

A Note on the Solute Dispersion in a Porous Medium

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Abstract In this paper, we study the solute transport through a semi-infinite channel filled with a fluid saturated sparsely packed porous medium. A small perturbation of magnitude ε is applied on the channel's walls on which the solute particles undergo a first-order chemical reaction. The effective model for solute concentration in the small-Péclet-number regime is derived using asymptotic analysis with respect to the small parameter ε . The obtained mathematical model clearly indicates the effects of porous medium, chemical reaction and boundary distortion. In particular, the effect of porous medium parameter on the dispersion coefficient is discussed.

Keywords Solute dispersion · Porous medium · Chemical reaction · Small boundary perturbation · Asymptotic analysis

1 Introduction

In this paper we consider the problem of the solute dispersion in the laminar flow in a sparsely packed porous medium. In view of that, the concentration $c^*(x^*, y^*, t^*)$ of a solute dissolved in a fluid satisfies a convection–dispersion equation of the form (see, e.g., [1]):

$$\frac{\partial c^*}{\partial t^*} + u_0^*(y^*) \frac{\partial c^*}{\partial x^*} - D^* \frac{\partial^2 c^*}{\partial (x^*)^2} - D^* \frac{\partial^2 c^*}{\partial (y^*)^2} = 0 \quad \text{in } \Omega^*. \quad (1)$$

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Here positive constant D^* denotes the dispersion coefficient, while Ω^* is a semi-infinite 2D symmetric channel given by

$$\Omega^* = \left\{ (x^*, y^*) \in \mathbf{R}^2 : 0 < x^* < +\infty, |y^*| < H - \lambda \varphi \left(\frac{x^*}{H} \right) \right\}. \quad (2)$$

Our intention is to consider a domain with slightly perturbed boundary so we take the ratio $\varepsilon = \frac{\lambda}{H}$ to be a small parameter, i.e., $0 < \varepsilon \ll 1$. φ is assumed to be an arbitrary smooth function of $\mathcal{O}(1)$ magnitude. We suppose that the solute particles do not react among themselves but they undergo a first-order chemical reaction at channel's walls, namely:

$$D^* \frac{\partial c^*}{\partial y^*} = -\beta^* c^* \quad \text{on } \Gamma^* = \left\{ (x^*, y^*) \in \mathbf{R}^2 : 0 < x^* < +\infty, |y^*| = H - \lambda \varphi \left(\frac{x^*}{H} \right) \right\}, \quad (3)$$

where constant β^* stands for the surface reaction coefficient. It is assumed that the chemical processes under consideration have no effect on the density and hence produce no buoyancy forces. Consequently, no coupling with the momentum equations is being modeled. For that reason, the fluid velocity $\mathbf{u}^* = u_0^*(y)\mathbf{e}_1$ is assumed to be known in (1) and given as the zero-order asymptotic solution of the Darcy–Brinkman equation provided in “Appendix”. Hereinafter, $(\mathbf{e}_1, \mathbf{e}_2)$ denotes the standard Cartesian basis.

Our main goal is to derive a simplified mathematical model for the solute concentration and to detect the effects of porous medium, chemical reaction and boundary perturbation on the solute dispersion. As presented, the problem is described by the convection–dispersion equation for solute concentration associated with Robin boundary condition describing the reaction mechanism. Naturally, one cannot expect to derive the exact solution of the governing problem, so we employ a singular perturbation technique with respect to the small parameter ε . We choose to work in a non-dimensional setting in which two characteristic numbers naturally appear, namely Péclet and Damkohler number. Motivated by the micro-fluidic applications (see, e.g., [8]), we address the regime under small Péclet number and identify three possible asymptotic models depending on the magnitude of the Damkohler number. Then we analyze the critical one in which all the above mentioned effects are balanced. To capture the effects of the boundary oscillations, we use the direct approach (see [9, 11, 12]) and expand the unknowns in Taylor series near the perturbed boundary. By doing that, we avoid tedious change of variable leading to a complicated rescaled equation. As a result, we obtain the effective system in the form of the 1D parabolic problem (see Sect. 4) being amenable for numerical simulations. In particular, the effects of the porous structure, surface reaction coefficient and boundary distortion are clearly visible. Moreover, the effective dispersion coefficient is deduced in the explicit form and its behavior is discussed with respect to the porous medium parameter k . To conclude, it must be emphasized that this study is not limited to periodic corrugations of the channel boundary, i.e., our result is valid for an arbitrary (smooth enough) boundary perturbation function. In view of that, we believe that the findings

presented in this paper could be instrumental from the practical point of view, namely in micro-fluidic applications where the flow could be significantly affected by the irregular wall roughness.

We finish this Sect. 1 by providing some bibliographic remarks on the subject. The problem of solute transport has been of considerable interest for many years due to its practical importance in chemical, mechanical and biological engineering. The pioneer researcher is Taylor [19] who first discussed the dispersion of a passive solute in a viscous fluid flowing through a circular pipe under laminar conditions. Rigorous derivation of the asymptotic model for a chemically reactive solute transport in a Poiseuille flow through a narrow 2-D channel was brought by Mikelić et al. [14]. The effects of the curved geometry on solute dispersion through a distorted pipe have been investigated by Nunge [16], Rosencrans [18] and Marušić-Paloka and Pažanin [10], etc.

In the context of porous medium flow, the breakthrough paper is due to Chandrasekhara et al. [6] in which deterministic model for the longitudinal dispersion is proposed following the analysis of Taylor [19]. Channel walls are assumed to be flat and no chemical reaction is considered in [6]. Afterward, numerous authors investigated the dispersion in porous media with and without reaction, both analytically and numerically. We refer the reader to the review paper by Rudraiah and Ng [21] (and the references therein) providing a nice overview of the results from the existing literature (see also Pal [17]). We also emphasize the paper by Valdes-Parada et al. [23] in which the upscaling process of mass transport with chemical reaction in porous media has been analyzed. In particular, the dependence of the dispersion coefficient on the particle Peclet number and Thiele moduli has been discussed in detail. Treatments of the dispersion problem in a domain with irregularities have remained rather neglected topic. It is due to the fact that introducing a small parameter as the perturbation quantity in the domain boundary makes analysis very complicated because of the tedious change of variable that needs to be performed. In case of Stokes flow ($k = 0$), two results by Bolster et al. [4] and Woollard et al. [24] should be mentioned. In those papers, the authors assume that the corrugations are described by a simple trigonometric function and study the problem numerically. To our knowledge, the present paper brings the first analytical result on the effect of boundary perturbation (described by an arbitrary shape function) on the solute dispersion in a porous medium. For that reason, we believe that it could improve the known engineering practice.

2 Setting of the Problem

As explained in the Sect. 1, we consider the following model of convection–dispersion with chemical reaction occurring at the walls:

$$\begin{cases} \frac{\partial c^*}{\partial t^*} + u_0^*(y^*) \frac{\partial c^*}{\partial x^*} - D^* \frac{\partial^2 c^*}{\partial (x^*)^2} - D^* \frac{\partial^2 c^*}{\partial (y^*)^2} = 0 & \text{in } \Omega^* \times (0, T^*), \\ D^* \frac{\partial c^*}{\partial y^*} = -\beta^* c^* & \text{on } \Gamma^* \times (0, T^*). \end{cases} \quad (4)$$

Here Ω^* denotes our domain with slightly perturbed boundary given by (2), while D^* (dispersion parameter) and k^* (surface reaction parameter) are given constants.

The unknown function is a solute concentration $c^*(x^*, y^*, t)$, while the fluid velocity $u_0^*(y^*)$ is assumed to be known. T^* is an arbitrarily chosen positive number.

For the purpose of our analysis, it is convenient to work in a non-dimensional setting. We adimensionalize system (4) in a standard way by dividing the space variables by H , while for all other quantities we use reference values denoted by the subscript R . Thus, we introduce

$$x = \frac{x^*}{H}, \quad y = \frac{y^*}{H}, \quad t = \frac{t^*}{T_R}, \quad c = \frac{c^*}{c_R}, \quad D = \frac{D^*}{D_R}, \quad \beta = \frac{\beta^*}{\beta_R}, \quad u = \frac{u^*}{u_R},$$

where $u_R = \frac{\delta H^2}{\mu_e}$ (see ‘‘Appendix’’). As a result, we obtain the problem in non-dimensional form

$$\begin{cases} \frac{\partial c}{\partial t} + \frac{u_R T_R}{H} u_0(y) \frac{\partial c}{\partial x} - \frac{D_R T_R}{H^2} D \left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} \right) = 0 & \text{in } \Omega \times (0, T), \\ \frac{D D_R}{H \beta_R} \frac{\partial c}{\partial t} = -\beta c & \text{on } \Gamma \times (0, T), \end{cases} \tag{5}$$

with

$$\Omega = \left\{ (x, y) \in \mathbf{R}^2 : 0 < x < +\infty, |y| < 1 - \varepsilon\varphi(x) \right\}, \tag{6}$$

$$\Gamma = \left\{ (x, y) \in \mathbf{R}^2 : 0 < x < +\infty, y = \pm 1 \mp \varepsilon\varphi(x) \right\}. \tag{7}$$

Two characteristic numbers (that could depend on ε) naturally appear in the above system, namely:

$$\mathbf{Pe}^\varepsilon = \frac{H u_R}{D_R} \quad (\text{Péclet number}), \quad \mathbf{Da}^\varepsilon = \frac{H \beta_R}{D_R} \quad (\text{Damkohler number}). \tag{8}$$

Following Chandrasekhara et al. [6], in the sequel we assume that the longitudinal dispersion is negligible with respect to its transverse counterpart, namely $\frac{\partial^2 c}{\partial x^2} \ll \frac{\partial^2 c}{\partial y^2}$. Such situation naturally arises if the domain (pore) under consideration is narrow or long (i.e., the longitudinal dimension is much larger than the transverse one), see, e.g., Mikelić et al. [14]. Choosing the timescale $T_R = \frac{H}{u_R}$, we arrive at the following problem:

$$\frac{\partial c^\varepsilon}{\partial t} + u_0(y) \frac{\partial c^\varepsilon}{\partial x} = \frac{D}{\mathbf{Pe}^\varepsilon} \frac{\partial^2 c}{\partial y^2} \quad \text{in } \Omega_+^\varepsilon \times (0, T), \tag{9}$$

$$D \frac{\partial c^\varepsilon}{\partial y} = -\beta \mathbf{Da}^\varepsilon c^\varepsilon \quad \text{on } \Gamma_+^\varepsilon \times (0, T), \tag{10}$$

$$\frac{\partial c^\varepsilon}{\partial y}(x, 0, t) = 0 \quad \text{for } (x, t) \in (0, +\infty) \times (0, T), \tag{11}$$

$$c^\varepsilon(0, y, t) = 0 \quad \text{for } (y, t) \in (0, 1) \times (0, T), \tag{12}$$

$$c^\varepsilon(x, y, 0) = 1 \quad \text{for } (x, y) \in \Omega_+^\varepsilon, \tag{13}$$

where $u_0(y) = \frac{1}{k^2} \left(1 - \frac{\cosh(ky)}{\cosh k} \right)$ (see ‘‘Appendix’’). Here we denote

$$\Omega_+^\varepsilon = \left\{ (x, y) \in \mathbf{R}^2 : 0 < x < +\infty, 0 < y < 1 - \varepsilon\varphi(x) \right\}, \tag{14}$$

$$\Gamma_+^\varepsilon = \left\{ (x, y) \in \mathbf{R}^2 : 0 < x < +\infty, y = 1 - \varepsilon\varphi(x) \right\}. \tag{15}$$

Note that the boundary condition (11) results from the y -symmetry of the solution. To close up the governing problem, we impose initial condition (13) and zero boundary condition at $x = 0$.

It is important to observe that Péclet and Damkohler numbers can be compared with small parameter ε in various ways leading to different asymptotic behaviors depending on their orders of magnitude. As emphasized in the Introduction, in this work we want to investigate the regime under small Péclet number. In view of that, we set

$$\mathbf{Pe}^\varepsilon = \varepsilon \tag{16}$$

to simplify the notation. Consequently, the Peclet number will only be implicitly included in the effective dispersion coefficient (46) through the appearance of the small parameter ε . As discussed in the forthcoming section, depending on the magnitude of Damkohler number \mathbf{Da}^ε , three different asymptotic models can be derived. It turns out (see Remark 2) that the most interesting case occur when $\mathbf{Da}^\varepsilon = \mathcal{O}(\varepsilon)$ since it leads to the macroscopic model in which the chemistry balances with the flow. Therefore, we restrict our attention to this situation and put

$$\mathbf{Da}^\varepsilon = \varepsilon. \tag{17}$$

3 Analysis

We first derive the effective boundary conditions at the upper boundary. To simplify the notation, we assume that $\varphi < 0$ on $(0, 1)$ implying $\Omega_+ = \left\{ (x, y) \in \mathbf{R}^2 : 0 < x < 1, 0 < y < 1 \right\} \subset \Omega_+^\varepsilon$. Consequently, the solution c^ε of (9)–(13) is defined on $\Omega_+ \times (0, T)$ so we are in position to directly expand velocity in Taylor series with respect to y near the upper boundary. Without such assumption, we would have to extend the solution to Ω_+ and pollute the notation.

Remark 1 It must be emphasized that $\varphi < 0$ is just a technical assumption and that the obtained results are valid for a general function φ . That is due to the fact that it can be proved that our approximation (constructed directly without the change of variables) is asymptotically the same as the one that could be built if we have first passed to the ε -independent domain $\Omega_+ = (0, 1)^2$ (by introducing the suitable change of variables), with no constraint imposed on φ . This part is straightforward and can be done following the same lines as in Marušić-Paloka [9] and Marušić-Paloka and Pažanin [12].

We expand the unknown solution in Taylor series with respect to y near the upper boundary, namely:

$$c^\varepsilon(x, y, t) = \sum_{i=0}^{\infty} \frac{1}{i!} \frac{\partial^i c^\varepsilon}{\partial y^i}(x, 1, t)(y - 1)^i. \tag{18}$$

Taking into account (17), from (10) we deduce:

$$\begin{aligned} & D \left[\frac{\partial c^\varepsilon}{\partial y}(x, 1, t) - \varepsilon \frac{\partial^2 c^\varepsilon}{\partial y^2}(x, 1, t)\varphi(x) + \frac{\varepsilon^2}{2} \frac{\partial^3 c^\varepsilon}{\partial y^3}(x, 1, t)\varphi(x)^2 - \dots \right] \\ &= -\beta\varepsilon \left[c^\varepsilon(x, 1, t) - \varepsilon \frac{\partial c^\varepsilon}{\partial y}(x, 1, t)\varphi(x) + \frac{\varepsilon^2}{2} \frac{\partial^2 c^\varepsilon}{\partial y^2}(x, 1, t)\varphi(x)^2 - \dots \right] \end{aligned} \tag{19}$$

On the other hand, we postulate the asymptotic expansion as follows

$$c^\varepsilon(x, y, t) = c_0(x, y, t) + \varepsilon c_1(x, y, t) + \varepsilon^2 c_2(x, y, t) + \dots. \tag{20}$$

Substituting the above expansion in (19) yields

$$\begin{aligned} & D \left[\frac{\partial c_0}{\partial y}(x, 1, t) + \varepsilon \left(\frac{\partial c_1}{\partial y}(x, 1, t) - \varphi(x) \frac{\partial^2 c_0}{\partial y^2}(x, 1, t) \right) \right. \\ & \quad \left. + \varepsilon^2 \left(\frac{\partial c_2}{\partial y}(x, 1, t) - \varphi(x) \frac{\partial^2 c_1}{\partial y^2}(x, 1, t) + \frac{1}{2} \varphi(x)^2 \frac{\partial^3 c_0}{\partial y^3}(x, 1, t) \right) + \dots \right] \\ &= -\beta \left[\varepsilon c_0(x, 1, t) + \varepsilon^2 \left(c_1(x, 1, t) - \varphi(x) \frac{\partial c_0}{\partial y}(x, 1, t) \right) + \dots \right]. \end{aligned} \tag{21}$$

Thus, we deduce the following boundary conditions at $y = 1$ satisfied by the unknown functions in (20):

$$1 : \frac{\partial c_0}{\partial y}(x, 1, t) = 0, \tag{22}$$

$$\varepsilon : D \frac{\partial c_1}{\partial y}(x, 1, t) = D\varphi(x) \frac{\partial^2 c_0}{\partial y^2}(x, 1, t) - \beta c_0(x, 1, t), \tag{23}$$

$$\varepsilon^2 : D \frac{\partial c_2}{\partial y}(x, 1, t) = D\varphi(x) \frac{\partial^2 c_1}{\partial y^2}(x, 1, t) - \frac{D}{2} \varphi(x)^2 \frac{\partial^3 c_0}{\partial y^3}(x, 1, t) + \tag{24}$$

$$+ \beta \varphi(x) \frac{\partial c_0}{\partial y}(x, 1, t) - \beta c_1(x, 1, t). \tag{25}$$

Considering (16), we substitute the expansion (20) into Eq. (9) to obtain

$$\varepsilon^{-1} : \frac{\partial^2 c_0}{\partial y^2} = 0 \text{ in } (0, 1), \text{ for } (x, t) \in (0, +\infty) \times (0, T). \tag{26}$$

Taking into account the boundary conditions (11) and (22), i.e.,

$$\frac{\partial c_0}{\partial y}(x, 1, t) = \frac{\partial c_0}{\partial y}(x, 1, t) = 0 \tag{27}$$

we conclude

$$c_0 = c_0(x, t). \tag{28}$$

Remark 2 From the above analysis it is clear that, if we had assumed $\mathbf{Da}^\varepsilon \gg \mathcal{O}(\varepsilon)$, we would obtain $c_0 = 0$. That would mean that the chemical reaction (taking place at the upper wall) dominates the process keeping almost all solute in a small region near the left entry. On the other hand, for $\mathbf{Da}^\varepsilon \ll \mathcal{O}(\varepsilon)$, the effects of the chemical reaction would be negligible. For that reason, we choose $\mathbf{Da}^\varepsilon = \mathcal{O}(\varepsilon)$ [see (17)] as the critical (and most interesting) case between those two cases.

The next term in the expansion (20) is given by [see (24)]:

$$1 : D \frac{\partial^2 c_1}{\partial y^2} = u_0(y) \frac{\partial c_0}{\partial x} + \frac{\partial c_0}{\partial t} \quad \text{in } (0, 1), \tag{29}$$

$$\varepsilon : D \frac{\partial c_1}{\partial y}(x, 1, t) = -\beta c_0(x, t), \tag{30}$$

$$\varepsilon : \frac{\partial c_1}{\partial y}(x, 0, t) = 0, \tag{31}$$

for every $(x, t) \in (0, +\infty) \times (0, T)$. Inserting the Darcy–Brinkman velocity (see “Appendix”) into Eq. (29), it can be rewritten as

$$D \frac{\partial^2 c_1}{\partial y^2} = \frac{1}{k^2} \left(\frac{\tanh k}{k} - \frac{\cosh(ky)}{\cosh k} \right) \frac{\partial c_0}{\partial x} + \frac{\partial c_0}{\partial t} + \frac{1}{k^2} \left(1 - \frac{\tanh k}{k} \right) \frac{\partial c_0}{\partial x} \quad \text{in } (0, 1). \tag{32}$$

The necessary condition for the existence of c_1 satisfying (30)–(32) gives the equation for c_0 :

$$\frac{1}{k^2} \left(1 - \frac{\tanh k}{k} \right) \frac{\partial c_0}{\partial x} + \frac{\partial c_0}{\partial t} + \beta c_0 = 0 \quad \text{in } (0, +\infty) \times (0, T). \tag{33}$$

This is a hyperbolic equation whose solution is discontinuous, due to the incompatibility of the initial and boundary data which need to be satisfied by c_0 . Consequently, we cannot use it for the purpose of our analysis, since the asymptotic approximation involves the derivatives of c_0 . To overcome this issue, we use the idea proposed by Rubinstein and Mauri [20] (extensively used afterward, see, e.g., [10, 14]) and assume that

$$\frac{1}{k^2} \left(1 - \frac{\tanh k}{k} \right) \frac{\partial c_0}{\partial x} + \frac{\partial c_0}{\partial t} + \beta c_0 = \mathcal{O}(\varepsilon) \quad \text{in } (0, +\infty) \times (0, T). \quad (34)$$

In view of that, Eq. (32) now becomes

$$D \frac{\partial^2 c_1}{\partial y^2} = \frac{1}{k^2} \left(\frac{\tanh k}{k} - \frac{\cosh(ky)}{\cosh k} \right) \frac{\partial c_0}{\partial x} - \beta c_0 \quad \text{in } (0, 1). \quad (35)$$

for every $(x_1, t) \in (0, +\infty) \times (0, T)$. Taking into account the boundary conditions (30)–(31), we can easily solve the above equation to obtain

$$c_1(x, y, t) = \frac{1}{Dk^2} \left(\frac{\tanh k}{k} \cdot \frac{y^2}{2} - \frac{1}{k^2} \frac{\cosh(ky)}{\cosh k} \right) \frac{\partial c_0}{\partial x} - \frac{\beta c_0}{D} \frac{y^2}{2} + A(x, t), \quad (36)$$

with $A(x, t)$ being an arbitrary function. We choose $A(x, t)$ such that $\int_0^1 c_1 dy = 0$ leading to

$$c_1(x, y, t) = \frac{1}{Dk^2} \left[\frac{\tanh k}{k} \cdot \frac{y^2}{2} - \frac{1}{k^2} \frac{\cosh(ky)}{\cosh k} + \frac{\tanh k}{k} \left(\frac{1}{k^2} - \frac{1}{6} \right) \right] \frac{\partial c_0}{\partial x} - \frac{\beta c_0}{D} \left(\frac{1}{6} - \frac{y^2}{2} \right), \quad (37)$$

for $(x_1, t) \in (0, +\infty) \times (0, T)$. By assuming $\int_0^1 c_1 dy = 0$, we simplify the resulting equation for c_0 , see (41).

In view of (34), the problem for c_2 is given by [see (25)]:

$$\begin{aligned} \varepsilon : D \frac{\partial^2 c_2}{\partial y^2} &= \frac{1}{k^2} \left(1 - \frac{\cosh(ky)}{\cosh k} \right) \frac{\partial c_1}{\partial x} + \frac{\partial c_1}{\partial t} \\ &+ \frac{1}{\varepsilon} \left[\frac{1}{k^2} \left(1 - \frac{\tanh k}{k} \right) \frac{\partial c_0}{\partial x} + \frac{\partial c_0}{\partial t} + \beta c_0 \right] \quad \text{in } (0, 1), \end{aligned} \quad (38)$$

$$\varepsilon^2 : D \frac{\partial c_2}{\partial y}(x, 1, t) = D\varphi(x) \frac{\partial^2 c_1}{\partial y^2}(x, 1, t) - \beta c_1(x, t), \quad (39)$$

$$\varepsilon^2 : \frac{\partial c_2}{\partial y}(x, 0, t) = 0, \quad (40)$$

for every $(x, t) \in (0, +\infty) \times (0, T)$. The system (38)–(40) will be solvable if and only if

$$\begin{aligned} & \frac{\partial}{\partial t} \int_0^1 c_1 \, dy + \frac{1}{k^2} \int_0^1 \left(1 - \frac{\cosh(ky)}{\cosh k} \right) \frac{\partial c_1}{\partial x} \, dy \\ & \quad + \frac{1}{\varepsilon} \left[\frac{1}{k^2} \left(1 - \frac{\tanh k}{k} \right) \frac{\partial c_0}{\partial x} + \frac{\partial c_0}{\partial t} + \beta c_0 \right] \\ & = D\varphi(x) \frac{\partial^2 c_1}{\partial y^2}(x, 1, t) - \beta c_1(x, t). \end{aligned} \tag{41}$$

It should be observed that the first term on the left-hand side in (41) vanishes since c_1 was computed such that $\int_0^1 c_1 \, dy = 0$. It remains to compute the second term by tedious, but direct integration using (37). Consequently, we obtain the effective equation satisfied by c_0 :

$$\begin{aligned} & \frac{\partial c_0}{\partial t} + \frac{1}{k^2} \left\{ \left(1 - \frac{\tanh k}{k} \right) (1 + \varepsilon\varphi(x)) + \varepsilon \frac{\beta}{Dk} \left[\tanh k \left(\frac{11}{6} + \frac{2}{k^2} \right) - \frac{2}{k} \right] \right\} \frac{\partial c_0}{\partial x} \\ & \quad + \beta \left\{ 1 - \varepsilon \frac{\beta}{3D} + \varepsilon\varphi(x) \right\} c_0 \\ & = \varepsilon \frac{1}{Dk^4} \left\{ \tanh^2 k \left(\frac{2}{k^2} + \frac{1}{k^4} - \frac{1}{6} \right) - \frac{3}{2k^3} \tanh k - \frac{1}{2k^2} \right\} \frac{\partial^2 c_0}{\partial x^2}. \end{aligned} \tag{42}$$

Endowing it with the initial and boundary condition at $x = 0$ [see (12)–(13)]:

$$c_0(x, 0) = 1, \quad c_0(0, t) = 0, \tag{43}$$

we deduce the effective problem describing the zero-order approximation for the solute concentration.

4 Main Results and Discussion

The effective behavior of the system (9)–(13) for small ε is given by the following parabolic problem satisfied by the solute concentration in $(0, +\infty) \times (0, T)$:

$$\begin{aligned} & \frac{\partial c_0}{\partial t} + \frac{1}{k^2} \left\{ \left(1 - \frac{\tanh k}{k} \right) (1 + \varepsilon\varphi(x)) + \varepsilon \frac{\beta}{Dk} \left[\tanh k \left(\frac{11}{6} + \frac{2}{k^2} \right) - \frac{2}{k} \right] \right\} \frac{\partial c_0}{\partial x} \\ & \quad + \beta \left\{ 1 - \varepsilon \frac{\beta}{3D} + \varepsilon\varphi(x) \right\} c_0 \\ & = \varepsilon \frac{1}{Dk^4} \left\{ \tanh^2 k \left(\frac{2}{k^2} + \frac{1}{k^4} - \frac{1}{6} \right) - \frac{3}{2k^3} \tanh k - \frac{1}{2k^2} \right\} \frac{\partial^2 c_0}{\partial x^2}, \end{aligned} \tag{44}$$

$$c_0(x, 0) = 1, \quad c_0(0, t) = 0, \quad \frac{\partial c}{\partial x} \in L^2((0, +\infty) \times (0, T)). \tag{45}$$

From the obtained simplified mathematical model (satisfied by the zero-order asymptotic approximation), we can clearly observe the effects of the porous medium parameter k , chemical reaction and small perturbation of the boundary. Though all

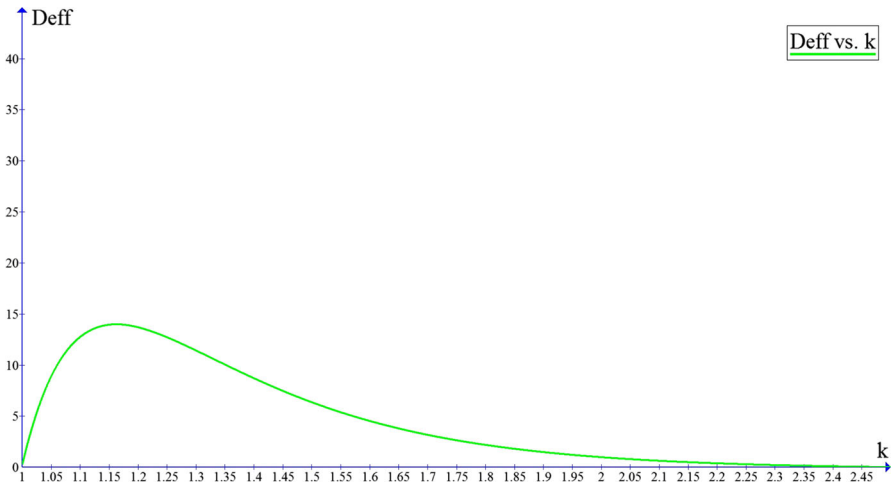


Fig. 1 Plot of dispersion coefficient versus porous medium parameter

the effects we seek for are present in the first part of the asymptotic solution, we can compute the correctors in the asymptotic expansion (20) leading to a higher order of accuracy. The first one c_1 has already been computed and it is given in the explicit form, see (37). The second one is described by the problem (38)–(40) which can be, again, explicitly solved using (42). We leave that to a reader as an easy exercise.

It is important to note that from (44) we can recover the effective dispersion coefficient [see (9)] as

$$D_{eff} = \frac{\varepsilon^2}{Dk^4} \left\{ \tanh^2 k \left(\frac{2}{k^2} + \frac{1}{k^4} - \frac{1}{6} \right) - \frac{3}{2k^3} \tanh k - \frac{1}{2k^2} \right\}. \quad (46)$$

Figure 1 is a plot of the dispersion coefficient (46) versus porous medium parameter k for $\varepsilon = 0.05$ and $D = 10^{-5}$ (see [7]). It can be observed that the increase in the dispersion coefficient is sharp in the early range of k , while the decrease is milder afterward. For $k > 2$, the values of the dispersion coefficient stabilize and become very small. Changing the magnitude of the small parameter ε dramatically influences on the range of the dispersion coefficient, as seen from (46).

5 Conclusion

Understanding solute transport in porous media is important from the practical point of view since we naturally come across such processes in biological, geological and artificial (industrial) media (see, e.g., [1] and references therein). In particular, the regimes under small Péclet number are of considerable interest, for instance in micro-fluidic applications (see, e.g., [8]). In this paper, we present a formal derivation of the effective model for enhanced dispersion through a 2D channel filled with a porous medium. The governing model is described by a (non-stationary) convection–dispersion equation

with known Darcy–Brinkman velocity. A chemical reaction (with surface reaction coefficient β) is considered at the channel boundary which has been perturbed by the product of the small parameter ε and arbitrary smooth function φ . The analysis addresses a regime under small Péclet number and employs a singular perturbation technique.

We believe the result presented here provides a good platform for understanding the influence of different parameters (porous structure, chemical reaction, boundary distortion) on the solute dispersion through porous media. The fact that we have derived the effective system in the form of the 1D parabolic problem is particularly important with regards to numerical simulations. To conclude, since the problem under consideration naturally appears in numerous applications, we hope that our result could have an impact on the current engineering practice.

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A Appendix: Darcy–Brinkman Velocity

It is well known that the stationary flow of an incompressible, viscous fluid through a porous media is described by the conservation of mass and conservation of linear momentum principles. Conservation of mass is expressed by the continuity equation

$$\operatorname{div} \mathbf{u}^* = 0 \quad \text{in } \Omega^*, \quad (47)$$

satisfied by the fluid velocity, while different models have been proposed over the past sixteen decades to describe the conservation of the linear momentum. Without any doubt, the Darcy law [3] is the most popular one stating that the filtration velocity is proportional to the driving pressure gradient. However, one of its major drawbacks is that it cannot sustain the (physically relevant) no-slip boundary condition imposed on an impermeable wall. Thus, if one wants to consider a sparse porous medium, the Darcy–Brinkman equation [5] would represent a suitable choice (see, e.g., [2, 13, 22]):

$$\mu_e \Delta \mathbf{u}^* - \frac{\mu}{K} \mathbf{u}^* = \nabla p^* \quad \text{in } \Omega^*. \quad (48)$$

Here \mathbf{u}^* and p^* denote (dimensional) filter velocity and pressure, μ is the physical viscosity of the fluid, K stands for the permeability of the porous medium, while μ_e denotes the effective viscosity for the Brinkman term. It should be mentioned that, in Chandrasekhara et al. [6], it is assumed that $\mu = \mu_e$. However, in general, those two viscosities are not equal (see, e.g., [13]). Being the second-order PDE for the velocity, Eq. (48) can handle the presence of a boundary on which the no-slip condition for the velocity can be imposed. Thus, the Darcy–Brinkman model represents an essential generalization of the Darcy law which is capable of successfully describing numerous situations naturally arising in industry and geophysical problems.

For the sake of reader’s convenience, let us derive the zero-order asymptotic solution of the system (47)–(48) entering in the starting convection–diffusion equation (1). It is natural to assume that the flow is governed by a constant pressure gradient $\frac{\partial p^*}{\partial x^*} = -\delta$ in the x^* -direction. Consequently, we deduce that the flow is purely in the longitudinal direction, i.e., $\mathbf{u}^* = u^*(x, y)\mathbf{e}_1$. Introducing

$$x = \frac{x^*}{H}, \quad y = \frac{y^*}{H}, \quad u = \frac{u^*}{\frac{\delta H^2}{\mu_e}}, \tag{49}$$

we get the dimensionless form of Eqs. (47)–(48) as

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \tag{50}$$

$$\Delta u - k^2 u = -1 \quad \text{in } \Omega, \tag{51}$$

Here

$$k = H \sqrt{\frac{\mu}{\mu_e K}} \tag{52}$$

is the non-dimensional parameter characterizing the porous medium which is proportional to the inverse square root of the Darcy number $Da = \frac{K}{H^2}$. Note that $k = 0$ corresponds to classical Stokes flow. Now, we plug the expansion

$$u(x, y) = u_0(x, y) + \varepsilon u_1(x, y) + \varepsilon^2 u_2(x, y) + \dots \tag{53}$$

in the Darcy–Brinkman Eq. (50) and also in the no-slip boundary condition

$$u = 0 \quad \text{for } y = \pm 1 \mp \varepsilon \varphi(x). \tag{54}$$

Using Taylor series approach (see, e.g., [15] for details), from (54), we deduce

$$0 = u|_{y=1-\varepsilon\varphi(x)} = u_0|_{y=1} + \varepsilon \left(u_1 - \varphi \frac{\partial u_0}{\partial y} \right) |_{y=1} + \mathcal{O}(\varepsilon^2), \tag{55}$$

$$0 = u|_{y=-1+\varepsilon\varphi(x)} = u_0|_{y=-1} + \varepsilon \left(u_1 + \varphi \frac{\partial u_0}{\partial y} \right) |_{y=-1} + \mathcal{O}(\varepsilon^2). \tag{56}$$

After collecting the terms with equal powers of ε , we obtain

$$\begin{cases} 1 : \Delta u_0 - k^2 u_0 = -1, \\ 1 : u_0 = 0 \quad \text{for } y = \pm 1. \end{cases} \tag{57}$$

Due to the divergence-free condition (50), the solution of (57) is independent of x and, thus, given by

$$u_0(y) = \frac{1}{k^2} \left(1 - \frac{\cosh(ky)}{\cosh k} \right). \tag{58}$$

Finally, applying (49) we can easily recover the dimensional velocity u_0^* which enters in the governing Eq. (1).

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