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Normal Families and Shared Functions Concerning Hayman's Question

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Abstract In this paper, we studied a normality criterion concerning Hayman's question and proved: let $n(\geq 2), k(\geq 1), m(\geq 0)$ be three integers, let $h(z)(\not\equiv 0)$ be a holomorphic function in a domain D with all zeros that have multiplicity at most m, and let $\mathcal{F}$ be a family of functions meromorphic in a domain D, all of whose zeros have multiplicity at least $k+m$. If, for any two functions $f, g \in \mathcal{F}$, $f^n f^{(k)}$ and $g^n g^{(k)}$ share $h(z)$ in D, then $\mathcal{F}$ is normal in D. The result gets rid of two conditions "all zeros of $h(z)$ have multiplicity divisible by $n+1$" and "all poles of $f(z)$ have multiplicity at least $m+1$" in the result due to Meng and Hu (Bull Malays Math Sci Soc 38:1331–1347, 2015).

Keywords Meromorphic function · Normal criterion · Shared function

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1 Introduction

In this paper, we assume the reader is familiar with Nevanlinna theory of meromorphic functions. Let D be a domain in $\mathbb{C}$ and let $\mathcal{F}$ be a family of meromorphic functions in D. We say that $\mathcal{F}$ is normal in D (in the sense of Montel) if each sequence $\{f_n\}$ in $\mathcal{F}$ has a subsequence $\{f_{n_j}\}$ that converges locally uniformly in D, with respect to the spherical metric, to a meromorphic function or ∞ (see [7,15,17]).

For simplicity, we take $\rightarrow$ to stand for convergence and $\Rightarrow$ for convergence spherically locally uniformly.

Let $f(z)$ and $g(z)$ be two meromorphic functions in a domain D, and let $h(z)$ be a holomorphic function in D. If $f(z) - h(z)$ and $g(z) - h(z)$ have the same zeros ignoring multiplicity (counting multiplicity), then we say that $f(z)$ and $g(z)$ share $h(z)$ IM (CM) in D.

The following normality criterion was conjectured by Hayman [8] and proved by several authors (see [1,4,6,10,16]).

Theorem 1 *Let n be a positive integer, and let $\mathcal{F}$ be a family of meromorphic functions in D. If, for each $f \in \mathcal{F}$, $f^n f' \neq 1$, then $\mathcal{F}$ is normal in D.*

For other related results, see Bergweiler and Langley [2], Pang and Zalcman [11], Wu and Xu [14] and Tan et al. [13].

In 2008, Zhang [18] considered the case of shared value and obtained.

Theorem 2 *Let $\mathcal{F}$ be a family of meromorphic functions in D, and let $n(\geq 2)$ be a positive integer. If, for any two functions $f, g \in \mathcal{F}$, $f^n f'$ and $g^n g'$ share a nonzero value a IM in D, then $\mathcal{F}$ is normal in D.*

In 2015, Meng and Hu [9] studied the case of $f^n f^{(k)} (n \geq 2)$ sharing a holomorphic function and obtained

Theorem 3 *Let $k(\geq 1), n(\geq 2), m(\geq 0)$ be three integers, let $h(z)(\not\equiv 0)$ be a holomorphic function in a domain D with all zeros that have multiplicity at most m and divisible by $n+1$, and let $\mathcal{F}$ be a family of meromorphic functions in domain D such that each $f \in \mathcal{F}$ has zeros of multiplicity at least $k+m$ and poles of multiplicity at least $m+1$. If, for any two functions $f, g \in \mathcal{F}$, $f^n(z) f^{(k)}(z)$ and $g^n(z) g^{(k)}(z)$ share $h(z)$ IM in D, then $\mathcal{F}$ is normal in D.*

By Theorems 2 and 3, it is nature to ask that: can we get rid of the condition "all zeros of $h(z)$ have multiplicity divisible by $n+1$" and "all poles of f have multiplicity at least $m+1$ in Theorem 3"?

In this paper, we studied the question and gave an affirmative answer to the question.

Theorem 4 *Let $k(\geq 1), n(\geq 2), m(\geq 0)$ be three integers, let $h(z)(\not\equiv 0)$ be a holomorphic function in a domain D with all zeros that have multiplicity at most m, and let $\mathcal{F}$ be a family of meromorphic functions in domain D such that each $f \in \mathcal{F}$ has zeros of multiplicity at least $k+m$. If, for any two functions $f, g \in \mathcal{F}$, $f^n(z) f^{(k)}(z)$ and $g^n(z) g^{(k)}(z)$ share $h(z)$ IM in D, then $\mathcal{F}$ is normal in D.*

In fact, we proved the following more general result:

Theorem 5 *Let $k(\geq 1)$, $n(\geq 2)$, $m(\geq 0)$ be three integers, let $h(z)(\not\equiv 0)$ be a holomorphic function in a domain D with all zeros that have multiplicity at most m, and let $\mathcal{F}$ be a family of meromorphic functions in a domain D such that each $f \in \mathcal{F}$ has zeros of multiplicity at least $k+m$. If, for any two functions $f, g \in \mathcal{F}$, $f^n(z)f^{(k)}(z) - h(z)$ has at most one distinct zero in D, then $\mathcal{F}$ is normal in D.*

The following examples show that the conditions in Theorem 5 are necessary.

Example 1 [9] Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$, let $h(z) \equiv 0$ and let

$$\mathcal{F} = \left\{ f_j(z) = e^{jz} \mid j = 1, 2, \ldots \right\}.$$

Obviously, $f_j^n(z) f_j^{(k)}(z) - h(z)$ does not have zero in D for each positive integer j. But the family $\mathcal{F}$ is not normal at $z = 0$. This shows that $h(z) \not\equiv 0$ is necessary Theorem 5.

Example 2 Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$, let $h(z) = \frac{1}{z^{n+k+1}}$ and let

$$\mathcal{F} = \left\{ f_j(z) = \frac{1}{jz} \mid j = 1, 2, \ldots, \text{and } j^{n+1} \neq (-1)^k k! \right\}.$$

Obviously, $f_j^n(z) f_j^{(k)}(z) - h(z)$ does not have zero in D for each positive integer j. But the family $\mathcal{F}$ is not normal at $z = 0$. This shows that Theorem 5 is not valid if $h(z)$ is a meromorphic function in D.

Example 3 Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$, let $h(z) = 1$ and let

$$\mathcal{F} = \left\{ f_j(z) = jz^{k-1} \mid j = 1, 2, \ldots \right\}.$$

Then $f_j^n(z) f_j^{(k)}(z) - h(z)$ does not have zero in D for each positive integer j. But the family $\mathcal{F}$ is not normal at $z = 0$. This shows that the condition "all zeros of f have multiplicity at least $k + m$ " in Theorem 5 is best.

Example 4 Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Let $h(z) = 1$ and

$$\mathcal{F} = \left\{ f_j(z) = jz \mid j = 1, 2, \ldots \right\}.$$

Obviously, $f_j^2(z) f_j'(z) - h(z) = j^3 z^2 - 1$ have exactly two distinct zeros in D for each positive integer j. But the family $\mathcal{F}$ is not normal at $z = 0$. This shows that the condition "$f^n(z) f^{(k)}(z) - h(z)$ has at most one distinct zero" in Theorem 5 is necessary.

2 Some Lemmas

For the proofs of our theorems, we require the following results.

Lemma 1 [12,17] *Let $\mathcal{F}$ be a family of meromorphic functions in the unit disk Δ such that all zeros of functions in $\mathcal{F}$ have multiplicity $\geq l$. Let α be a real number satisfying $-l < \alpha < 1$. Then $\mathcal{F}$ is not normal in any neighborhood of $z_0 \in \Delta$ if and only if there exist*

(a) *points $z_j \in \Delta$, $z_j \to z_0$;*
(b) *functions $f_j \in \mathcal{F}$; and*
(c) *positive numbers $\rho_j \to 0$*

such that $g_j(\xi) = \rho_j^{\alpha} f_j(z_j + \rho_j \xi) \Rightarrow g(\xi)$ spherically uniformly on compact subsets of $\mathbb{C}$, where $g(\xi)$ is a non-constant meromorphic function in $\mathbb{C}$ satisfying that all zeros of g have multiplicity at least l.

Lemma 2 [15] *Let f_1 and f_2 be two non-constant meromorphic functions in $\mathbb{C}$, then*

$$N(r, f_1 f_2) - N\left(r, \frac{1}{f_1 f_2}\right) = N(r, f_1) + N(r, f_2) - N\left(r, \frac{1}{f_1}\right) - N\left(r, \frac{1}{f_2}\right).$$

The following Lemma was proved by Zhang and Li [19] when f is a transcendental meromorphic function, and by Meng and Hu [9] when f is a rational function.

Lemma 3 *Let $n(\geq 2)$, $k(\geq 1)$ be three integers, let $a \neq 0$ be a finite complex number, and let $f(z)$ be a non-constant meromorphic in $\mathbb{C}$ with all zeros that have multiplicity at least k. Then $f^n(z)f^{(k)}(z) - a$ have at least two distinct zeros.*

Lemma 4 *Let $n(\geq 1)$, $k(\geq 1)$, $M(\geq 1)$ be three integers, let $p(z)$ be a polynomial with* $\deg p = M$, *and let $f(z)$ be a non-constant rational function in $\mathbb{C}$ with $f(z) \neq 0$. Then $f^n(z)f^{(k)}(z) - p(z)$ has at least $n + k + 1$ distinct zeros.*

The proof of Lemma 4 is almost the same with Chang [3] and Lemma 11 in Deng etc. [5], we omit the detail.

Lemma 5 *Let $n(\geq 2)$, $k(\geq 1)$, $m(\geq 1)$ be three integers, let $p(z)$ be a polynomial with* $\deg p = m$, *and let $f(z)$ be a non-constant meromorphic in $\mathbb{C}$ with all zeros that have multiplicity at least $k + m$. Then $f^n(z)f^{(k)}(z) - p(z)$ has at least two distinct zeros.*

Proof Set

$$\frac{1}{f^{n+1}} = \frac{f^n f^{(k)}}{p f^{n+1}} - \frac{p[f^n f^{(k)}]' - p' f^n f^{(k)}}{p f^{n+1}} \frac{f^n f^{(k)} - p}{p[f^n f^{(k)}]' - p' f^n f^{(k)}}.$$

Then by $m(r, \frac{f^{(i)}}{f}) = S(r, f)(i \geq 1)$, $m(r, p) = m \log r + O(1)$, $m\big(r, \frac{1}{p}\big) = O(1)$, Lemma 2 and Nevanlinna's elementary theory, we get

$$
\begin{aligned}
(n+1)\, m\left(r, \frac{1}{f}\right) &\leq m\left(r, \frac{f^n f^{(k)}}{p f^{n+1}}\right) + m\left(r, \frac{p\left(f^n f^{(k)}\right)' - p' f^n f^{(k)}}{p f^{n+1}}\right) \\
&\quad + m\left(r, \frac{f^n f^{(k)} - p}{p\left[f^n f^{(k)}\right]' - p' f^n f^{(k)}}\right) + S(r, f) \\
&\leq T\left(r, \frac{f^n f^{(k)} - p}{p\left[f^n f^{(k)}\right]' - p' f^n f^{(k)}}\right) \\
&\quad - N\left(r, \frac{f^n f^{(k)} - p}{p\left[f^n f^{(k)}\right]' - p' f^n f^{(k)}}\right) + S(r, f) \\
&= m\left(r, \frac{p\left[f^n f^{(k)}\right]' - p' f^n f^{(k)}}{f^n f^{(k)} - p}\right) \\
&\quad + N\left(r, \frac{p\left[f^n f^{(k)}\right]' - p' f^n f^{(k)}}{f^n f^{(k)} - p}\right) \\
&\quad - N\left(r, \frac{f^n f^{(k)} - p}{p\left[f^n f^{(k)}\right]' - p' f^n f^{(k)}}\right) + S(r, f) \\
&= m\left(r, \frac{p\left[\frac{f^n f^{(k)}}{p} - 1\right]'}{\frac{f^n f^{(k)}}{p} - 1}\right) \\
&\quad + N\left(r, p\left[f^n f^{(k)}\right]' - p' f^n f^{(k)}\right) + N\left(r, \frac{1}{f^n f^{(k)} - p}\right) \\
&\quad - N\left(r, \frac{1}{p\left[f^n f^{(k)}\right]' - p' f^n f^{(k)}}\right) \\
&\quad - N\left(r, f^n f^{(k)} - p\right) + S(r, f) \\
&\leq \overline{N}(r, f) + N\left(r, \frac{1}{f^n f^{(k)} - p}\right) \\
&\quad - N\left(r, \frac{1}{p\left[f^n f^{(k)}\right]' - p' f^n f^{(k)}}\right) + m \log r + S(r, f).
\end{aligned}
$$

Let z_1 is a zero of f with multiplicity $l_1 \geq k+m$. Then z_1 is a zero of $p[f^n f^{(k)}]' - p' f^n f^{(k)}$ with multiplicity at least $(n+1)l_1 - k - 1$.

Let z_2 is a zero of $f^n f^{(k)} - p$ with multiplicity l_2. Obviously, we have

$$p[f^n f^{(k)}]' - p' f^n f^{(k)} = p[f^n f^{(k)} - p]' - p'[f^n f^{(k)} - p].$$

Then z_2 is a zero of $p[f^n f^{(k)}]' - p' f^n f^{(k)}$ with multiplicity at least $l_2 - 1$.

Hence, we have

$$\begin{aligned}(n+1)\,T\,(r,f) &\le \overline{N}\,(r,f) + (n+1)\,N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^n f^{(k)} - p}\right)\\ &\quad + m\log r - N\left(r,\frac{1}{p\left[f^n f^{(k)}\right]' - p' f^n f^{(k)}}\right) + S\,(r,f)\\ &\le \overline{N}\,(r,f) + (k+1)\,\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^n f^{(k)} - p}\right)\\ &\quad + m\log r + S\,(r,f)\\ &\le \overline{N}\,(r,f) + \frac{k+1}{k+m}N\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^n f^{(k)} - p}\right)\\ &\quad + m\log r + S\,(r,f)\,. \end{aligned} \tag{2.1}$$

Suppose that $f^n(z)f^{(k)}(z) - p(z)$ has at most one distinct zero.

Next we consider two cases.

Case 1 $m \ge 2$. Then by (2.1), we have

$$T(r,f) < \left(n - \frac{k+1}{k+m}\right)T(r,f) \le (m+1)\log r + S(r,f).$$

Thus, f is a rational function with $\deg f < m+1$. Since all zeros of f have multiplicity at least $k+m \ge 1+m$, we deduce that $f(z) \ne 0$. Then by Lemma 4, we obtain that $f^n(z)f^{(k)}(z) - p(z)$ has at least $n+k+1 > 2$ distinct zeros, a contradiction.

Case 2 $m = 1$.

If $f^n(z)f^{(k)}(z) - p(z) \ne 0$, then by (2.1), we get $T(r,f) \le \log r + S(r,f)$. It follows that f is a rational function with $\deg f \le 1$. We deduce that $f(z) \ne 0$, since all zeros of f have multiplicity at least $k+m \ge 2$, Then by Lemma 4, we get $f^n(z)f^{(k)}(z) - p(z)$ has at least $n+k+1 > 2$ distinct zeros, a contradiction.

Thus $f^n(z)f^{(k)}(z) - p(z)$ has exactly one distinct zero. By (2.1), we have

$$nT(r,f) \le 2\log r + \overline{N}(r,f) + S(r,f). \tag{2.2}$$

If $n \ge 3$, by (2.2), we obtain $T(r,f) \le \log r + S(r,f)$. It follows that f is a rational function with $\deg f \le 1$. Since all zeros of f have multiplicity at least $k+m \ge 2$, we obtain $f(z) \ne 0$, then by Lemma 4, we get $f^n(z)f^{(k)}(z) - p(z)$ has at least $n+k+1 > 2$ distinct zeros, a contradiction.

Thus $n = 2$. By (2.2) again, we get $T(r,f) \le 2\log r + S(r,f)$. It follows that f is a rational function with $\deg f \le 2$. If $k \ge 2$, since all zeros of f have multiplicity at least $k+m \ge 3$, we get $f(z) \ne 0$, then by Lemma 4, we get a contradiction. Hence $k = 1$, then f has one zero with multiplicity 2 at most. If f has no zero, then by

Lemma 4, a contradiction. Thus, $f(z)$ has exactly one distinct zero with multiplicity 2, and of the following forms:

$$A_1\colon f(z) = a(z-\alpha)^2; \quad A_2\colon f(z) = a\frac{(z-\alpha)^2}{z-\beta};$$
$$A_3\colon f(z) = \frac{a(z-\alpha)^2}{(z-\beta_1)(z-\beta_2)}; \quad A_4\colon f(z) = a\frac{(z-\alpha)^2}{(z-\beta)^2}.$$

If $f(z)$ has the form A_1 or A_2 or A_4, we have $\overline{N}(r, f) \le \log r = 1/2T(r, f) + O(1)$. Then by (2.2), we get $T(r, f) \le 4/3 \log r + S(r, f)$, this contradicts with $T(r, f) = 2\log r + O(1)$.

Then

$$f(z) = \frac{a(z-\alpha)^2}{(z-\beta_1)(z-\beta_2)}. \tag{2.3}$$

It follows from (2.3) that

$$f'(z) = \frac{a(z-\alpha)\left[(2\alpha-\beta_1-\beta_2)z + 2\beta_1\beta_2 - \alpha(\beta_1+\beta_2)\right]}{(z-\beta_1)^2(z-\beta_2)^2}. \tag{2.4}$$

By (2.3) and (2.4), we get

$$f^2(z)f'(z) = \frac{a^3(z-\alpha)^5\left[(2\alpha-\beta_1-\beta_2)z + 2\beta_1\beta_2 - \alpha(\beta_1+\beta_2)\right]}{(z-\beta_1)^4(z-\beta_2)^4}. \tag{2.5}$$

Since $\deg p = m = 1$, we may set $p(z) = b(z-z_0)$, where $b \ne 0$ is a constant. Since $f^n(z)f^{(k)}(z) - p(z)$ has exactly one distinct zero, by (2.5), we may set

$$f^2(z)f'(z) = b(z-z_0) - \frac{b(z-w)^9}{(z-\beta_1)^4(z-\beta_2)^4}, \tag{2.6}$$

where $w \ne \alpha$. Otherwise, if $w = \alpha$, then by (2.5), we get α is a zero of $(f^2(z)f'(z))''$ with multiplicity 3. But from (2.6), we get α is a zero of $(f^2(z)f'(z))''$ with multiplicity 7, a contradiction.

Differentiating (2.5) two times, we obtain,

$$[f^2(z)f'(z)]'' = \frac{(z-\alpha)^3 g_1(z)}{(z-\beta_1)^6(z-\beta_2)^6}, \tag{2.7}$$

where $g_1(z)$ is a polynomial with $\deg g_1 \le 5$.

On the other hand, differentiating (2.6) two times, we obtain,

$$[f^2(z)f'(z)]'' = \frac{(z-w)^7 g_2(z)}{(z-\beta_1)^6(z-\beta_2)^6}, \tag{2.8}$$

where $g_2(z)$ is a polynomial with $\deg g_2 \le 4$.

From (2.7)–(2.8), and $w \ne \alpha$, we get $7 \le \deg g_1 \le 5$, a contradiction.

This completes the proof of Lemma 5.

Lemma 6 *Let $n(\geq 2)$, $k(\geq 1)$ be three integers, and let $\{f_j\}$ be a sequence of meromorphic functions in domain D, $\{h_j(z)\}$ be a sequence of holomorphic functions in D such that $h_j(z) \Rightarrow h(z)$, where $h(z) \neq 0$ be a holomorphic function. If, for each $j \in N^+$, all zeros of function $f_j(z)$ have multiplicity at least k, and $f_j^n(z) f_j^{(k)}(z) - h_j(z)$ has at most one distinct zero in D, then $\{f_j\}$ is normal in D.*

Proof Suppose that $\{f_j\}$ is not normal at $z_0 \in D$. By Lemma 1, there exists a sequence z_j of complex numbers $z_j \to z_0$, a sequence ρ_j of positive numbers $\rho_j \to 0$, and a subsequence of $\{f_j\}$ (we may still denote by $\{f_j\}$) such that

$$g_j(\xi) = \frac{f_j(z_j + \rho_j \xi)}{\rho_j^{\frac{k}{n+1}}} \Rightarrow g(\xi)$$

locally uniformly on compact subsets of $\mathbb{C}$, where $g(\xi)$ is a non-constant meromorphic function in $\mathbb{C}$. By Hurwitz's theorem, all zeros of $g(\xi)$have multiplicity at least k. Then, we have

$$\begin{aligned} g_j^n(\xi) g_j^{(k)}(\xi) - h_j(z_j + \rho_j \xi) &= f_j^n(z_j + \rho_j \xi) f_j^{(k)}(z_j + \rho_j \xi) - h_j(z_j + \rho_j \xi) \\ &\Rightarrow g^n(\xi) g^{(k)}(\xi) - h(z_0). \end{aligned}$$

for all $\xi \in \mathbb{C}/\{g^{-1}(\infty)\}$.

Obviously, $g^n(\xi) g^{(k)}(\xi) - h(z_0) \not\equiv 0$. Otherwise, suppose that

$$g^n(\xi) g^{(k)}(\xi) - h(z_0) \equiv 0, \tag{2.9}$$

then we have $g(\xi) \neq 0$ since $h(z_0) \neq 0$.

It follows from (2.9) that

$$\frac{1}{g^{n+1}(\xi)} \equiv \frac{g^{(k)}(\xi)}{h(z_0) g(\xi)}.$$

Then, we get

$$(n+1) m\left(r, \frac{1}{g}\right) = m\left(r, \frac{g^{(k)}}{h(z_0) g}\right) = S(r, g).$$

It follows that $T(r, g) = S(r, g)$ since $g \neq 0$. Hence g is a constant, a contradiction.

We claim that $g^n(\xi) g^{(k)}(\xi) - h(z_0)$ has at most one distinct zero. Otherwise, suppose that ξ_1, ξ_2 are two distinct zeros of $g^n(\xi) g^{(k)}(\xi) - h(z_0)$. We choose a positive number σ small enough such that $D_1 \cap D_2 = \emptyset$ and $g^n(\xi) g^{(k)}(\xi) - h(z_0)$ has no other zeros in $D_1 \bigcup D_2$ except for ξ_1 and ξ_2, where $D_1 = \{\xi \colon \mid \xi - \xi_1 \mid < \sigma\}$ and $D_2 = \{\xi \colon \mid \xi - \xi_2 \mid < \sigma\}$.

By Hurwitz's theorem, for sufficiently large j there exist points $\xi_{1,j} \to \xi_1$ and $\xi_{2,j} \to \xi_2$ such that

$$f_j^n(z_j + \rho_j\xi_{1,j})f_j^{(k)}(z_j + \rho_j\xi_{1,j}) - h_j(z_j + \rho_j\xi_{1,j}) = 0;$$
$$f_j^n(z_j + \rho_j\xi_{2,j})f_j^{(k)}(z_j + \rho_j\xi_{2,j}) - h_j(z_j + \rho_j\xi_{2,j}) = 0.$$

By the assumption in Lemma 6, $f_j^n f_j^{(k)}(z) - h_j(z)$ has at most one zero in D, it follows that $z_j + \rho_j\xi_{1,j} = z_j + \rho_j\xi_{2,j}$, that is $\xi_{1,j} = \xi_{2,j} = (z_0 - z_j)/\rho_j$, which contradicts with the facts $D_1 \cap D_2 = \emptyset$.

The claim is proved. On the other hand, it follows from Lemma 3 that $g^n(\xi)g^{(k)}(\xi) - h(z_0)$ has at least two distinct zeros, a contradiction. Thus $\{f_j\}$ is normal in D.

3 Proof of Theorems

Proof of Theorem 5 By Lemma 6, it is enough to prove that $\mathcal{F}$ is normal at the point z_0, where $h(z_0) = 0$. By making standard normalization, we may assume that $z_0 = 0$, and $h(z) = z^t b(z)$ where $1 \le t \le m$ is a positive integer, and $b(0) = 1$.

Suppose that $\mathcal{F}$ is not normal at $z_0 = 0$. By Lemma 1, there exists a sequence z_j of complex numbers $z_j \to 0$, a sequence ρ_j of positive numbers $\rho_j \to 0$, and a sequence of functions $\{f_j\} \subseteq \mathcal{F}$ such that

$$g_j(\xi) = \frac{f_j(z_j + \rho_j\xi)}{\rho_j^{\frac{k+t}{n+1}}} \Rightarrow g(\xi) \tag{3.1}$$

locally uniformly on compact subsets of $\mathbb{C}$, where $g(\xi)$ is a non-constant meromorphic function in $\mathbb{C}$. By Hurwitz's theorem, all zeros of $g(\xi)$ have multiplicity at least $k+m$. Next we consider two cases.

Case 1 $z_j/\rho_j \to \infty$. Set

$$F_j(\xi) = \frac{f_j(z_j + z_j\xi)}{z_j^{\frac{k+t}{n+1}}}.$$

Then, we have

$$F_j^n(\xi)F_j^{(k)}(\xi) - (1+\xi)^t b(z_j + z_j\xi) = \frac{f_j^n(z_j + z_j\xi)f_j^{(k)}(z_j + z_j\xi) - h(z_j + z_j\xi)}{z_j^t}.$$

As the same argument as in Lemma 6, we deduce that $F_j^n(\xi)F_j^{(k)}(\xi) - (1+\xi)^t b(z_j + z_j\xi)$ has at most one distinct zero in $\Delta = \{\xi : |\xi| < 1\}$.

Since all zeros of F_j have multiplicity at least $k+m$, and $(1+\xi)^t b(z_j + z_j\xi) \to (1+\xi)^t \ne 0$ when $\xi \in \Delta$. Then by Lemma 6, $\{F_j\}$ is normal in Δ.

So, there exists a subsequence of functions [we still denote as $F_j(\xi)$] and a function $F(\xi)$ (a meromorphic function or ∞), such that $F_j(\xi) \Rightarrow F(\xi)$.

If $F(0) \neq \infty$, then it follows from $k + m - 1 - \frac{k+t}{n+1} > 0$ that

$$g^{(k+m-1)}(\xi) = \lim_{j\to\infty} g_j^{(k+m-1)}(\xi) = \lim_{j\to\infty} \frac{f_j^{(k+m-1)}\left(z_j + \rho_j \xi\right)}{\rho_j^{\frac{k+t}{n+1}-(k+m-1)}}$$
$$= \lim_{j\to\infty} \left(\frac{\rho_j}{z_j}\right)^{k+m-1-\frac{k+t}{n+1}} F_j^{(k+m-1)}\left(\frac{\rho_j}{z_j}\xi\right) = 0,$$

for all $\xi \in \mathbb{C}/\{g^{-1}(\infty)\}$.

Thus we deduce that $g^{(k+m-1)} \equiv 0$. Hence g is a polynomial of degree at most $k+m-1$. Since all zeros of g have multiplicity at least $k+m$, it follows that $g(\xi)$ is a constant, a contradiction.

If $F(0) = \infty$, then by

$$\frac{1}{F_j\left(\frac{\rho_j}{z_j}\xi\right)} = \frac{z_j^{\frac{k+t}{n+1}}}{f_j\left(z_j + \rho_j\xi\right)} \to \frac{1}{F(0)} = 0,$$

when $\xi \in \mathbb{C}/\{g^{-1}(0)\}$, we obtain that,

$$\frac{1}{g(\xi)} = \lim_{j\to\infty} \frac{\rho_j^{\frac{k+t}{n+1}}}{f_j\left(z_j + \rho_j\xi\right)} = \lim_{j\to\infty} \left(\frac{\rho_j}{z_j}\right)^{\frac{k+t}{n+1}} \frac{z_j^{\frac{k+t}{n+1}}}{f_j\left(z_j + \rho_j\xi\right)} = 0.$$

Thus $g(\xi) \equiv \infty$, which contradicts that $g(\xi)$ is a non-constant meromorphic function.

Case 2 $z_j/\rho_j \to \alpha$, where α is a finite complex number. Then by (3.1), we have

$$g_j^n(\xi)\, g_j^{(k)}(\xi) - \left(\xi + \frac{z_j}{\rho_j}\right)^t b\left(z_j + \rho_j\xi\right)$$
$$= \frac{f_j^n\left(z_j + \rho_j\xi\right) f_j^{(k)}\left(z_j + \rho_j\xi\right) - h\left(z_j + \rho_j\xi\right)}{\rho_j^t}$$
$$\Rightarrow g^n(\xi)\, g^{(k)}(\xi) - (\xi + \alpha)^t$$

for all $\xi \in \mathbb{C}/\{g^{-1}(\infty)\}$.

Since for sufficiently large j, $f_j^n(z_j + \rho_j\xi) f_j^{(k)}(z_j + \rho_j\xi) - h(z_j + \rho_j\xi)$ has one distinct zero, it follows from the proof of Lemma 6 that $g^n(\xi) g^{(k)}(\xi) - (\xi + \alpha)^t$ has at most one distinct zero.

But from Lemma 5, $g^n(\xi) g^{(k)}(\xi) - (\xi + \alpha)^t$ have at least two distinct zeros. Hence $g(\xi)$ is a constant, a contradiction.

This completes the proof of Theorem 5.

Proof of Theorem 4 Let $z_0 \in D$. We show that $\mathcal{F}$ is normal at z_0. Let $f \in \mathcal{F}$.

We consider two cases.

Case 1 $f^n(z_0)f^{(k)}(z_0) \neq h(z_0)$. Then there exists a disk $D_\delta(z_0) = \{z\colon \mid z - z_0 \mid < \delta\}$ such that $f^n(z)f^{(k)}(z) \neq h(z)$ in $D_\delta(z_0)$. Since for each pair of functions $(f, g) \in \mathcal{F}$, $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}(z)$ share $h(z)$ in D. Thus, for every $g \in \mathcal{F}$, $g^n(z)g^{(k)} \neq h(z)$ in $D_\delta(z_0)$. By Theorem 5, $\mathcal{F}$ is normal in $D_\delta(z_0)$. Hence $\mathcal{F}$ is normal at z_0.

Case 2 $f^n(z_0)f^{(k)}(z_0) = h(z_0)$. Then there exists a disk $D_\delta(z_0) = \{z\colon \mid z - z_0 \mid < \delta\}$ such that $f^n(z)f^{(k)}(z) \neq h(z)$ in $D^0_\delta(z_0)$. Since for each pair of functions $(f, g) \in \mathcal{F}$, $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}$ share $h(z)$ in D. Thus, for every $g \in \mathcal{F}$, $g^n(z)g^{(k)} \neq h(z)$ in $D^0_\delta(z_0)$ and $g^n(z_0)g^{(k)}(z_0) = h(z_0)$. So, $g^n(z)g^{(k)} - h(z)$ have only distinct zero in $D_\delta(z_0)$. By Theorem 5, $\mathcal{F}$ is normal in $D_\delta(z_0)$. Hence $\mathcal{F}$ is normal at z_0.

This completes the proof of Theorem 4.

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