set-valued bonding functions* $f_n : X_{n+1} \multimap X_n$. *If there is* $n_0 \in \{1, 2, 3, \ldots\} \cup \{\infty\}$ *such that*

1. *for each integer* $n > n_0$, f_n^{-1} *is single-valued, and*

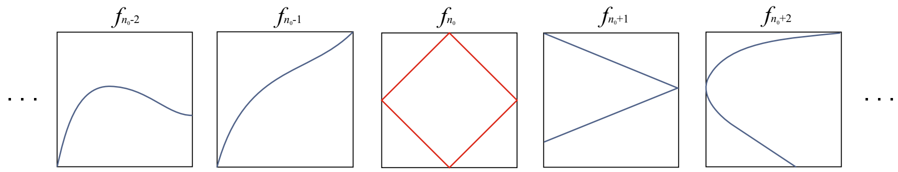

Fig. 2 An inverse sequence satisfying conditions of Corollary 2

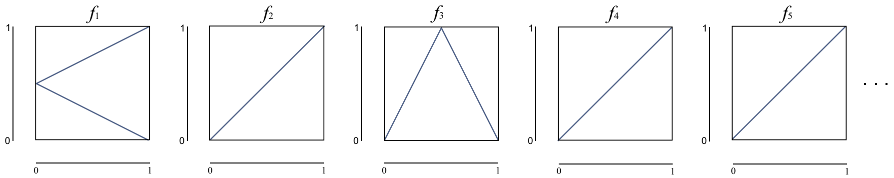

Fig. 3 Graphs of the bonding functions from Example 2

2. *for each positive integer $n < n_0$, f_n is single-valued,*

then the inverse sequence $\{X_n, f_n\}_{n=1}^{\infty}$ has ILP.

Proof Obviously, the sequence $\{f_n\}_{n=1}^{\infty}$ does not produce a non-trivial crossover. Therefore, the inverse sequence $\{X_n, f_n\}_{n=1}^{\infty}$ has ILP by Theorem 5. □

Figure 2 shows an example of an inverse sequence satisfying conditions of Corollary 2.

Theorem 1 can be reformulated as follows (by Lemma 1) and obtained as a special case $n_0 = \infty$ of Corollary 2:

Corollary 3 *Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of compact metric spaces with upper semicontinuous set-valued bonding functions $f_n : X_{n+1} \multimap X_n$. If each of the bonding functions is single-valued, then the inverse sequence $\{X_n, f_n\}_{n=1}^{\infty}$ has ILP.*

As a special case $n_0 = 1$ of Corollary 2 we obtain also the following corollary:

Corollary 4 *Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of compact metric spaces with upper semicontinuous set-valued bonding functions $f_n : X_{n+1} \multimap X_n$. If each f_n^{-1} is single-valued, then the inverse sequence $\{X_n, f_n\}_{n=1}^{\infty}$ has ILP.*

We continue with two illustrative examples.

Example 2 Let $\{[0, 1], f_n\}_{n=1}^{\infty}$ be the inverse sequence of closed unit intervals with bonding functions, as seen in Fig. 3.

The points $\mathbf{x} = (\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \ldots)$ and $\mathbf{y} = (\frac{3}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \ldots)$ form a non-trivial crossover. Therefore, this inverse sequence does not have ILP.

Example 3 The inverse sequence from Ingram's example [2, Example 3.4, p. 51] has ILP since it does not produce a non-trivial crossover.

Recall that we have proved that the statements 1 to 6 in Theorem 5 are equivalent using the assumption of the surjectivity of the graphs of the bonding functions. The following simple example shows that without surjectivity there are such inverse sequences that satisfy the statement 4 but do not satisfy the statements 5 and 6.

Example 4 For each positive integer n, let $X_n = [0, 1]$ and let $f_n : X_{n+1} \multimap X_n$ be defined by $f_1(0) = \{0, 1\}$ and $f_1(x) = \{0\}$ for any $x > 0$, and $f_n(x) = \{0\}$ for all x for each $n > 1$. Then $\varprojlim\{X_n, f_n\}_{n=1}^{\infty} = \{(0, 0, 0, \ldots), (1, 0, 0, \ldots)\}$ and the statement 4 of Theorem 5 holds true but the statements 5 and 6 don't.

The general case when the bonding functions f_n do not necessarily have surjective graphs can be reduced to the surjective case by replacing the original spaces and bonding functions by projections. In this way we obtain the following final theorem.

Theorem 6 *Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of compact metric spaces with upper semicontinuous set-valued bonding functions $f_n : X_{n+1} \multimap X_n$ (whose graphs $\Gamma(f_n)$ are not necessarily surjective) and let*

$$X = \varprojlim\{X_n, f_n\}_{n=1}^{\infty}.$$

Also, for each positive integer $\mathbf{\Theta}_n$, let $\mathbf{\Theta}_n = \pi_n(X)$ and $\Gamma(\mathbf{\Psi}_n) = \pi_{n+1,n}(X)$. The following statements are equivalent.

1. *The inverse sequence $\{X_n, f_n\}_{n=1}^{\infty}$ has ILP.*
2. *For each positive integer n and for each $x_n \in \mathbf{\Theta}_n$, $|\mathbf{\Psi}_n^{-1}(x_n)| > 1$ implies $|\mathbf{\Psi}_{i,n}(x_n)| = 1$ for each $i \le n - 1$.*
3. *For each positive integer n and for each $x_{n+1} \in \mathbf{\Theta}_{n+1}$, $|\mathbf{\Psi}_n(x_{n+1})| > 1$ implies $|\mathbf{\Psi}_{n+1,i}^{-1}(x_{n+1})| = 1$ for each $i > n + 1$.*

Proof By Banič et al. [3, Theorem 5.1, p. 165], $X = L(X) = \varprojlim\{\mathbf{\Theta}_n, \mathbf{\Psi}_n\}_{n=1}^{\infty}$. Note that $\mathbf{\Psi}_n : \mathbf{\Theta}_{n+1} \multimap \mathbf{\Theta}_n$ all have surjective graphs. By Theorem 5, 2 and 3 of Theorem 6 are equivalent to

4. The inverse sequence $\{\mathbf{\Theta}_n, \mathbf{\Psi}_n\}_{n=1}^{\infty}$ has ILP.

Obviously, $\{\mathbf{\Theta}_n, \mathbf{\Psi}_n\}_{n=1}^{\infty}$ has ILP if and only if $\{X_n, f_n\}_{n=1}^{\infty}$ has ILP, since $\varprojlim\{X_n, f_n\}_{n=1}^{\infty} = \varprojlim\{\mathbf{\Theta}_n, \mathbf{\Psi}_n\}_{n=1}^{\infty}$. □

5 Open Problems

Since two-point spaces are not connected, one may expect that there are inverse sequences $\{X_n, f_n\}_{n=1}^{\infty}$ having ILP($C(\varprojlim\{X_n, f_n\}_{n=1}^{\infty})$) but without ILP. The following example shows that this is indeed the case.

Example 5 For each positive integer n, let $X_n = [0, 1]$ and let $f_n : X_{n+1} \multimap X_n$ be defined as follows. Let $C \subseteq [0, 1]$ be the standard ternary Cantor set and let $g : C \to [0, 1]$ be a continuous surjection. Let $\Gamma(f_n) = \Gamma(g) \cup \Gamma(g^{-1})$ for each n. Note that f_n :

$X_{n+1} \multimap X_n$ is an upper semicontinuous function with surjective graph and $\Gamma(f_n)$ is a Cantor set. Then $\varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ is a nondegenerate totally disconnected compactum, hence $\{X_n, f_n\}_{n=1}^{\infty}$ has ILP($C(\varprojlim\{X_n, f_n\}_{n=1}^{\infty})$). Let n be a positive integer and let $t \in [0, 1]$ be such that $|f_n(t)| > 1$ (If such a t did not exist, then f_n would be a single-valued upper semicontinuous function on [0, 1], hence f_n would be a continuous mapping on [0, 1]. Therefore $\Gamma(f_n)$ would be connected—a contradiction.). Let $x, y \in f_n(t)$ such that $x \neq y$. Since $f_{n+1}^{-1}(t) = f_n(t)$, it follows that $x, y \in f_{n+1}^{-1}(t)$. Hence $\{f_n\}_{n=1}^{\infty}$ produces a non-trivial crossover. Therefore $\{X_n, f_n\}_{n=1}^{\infty}$ does not have ILP by Theorem 5.

Therefore it may be interesting to solve the following variants of Ingram's problems that still remain open.

Problem 8 Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of compact metric spaces X_n with upper semicontinuous set-valued bonding functions $f_n : X_{n+1} \multimap X_n$. Find sufficient conditions on f_n under which $\{X_n, f_n\}_{n=1}^{\infty}$ has ILP($C(\varprojlim\{X_n, f_n\}_{n=1}^{\infty})$) (i.e., under which $A = L(A)$ for any continuum A in $\varprojlim\{X_n, f_n\}_{n=1}^{\infty}$).

Problem 9 Solve Problem 8 on [0, 1].

Problem 10 Solve Problem 8 for connected inverse limits.

Problem 11 Solve Problem 10 on [0, 1].

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