Bull. Malays. Math. Sci. Soc. (2019) 42:697–706
https://doi.org/10.1007/s40840-017-0510-z

BULLETIN of the MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Generalized Symmetries and Recursive Operators of Some Diffusive Equations

Sameerah Jamal[1] · **A. Mathebula**[1]

Received: 22 February 2017 / Revised: 22 April 2017 / Published online: 20 May 2017
© Malaysian Mathematical Sciences Society and Penerbit Universiti Sains Malaysia 2017

Abstract This paper considers different routes to generalized symmetries for some ecological equations that arise in spatial theory. Two primary methods for the derivation of generalized symmetries are the standard Lie invariance condition with vector fields dependent on derivatives and, secondly, a recursive operator. The former is less efficient especially if it includes derivatives that become increasingly higher in order, and this necessarily complicates the nature of the computations. The latter involves a nontrivial analysis to define a recursion operator, if one exists, but is successful in providing higher-order analogs of the equation or equivalently, higher-order symmetries. A linear Kierstead–Slobodkin and Skellam model is shown to possess a recursion operator that renders the equation completely integrable, by verifying the presence of infinitely many higher-order symmetries. Moreover, we apply the scheme of the characteristic approach to establish nontrivial conserved vectors from multipliers $\Lambda(t, x, u, u_x, u_t)$, that are analogous to integrating factors.

Keywords Diffusion equations · Lie symmetries · Higher-order symmetries · Recursion operators

Mathematics Subject Classification 37L20 · 35K57 · 70G65 · 58J72

Communicated by Shangjiang Guo.

SJ would like to acknowledge the financial support from the National Research Foundation of South Africa with Grant Number 99279 and AM acknowledges the SANHARP programme for financial support.

✉ Sameerah Jamal
sameerah.jamal@wits.ac.za

[1] School of Mathematics and Centre for Differential Equations, Continuum Mechanics and Applications, University of the Witwatersrand, Johannesburg, South Africa

1 Introduction

In 1977, Olver [1] pioneered a study of recursion operators for several evolution equations that possess infinitely many symmetries. One such discussion revolved around the higher-order analogs of the KdV equation

$$u_t = u_{xxx} + uu_x,$$

which could be reinterpreted as "higher-order symmetries." The operator itself was due to Lenard [2]

$$\mathcal{R} = D + \frac{1}{2}u + \frac{1}{2}u_x D^{-1},$$

where D denotes the total derivative with respect to x unless stated otherwise, namely $D = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + u_{xxx}\partial_{u_{xx}} + \cdots$. Inspired by these works, we obtain higher-order analogs of a special diffusion equation by formulating a recursion operator.

Symmetry methods feature in the analysis of phenomena that arise in physical and mathematical fields. For example, several studies were devoted to fundamental equations in physics, general relativity, biology and financial mathematics, see [3–6] and references therein. These days the determination of point symmetries can be done mechanically by a number of symbolic computer programs, e.g., to list just a few: Macsyma [7], Mathematica [8,9] and Maple have an automated routine as well. We point out here that point symmetries depend only on the independent variables ($\underline{x^k} = (x^1, \ldots x^p)$) of a system and its associated dependent variables only ($\underline{u^l} = (u^1, \ldots u^q)$) and excludes derivatives of the dependent variable, that is $X = \xi^k(\underline{x}, \underline{u}) + \eta^l(\underline{x}, \underline{u})$, while generalized symmetries, also commonly referred to as Lie Bäcklund symmetries, include derivatives of the dependent variable. The aim of this paper is also to study some properties surrounding generalized symmetries that arise from techniques apart from recursions.

At this point it is appropriate to summarize several strategies to construct higher-order symmetries: a) The standard way of computing Lie point symmetries can be extended to include derivative dependent terms, thereby resulting in higher-order Lie Bäcklund symmetries. b) Implement the multiplier or characteristic approach [10] to construct multipliers (integrating factors) of an equation. In this case the multiplier itself is a symmetry of the underlying equation and can be defined in such a way as to consist of higher-order terms. The advantage of this method is that every multiplier provides a conserved quantity. Obviously such a conservation law is not Noetherian and does not stem from a variational principle. For further discussion of this method we refer the reader to recent literature on the subject [11–15]. Last but not least, (c) a recursion operator, if it exits, generates infinite hierarchies of higher-order symmetries. Historically, recursive operators were found by guesswork [1], although it can debated that this is simply a practice of searching for a pattern in a perceptive fashion. By far, (c) is exceedingly sought after, more useful and increasingly more efficient.

An important problem in analytical theories is the integrability of partial differential equations (PDEs). From the mathematical point of view their importance is due to the following circumstances. Complete integrability provides important information about the nature of the solutions of PDEs. Once ascertained that a PDE is completely integrable, numerous methods then exist for solving, for instance the method of inverse scattering transform. These reasons have motivated many investigations regarding integrability, including this work.

2 Some Diffusive Equations

We now mention some important PDEs that have received much attention in the literature and also the equations that underpin this study. Reaction–diffusion problems have been studied extensively in fundamental areas of engineering, mathematics, biology, population ecology and many others. Numerous models exist to explain spatial theory and population dynamics alone. The expansion of muskrat populations in Europe instigated mathematical attempts to model the problem which lead to reaction–diffusion equations [16]. A typical reaction–diffusion model, with u as the population density at spatial coordinates and time t, is

$$u_t = (\mathcal{C}(u)u_x)_x + (\mathcal{C}(u)u_y)_y + f(u), \tag{1}$$

where $\mathcal{C}(u)$ describes the diffusive movement, while reaction and population dynamics is represented by $f(u)$. Emphasis is also placed on reaction–diffusion models in which a combination of population dynamics such as movement and multi-species interactions is considered [17]. The notion of a correlated random walk [17] by species gave rise to a telegraph PDE model [18]:

$$u_t = \frac{s^2}{2\lambda}(u_{xx} + u_{yy}) - \frac{1}{2\lambda}u_{tt}, \tag{2}$$

where 2λs a measure of the correlation between directions of travel from one step to the next and s is the velocity. The Fisher model [19] is arguably the most important reaction–diffusion model, which represents Brownian random dispersal and logistic population growth:

$$u_t = ru\left(1 - \frac{u}{K}\right) + B(u_{xx} + u_{yy}), \tag{3}$$

where r is the population's growth rate, B is the diffusion constant that measures the rate of dispersal and the carrying capacity is represented by K. Prior to these models, early PDE models of population ecology such as

$$u_t = B(u_{xx} + u_{yy}), \tag{4}$$

were used to analyze the dispersion of numerous organisms in mark-recapture studies (e.g., [20]), whereby the simplistic conjecture is assumed that expects organisms to have Brownian movement, the rate of which is invariant in space and time [17,21]. To build on this model one considers when organisms adapt to external stimuli or are moved by rainwater or airstream currents, and therefore, convection or drift terms are added to (4), which leads to the model [22]

$$u_t = B(u_{xx} + u_{yy}) - w_x u_x - w_y u_y, \tag{5}$$

where w_x and w_y are convection velocities. The Lie point symmetries of (5) are easily obtainable and are surprisingly many, viz. we find a ten-dimensional algebra:

$$\begin{aligned}
\Gamma_1 &= \partial_y,\ \Gamma_2 = \partial_x,\ \Gamma_3 = \partial_t,\ \Gamma_4 = u\partial_u,\\
\Gamma_5 &= t\partial_x + \frac{1}{2}\frac{u\left(tw_y - x\right)}{B}\partial_u,\ X_6 = t\partial_y + \frac{1}{2}\frac{u\left(tw_x - y\right)}{B}\partial_u,\\
\Gamma_7 &= y\partial_x - x\partial_y - \frac{1}{2}\frac{u\left(w_x x - w_y y\right)}{B}\partial_u,\\
\Gamma_8 &= \frac{1}{2}x\partial_x + \frac{1}{2}y\partial_y + t\partial_t - \frac{1}{4}\frac{u\left(\left(w_x^2 + w_y^2\right)t - w_x y - w_y x\right)}{B}\partial_u,\\
\Gamma_9 &= \frac{1}{2}tx\partial_x + \frac{1}{2}ty\partial_y + \frac{1}{2}t^2\partial_t\\
&\quad -\frac{1}{8}\frac{u\left(t^2w_x^2 + t^2w_y^2 - 2tw_x y - 2tw_y x + 4Bt + x^2 + y^2\right)}{B}\partial_u,\\
\Gamma_\infty &= F(t,x,y)\partial_u,
\end{aligned} \tag{6}$$

where $F(t, x, y)$ is the infinite symmetry that is the infinite-dimensional abelian subalgebra of solutions which is a solution of Eq. (5). We remark that since Eq. (5) is linear, it naturally admits the linear symmetry Γ_4 and the infinite symmetry Γ_∞ [23]. Several PDEs have been designed to model interactions between conspecifics, whereby attraction or repellence between species leads to a simple diffusion equation being replaced by a biased nonlinear diffusive equation [17,24]:

$$u_t = Bu_{xx} + (kuu_x)_x, \tag{7}$$

where again $u(t, x)$ is the density of population, and k is a measure of the tendency to travel away from conspecifics ($k > 0$) and is a measure of the tendency to travel near conspecifics ($k < 0$). Such a model admits a four-dimensional Lie algebra of point symmetries,

$$\Sigma_1 = \partial_t,\ \Sigma_2 = \partial_x,\ \Sigma_3 = \tfrac{1}{2}x\partial_x + t\partial_t,\ \Sigma_4 = \tfrac{1}{2}kx\partial_x + (ku + B)\,\partial_u. \tag{8}$$

In the next section we consider the higher-order properties of (7) using the idea of multipliers and Lie Bäcklund symmetries. Lastly in Sect. 4 we find and apply a recursion operator to a second ecological equation to prove that it is completely integrable.

3 Generalized Symmetries of a Nonlinear Diffusion Equation

If a vector field with components ξ^j, η^i relies on the following derivatives, where $u^{(s)}$ represents the s^{th} derivative of u with respect to x

$$
\begin{aligned}
(\bar{x})^j &= x^j + \epsilon \xi^j \left(x, u^{(s)}\right) + \mathcal{O}(\epsilon^2),\ j = 1, \ldots, n,\\
(\bar{x})^i &= u^i + \epsilon \eta^i \left(x, u^{(s)}\right) + \mathcal{O}(\epsilon^2),\ i = 1, \ldots, m
\end{aligned}
\tag{9}
$$

then the resulting symmetries are said to be of higher order. Without loss of generality, the symmetry generators in a higher-order context are usually expressed in the evolutionary or characteristic form, videlicet

$$
\bar{X} = \phi^i(x, u^{(s)})\partial_{u^i},\ \text{assuming that}\ \xi^j = 0. \tag{10}
$$

Hence, suppose we restrict our vector field to the form

$$
\bar{X} = \phi(x, t, u, u_x, u_{xx}), \tag{11}
$$

which is to be the higher-order symmetry generator of Eq. (7). The infinitesimal criterion of invariance is given by

$$
\bar{X}\left[\text{Eq. (7)}\right] |_{\text{Eq.}(7)} = 0, \tag{12}
$$

and therefore, higher-order symmetries of our equation are given by the determining equations, for the simplest scenario $B = k = 1$,

$$
\begin{aligned}
&\phi_{u_{xx},u_{xx}} + \phi_{u_{xx},u_{xx}u} = 0,\\
&2\phi_{u_{xx},u_{u_x}} u_{xx} + 2\phi_{u_{xx},u} u u_x + 2\phi_{u_{xx},u_x} u u_{xx} - 2\phi_{u_{xx}} u_x + 2\phi_{x,u_{xx}} u\\
&\quad + 2\phi_{x,uxx} + 2\phi_{u_{xx},u} u_x = 0,\\
&2\phi_{u_x,u} u u_x u_{xx} - \phi_{u_x} u_x u_{xx} + \phi_{u,u} u_x^2 - 3\phi_{u_{xx}} u_{xx}^2 + \phi_{x,x} + \phi u_{xx}\\
&+ 2\phi_{x,u} u u_x + \phi_{u,u} u u_x^2 + \phi_{u_x,u_x} u u_{xx}^2\\
&\quad + 2\phi_{u_x,u} u_x u_{xx} + 2\phi_x u_x + 2\phi_{x,u_x} u u_{xx} + \phi_{x,x} u + \phi_u u_x^2 - \phi_t + 2\phi_{x,u} u_x +\\
&\phi_{u_x,u_x} u_{xx}^2 + 2\phi_{x,u_x} u_{xx} = 0.
\end{aligned}
\tag{13}
$$

The solution of the system (13) gives the following four higher-order symmetries:

$$
\begin{aligned}
\bar{X}_1 &= u_x \partial_u,\\
\bar{X}_2 &= (u_{xx} - 2u - 2)\partial_u,\\
\bar{X}_3 &= (u u_{xx} + u_x^2 + u_{xx})\partial_u,\\
\bar{X}_4 &= (u u_{xx} t + u_x^2 t + u + 1)\partial_u.
\end{aligned}
\tag{14}
$$

As an alternative, we now utilize the characteristic approach to determine whether such an approach yields a larger collection of higher-order symmetries. Note that the evolutionary symmetries $\bar{X}_{1-4}$ do not normally yield conservation laws. To this end an exploration of multipliers has the benefit of leading to conserved vectors.

3.1 The Characteristic Approach

Information about conservation laws is important to any symmetry study of a PDE. These conservation laws are of paramount importance, and it is well known that they show a vital part in mathematical physics as they define critical physical properties of the modeled process. Conservation laws are also applicable when eliminating numerical errors of PDEs [25]. Once a multiplier is found, conserved vectors may be derived systematically by using a homotopy operator (see details and references in [26,27]); however, in some cases it is simple to construct the conserved vectors by elementary manipulations. The explicit relation between multipliers and conserved densities is summarized by Anco and Bluman [28]. To apply this approach consider a multiplier that contains the dependent variable, the independent variables and derivatives of dependent variables up to some fixed order, i.e., let $\Lambda = \Lambda(t, x, u, u_x, u_t)$ of Eq. (7) have the property that

$$\Lambda \left[\text{Eq. (7)}\right] = D_x T^x + D_t T^t, \tag{15}$$

for all functions $u(t, x)$, where the total derivative operative is defined as

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots . \tag{16}$$

The right-hand side of (15) is a divergence expression, and the conserved vector $T = (T^x, T^t)$ has components $T^j \ (j = x, t)$. The determining equations for the multipliers Λ are obtained from the expressions

$$\frac{\delta}{\delta u}\left[\Lambda(\text{Eq. 7})\right] = 0, \tag{17}$$

where $\frac{\delta}{\delta u}$ are the Euler–Lagrange operators given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}}, \tag{18}$$

which annihilate divergence expressions. Solving Eq. (17) yields,

$$\begin{aligned} &u_x^2(ku + B)\Lambda_{uu} + 2u_x(ku + B)\Lambda_{ux} + (ku + B)\Lambda_{xx} \\ &+ \left(ku_x^2 + 2u_{xx}(ku + B)\right) \Lambda_u + \Lambda_t = 0. \end{aligned} \tag{19}$$

Solving for Λ, we find the solution (C_1, C_2 are arbitrary constants):

$$\Lambda(t, x, u, u_x, u_t) = C_1 x + C_2. \tag{20}$$

Hence, the solutions of the determining system are the multipliers,

$$\Lambda_1 = 1 \quad \text{and} \quad \Lambda_2 = x,$$

which we note are not higher order in the end, but each yields a nontrivial generalized biased diffusion conservation law

$$T_1^t = -u \quad \text{and} \quad T_1^x = (B + ku)u_x,$$

and

$$T_2^t = -xu \quad \text{and} \quad T_2^x = -\frac{1}{2}ku^2 + Bxu_x + u(kxu_x - B),$$

respectively. Next we shift our focus to a special evolution equation for which we prove that it possesses an infinite hierarchy of symmetries.

4 The Kierstead–Slobodkin and Skellam Problem

In this section we consider the linear Kierstead–Slobodkin [29] and Skellam [16] problem, commonly known as the KiSS model, and extend our study to the nonlinear model. Such a model is also described as the basic critical patch equation. Determination of critical patch size to guarantee the sustenance of the population is an important study. The rate at which a population exits the area, the population dynamics in the patch, the spatial area and the region surrounding the patch are some of the factors that influence the critical patch size.

A generalized $(1+2)$ KiSS model is expressed as

$$u_t = B\left(u_{xx} + u_{yy}\right) + rF(u)^{\rho}. \tag{21}$$

Here, $\rho > 0$ is the critical exponent parameter that determines whether the model is linear ($\rho = 1$) or nonlinear ($\rho \geq 2$), and r is the growth rate. An investigation of several special cases that produce interesting symmetries is presented in Table 1.

As mentioned before, the linear symmetry and the infinite symmetry are added to Table 1 whenever Eq. (21) is a linear model. Our interest lies in the higher-order symmetries, and it turns out that we are able to find a infinite sequence of symmetries for this particular model.

4.1 Higher-Order Symmetries Via Recursion Operators

In this subsection we are concerned about determining an infinite series of higher-order symmetries by defining a recursion operator. A study by [30] studied higher-order symmetries as the fundamental feature of completely integrable equations and proposed that an equation is completely integrable if and only if it admits infinitely many time-independent Lie Bäcklund symmetries. Motivated by this consideration, we define a recursion operator for the diffusive KiSS model to prove its complete integrability. In practice, and for the sake of simplicity, we study the PDE in $(1+1)$ dimensions. We stipulate the form of the model by the selection of the free function $F(u) = u$ and assume that all parameters are nonzero, specifically $B = r = \rho = 1$.

For the convenience of the reader we present the basic theoretical framework of recursive operators. For a polynomial system that arises from evolution equations

$$u_t = A(u^{(s)}), \tag{22}$$

Table 1 Classification of Lie point symmetries of model (21)

Case	Lie symmetry	ρ	$F(u)$
I	$X_1 = \partial_t, X_2 = \partial_x,\ X_3 = \partial_y,\ X_4 = u\partial_u,$		
	$X_5 = y\partial_x - x\partial_y,$	1	u
	$X_6 = t\partial_x - \frac{1}{2}\frac{ux}{B}\partial_u,\ X_7 = t\partial_y - \frac{1}{2}\frac{uy}{B}\partial_u,$		
	$X_8 = t\partial_t + \frac{1}{2}x\partial_x + \frac{1}{2}y\partial_y + urt\partial_u,$		
	$X_9 = \frac{1}{2}t^2\partial_t + \frac{1}{2}xt\partial_x + \frac{1}{2}yt\partial_y +$		
	$\frac{1}{8}\frac{u\left(Brt^2 4 - 4Bt - x^2 - y^2\right)}{B}\partial_u$		
II	$X_1, X_2, X_3, X_5,$	n	u
	$X_{10} = t\partial_t + \frac{1}{2}x\partial_x + \frac{1}{2}y\partial_y - \frac{u}{n-1}\partial_u$	$(n \neq 1)$	
III	X_1, X_2	1	Arbitrary
IV	$X_1,\ X_2,$	1	e^{bu}
	$X_{11} = t\partial_t + \frac{x}{2}\partial_x - \frac{1}{b}\partial_u$		($b \neq 0$ is const.)
V	$X_1,\ X_2,\ X_4,$	–	0^a
	$X_{12} = t\partial_t + \frac{x}{2}\partial_x - 2Bt\partial_x + xu\partial_u$		
	$X_{13} = \frac{1}{2}t^2\partial_t + \frac{1}{2}tx\partial_x + \left(\frac{t}{4} - \frac{x^2}{8B}\right)u\partial_u$		

[a] This is equivalent to Eq. (4)

a higher-order symmetry $Y(u)$ leaves the above PDE invariant under the substitution $u \to u + \epsilon Y$ up to order ϵ, and must satisfy the relation [1]

$$D_t Y(u) = A'(u)[Y(u)],$$

where the right-hand side is equivalent to the adjoint Fréchet derivative

$$\frac{\partial}{\partial \epsilon} A\left(u + \epsilon Y\right)|_{\epsilon=0}\ .$$

A recursion operator, $\mathcal{R}$, links higher-order symmetries [1]

$$Y^{(p+q)} = \mathcal{R} Y^{(p)}, \quad p = 1, 2, \ldots \tag{23}$$

where $q = 1$ and $Y^{(p)}$ is the $p-$th higher-order symmetry.

To return to our model, we let

$$u_t = Y(u) = u_{xx} + u. \tag{24}$$

We define the recursion operator to be $\mathcal{R} = D$; therefore, the infinite series of generalized symmetries

$$Y^{(p)}(u) = \mathcal{R}^p Y,$$

can be written in evolution form $u_t = Y^{(p)}(u)$. The first few of these are

$$\begin{aligned} u_t &= Y^{(0)}(u) = u_{xx} + u, \\ u_t &= Y^{(1)}(u) = u_{xxx} + u_x, \\ u_t &= Y^{(2)}(u) = u_{xxxx} + u_{xx}, \text{ etc.,} \end{aligned} \tag{25}$$

which preserve the flows of the KiSS equation, and we conclude that we have infinite symmetries of the equation.

5 Comments and Conclusions

A recursion operator not only provides a connection between the generalized symmetries of an equation but is also an important tool to prove the existence of an infinite series of flows—a strong indicator of complete integrability. In fact any equation that passes the Painlevé test or possesses a recursion operator is a candidate for being solvable by the inverse scattering transformation [30]. However, it is worth mentioning that recursive operators do not yield an exhaustive list of all possible higher-order symmetries. In this work we verified the presence of infinitely many generalized symmetries, all of which preserve the linear KiSS equation and thus proved that it belongs to a class of evolution equations that are completely integrable.

Furthermore, multipliers and Lie Bäcklund transformations were obtained for a nonlinear diffusion equation. The multipliers were defined to contain terms up to first order in derivatives, and we found two multipliers that lead to two nontrivial conservation laws.

References

1. Olver, P.J.: Evolution equations possessing infinitely many symmetries. J. Math. Phys. **18**(6), 1212–1215 (1977)
2. Lax, P.D.: Periodic solutions of the KdV equation. Commun. Pure Appl. Math. **28**, 141–188 (1975)
3. Paliathanasis, A., Krishnakumar, K., Tamizhmani, K.M., Leach, P.G.L.: Lie symmetry analysis of the Black–Scholes–Merton model for European options with stochastic volatility. Mathematics **4**(2), 1–14 (2016)
4. Nucci, M.C., Sanchini, G.: Noether symmetries quantization and superintegrability of biological models. Symmetry **8**, 1–9 (2016)
5. Paliathanasis, A., Tsamparlis, M.: The reduction of Laplace equation in certain Riemannian spaces and the resulting Type II hidden symmetries. J. Geom. Phys. **76**, 107–123 (2014)
6. Belmonte-Beitia, J., Pérez-García, V.M., Vekslerchik, V., Torres, P.J.: Lie symmetries and solitons in nonlinear systems with spatially inhomogeneous nonlinearities. Phys. Rev. Lett. **98**, 064102 (2007)
7. Champagne, B., Hereman, W., Winternitz, P.: The computer calculation of Lie point symmetries of large systems of differential equations. Comput. Phys. Commun. **66**, 319–340 (1991)
8. Baumann, G.: Symmetry Analysis of Differential Equations with Mathematica. Springer, New York (2000)
9. Dimas, S., Tsoubelis, D.: SYM: A New Symmetry-Finding Package for Mathematica in Group Analysis of Differential Equations. University of Cyprus, Nicosia (2005)
10. Steudel, H.: Über die Zuordnung zwischen Invarianzeigenschaften und Erhaltungssätzen. Zeitschrift für Naturforschung **17**, 129–132 (1962)
11. Jamal, S., Kara, A.H.: New higher-order conservation laws of some classes of wave and Gordon-type equations. Nonlinear Dyn. **67**, 97–102 (2012)

12. Morris, R., Kara, A.H., Biswas, A.: Soliton solution and conservation laws of the Zakharov equation in plasmas with power law nonlinearity. Nonlinear Anal.: Model. Control **18**(2), 153–159 (2013)
13. Jamal, S., Kara, A.H., Bokhari, A.H., Zaman, F.D.: The symmetries and conservation laws of some Gordon-type equations in Milne space-time. Pramana J. Phys. **80**(5), 739–755 (2013)
14. Naz, R.: Conservation laws for some systems of nonlinear partial differential equations via multiplier approach. J. Appl. Math. **871253**, 1–13 (2012)
15. Jamal, S., Kara, A.H.: Higher-order symmetries and conservation laws of multi-dimensional Gordon-type equations. Pramana J. Phys. **77**(3), 1–14 (2011)
16. Skellam, J.G.: Random dispersal in theoretical populations. Biometrika **38**, 196–218 (1951)
17. Holmes, E.E., Lewis, M.A., Banks, J.E., Veit, R.R.: Partial differential equations in ecology: spatial interactions and population dynamics. Ecology **75**(1), 17–29 (1994)
18. Goldstein, S.: On diffusion by discontinuous movements, and on the telegraph equation. Q. J. Mech. Appl. Mech. **6**, 129–156 (1951)
19. Fisher, R.A.: The wave of advance of advantageous genes. Ann. Eugen. **7**, 355–369 (1937)
20. Dobzhansky, T., Wright, S.: Genetics of natural populations. X. Dispersion rates in Drosophila pseudoobscura. Genetics **28**, 304–340 (1943)
21. Okubo, A.: Diffusion and Ecological Problems: Mathematical Models. Springer, Berlin (1980)
22. Helland, I.S., Hoff, J.M., Anderbrant, G.: Attraction of bark beetles (Coleoptera: Scolytidae) to a pheromone trap: experiment and mathematical models. J. Chem. Ecol. **10**, 723–752 (1984)
23. Bluman, G.W.: Simplifying the form of Lie groups admitted by a given differential equation. J. Math. Anal. Appl. **145**, 52–62 (1990)
24. Gurney, W.S.C., Nisbet, R.M.: The regulation of inhomogeneous populations. J. Theor. Biol. **52**, 441–457 (1975)
25. LeVeque, R.J.: Numerical Methods for Conservation Laws. Birkhauser-Verlag, Basel (1992)
26. Hereman, W.: Symbolic computation of conservation laws of nonlinear partial differential equations in multidimensions. Int. J. Quantum Chem. **106**, 278–299 (2006)
27. Kara, A.H.: An analysis of the symmetries and conservation laws of the class of Zakharov–Kuznetsov equations. Math. Comput. Appl. **15**(4), 658–664 (2010)
28. Anco, S., Bluman, G.: Direct construction method for conservation laws of partial differential equations Part I: examples of conservation law classifications. Eur. J. Appl. Math. **13**, 545–566 (2002)
29. Kierstead, H., Slobodkin, L.B.: The size of water masses containing plankton blooms. J. Mar. Res. **12**, 141–147 (1953)
30. Fokas, A.S.: Symmetries and integrability. Stud. Appl. Math. **77**, 253–299 (1987)