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On σ-Tripartite Labelings of Odd Prisms and Even Möbius Ladders

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Abstract A common question in the study of graph decompositions is when does a graph G decompose the complete graph or the complete graph with a 1-factor removed or added. It is known that a σ-tripartite labeling of a tripartite graph G with n edges can be used to obtain a cyclic G-decomposition of K_{2nt+1} for every positive integer t. Moreover, it can be used to obtain a cyclic G-decomposition of both $K_{2nt+2} - I$ and $K_{2nt} + I$, where I is a 1-factor. We show that if G is an odd prism on 10 or more vertices or an even Möbius ladder, then G admits a σ-tripartite labeling.

Keywords Cyclic G-designs · Cubic Tripartite graphs · σ-tripartite labelings

Mathematics Subject Classification 05C78

1 Introduction

For integers r and s, we denote the set $\{r, r+1, \ldots, s\}$ by $[r, s]$ (if $r > s$, then $[r, s] = \emptyset$). Let $\mathbb{N}$ denote the set of nonnegative integers, $\mathbb{Z}^+$ denote the set of positive integers, and $\mathbb{Z}_n$ denote the group of integers modulo n. Call a graph G *tripartite* if the chromatic number of G is at most 3. Thus, bipartite graphs can be considered tripartite.

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Let m be a positive integer and let $V(K_m) = [0, m-1]$. The *length* of an edge $\{i, j\}$ in K_m is $\min\{|i-j|, m-|i-j|\}$. Note that if m is odd, then K_m consists of m edges of length i for $i \in [1, \frac{m-1}{2}]$. However, if m is even, then K_m consists of m edges of length i for $i \in [1, \frac{m}{2}-1]$ and of only $\frac{m}{2}$ edges of length $\frac{m}{2}$. In this case, the edges of length $\frac{m}{2}$ form a 1-factor in K_m. Throughout this manuscript, if m is even, we will denote the 1-factor formed by the set of edges of length $\frac{m}{2}$ in K_m by I.

Let $V(K_m) = \mathbb{Z}_m$ and let G be a subgraph of K_m. By *clicking* G, we mean applying the permutation $i \mapsto i+1$ to $V(G)$. Note that clicking an edge does not change its length. Let H and G be graphs such that G is a subgraph of H. A G-*decomposition* of H is a set $\Delta = \{G_1, G_2, \ldots, G_t\}$ of pairwise edge-disjoint subgraphs of H each of which is isomorphic to G and such that $E(H) = \bigcup_{i=1}^{t} E(G_i)$. A G-decomposition of K_m is also known as a (K_m, G)-*design*. A (K_m, G)-design Δ is *cyclic* if clicking is an automorphism of Δ. The study of graph decompositions is generally known as the study of graph designs, or G-designs. For surveys on G-designs, see [1] and [2].

Let G be a graph with n edges. A primary question in the study of graph designs is: *for what values of* v *does there exist a* (K_v, G)-*design*? Another question of interest is the existence of $(K_v \pm I, G)$-designs where v is even. For most studied graphs G, it is often the case that if $v \equiv 1 \pmod{2n}$, then there exists a (K_v, G)-design. Similarly, if $v \equiv 2 \pmod{2n}$ or $v \equiv 0 \pmod{2n}$, then there often exists a $(K_v - I, G)$-design in the former and a $(K_v + I, G)$-design in the latter. A common approach to finding these designs is through the use of graph labelings.

1.1 Graph Labelings

For a graph G, a one-to-one function $f\colon V(G) \to \mathbb{N}$ is called a *labeling* (or a *valuation*) of G. In a seminal paper on the topic [11], Rosa introduced a hierarchy of labelings. Let G be a graph with n edges and no isolated vertices and let f be a labeling of G. Let $f(V(G)) = \{f(u) : u \in V(G)\}$. Define a function $\bar{f}\colon E(G) \to \mathbb{Z}^+$ by $\bar{f}(e) = |f(u) - f(v)|$, where $e = \{u, v\} \in E(G)$. We will refer to $\bar{f}(e)$ as the *label* of e. Let $\bar{f}(E(G)) = \{\bar{f}(e) : e \in E(G)\}$. Consider the following conditions:

(ℓ1) $f(V(G)) \subseteq [0, 2n]$,
(ℓ2) $f(V(G)) \subseteq [0, n]$,
(ℓ3) $\bar{f}(E(G)) = \{x_1, x_2, \ldots, x_n\}$, where for each $i \in [1, n]$ either $x_i = i$ or $x_i = 2n + 1 - i$,
(ℓ4) $\bar{f}(E(G)) = [1, n]$.

If in addition G is bipartite with bipartition $\{A, B\}$ of $V(G)$ consider also

(ℓ5) for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $f(a) < f(b)$,
(ℓ6) there exists an integer λ such that $f(a) \leq \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$.

Then a labeling satisfying the conditions:

(ℓ1), (ℓ3) is called a ρ-*labeling*;
(ℓ1), (ℓ4) is called a σ-*labeling*;
(ℓ2), (ℓ4) is called a β-*labeling*.

A β-labeling is necessarily a σ-labeling, which in turn is a ρ-labeling. Suppose G is bipartite. If a ρ-, σ-, or β-labeling of G satisfies condition (ℓ5), then the labeling is *ordered* and is denoted by ρ^+, σ^+, or β^+, respectively. If in addition (ℓ6) is satisfied, the labeling is *uniformly ordered* and is denoted by ρ^{++}, σ^{++}, or β^{++}, respectively.

A β-labeling is better known as a *graceful* labeling, and a uniformly ordered β-labeling is an *α-labeling* as introduced in [11]. Labelings of the types above are called *Rosa-type labelings* because of Rosa's original article [11] on the topic. (See [4] for a recent comprehensive survey of Rosa-type labelings.) A dynamic survey on general graph labelings is maintained by Gallian [8].

Labelings are critical to the study of cyclic graph decompositions as seen in the following theorem from [11].

Theorem 1 *Let G be a graph with n edges. There exists a cyclic G-decomposition of K_{2n+1} if and only if G admits a ρ-labeling.*

If G admits a σ-labeling instead, then cyclic G-decompositions of $K_{2n+2} - I$ and of $K_{2n} + I$ can also obtained. Theorem 2 appears as Theorem 3.5 in [4]. We provide a proof of Theorem 3 for the sake of completeness.

Theorem 2 *Let G be a graph with n edges. If G admits a σ-labeling, then there also exists a cyclic G-decomposition of $K_{2n+2} - I$.*

Theorem 3 *Let G be a graph with n edges. If G admits a σ-labeling, then there exists a cyclic G-decomposition of $K_{2n} + I$.*

Proof Let G, n and I be as in the statement of the theorem. Let $V(K_{2n}) = \mathbb{Z}_{2n}$. Note that $K_{2n} + I$ is the multigraph obtained form K_{2n} by making each of the edges of length n have multiplicity 2. Thus, for each $i \in [1, n]$, the number of edges of length i in $K_{2n} + I$ is $2n$. Let h be a σ-labeling of G. Let G_0 be the subgraph of $K_{2n} + I$ obtained by identifying vertex $v \in V(G)$ with $i \in V(K_{2n})$ if $h(v) = i$. Thus, G_0 is an embedding of G in K_{2n} so that there is an edge in G_0 of length i for each $i \in [1, n]$. For $t \in [1, 2n - 1]$, let G_t be the subgraph of $K_{2n} + I$ obtained by clicking G_0 a total of t times. Then $\Delta = \{G_t : t \in \mathbb{Z}_{2n}\}$ is a cyclic G-decomposition of $K_{2n} + I$. □

If G admits an α-labeling, then we have the following powerful result of Rosa [11].

Theorem 4 *Let G be a bipartite graph with n edges. If G admits an α-labeling, then there exists a cyclic G-decomposition of K_{2nt+1} for all positive integers t.*

We illustrate how Theorem 4 works. Let h be an α-labeling of a graph G with n edges and bipartition (A, B). Let $A = \{u_1, u_2, \ldots, u_r\}$ and $B = \{v_1, v_2, \ldots, v_s\}$. Let t be a positive integer. For $1 \leq i \leq t$, let G_i be a copy of G with bipartition (A, B_i) where $B_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,s}\}$ and $v_{i,j}$ corresponds to v_j in B. Let $G(t) = G_1 \cup G_2 \cup \ldots \cup G_t$. Thus, $G(t)$ has nt edges and is bipartite with bipartition $(A, B_1 \cup B_2 \cup \ldots \cup B_t)$. Define a labeling f' of $G(t)$ as follows: $f'(u_j) = f(u_j)$ for each $u_j \in A$ and $f'(v_{i,j}) = f(j) + (i - 1)n$ for $1 \leq i \leq t$ and $1 \leq j \leq s$. It is easy to see that f' is an α-labeling of $G(t)$, and thus, Theorem 1 applies. Since f' is necessarily a σ-labeling, Theorems 2 and 3 also apply, and we have the following.

Corollary 5 *Let G be a bipartite graph with n edges. If G admits an α-labeling, then there exist cyclic G-decompositions of $K_{2nt+2} - I$ and of $K_{2nt} + I$ for all positive integers t.*

From a graph decompositions perspective, Theorem 2 offers a slight advantage over Theorem 1. In the case G is bipartite, Theorem 4 offers a great advantage over the first two. However, there are many classes of bipartite graphs (see [4]) that do not admit α-labelings. Theorem 4 was extended to cover graphs that admit ρ^+-labelings in [5].

Theorem 6 *Let G be a bipartite graph with n edges. If G admits a ρ^+-labeling, then there exists a cyclic G-decomposition of K_{2nt+1} for all positive integers t.*

Again, since Theorem 6 is ρ-labeling based, it does not guarantee the existence of decomposition results involving the addition or removal of a 1-factor. Two labelings that lead to results similar to those of Theorems 4 and 6 were recently introduced for tripartite graphs in [3]. One of them is called a ρ-tripartite labeling and the other a σ-tripartite labeling. Both lead to cyclic G-decompositions of K_{2nt+1}, but only the σ-labeling-based one leads to results involving the addition or removal of a 1-factor. In this manuscript, we focus on the σ-labeling-based one.

Let G be a tripartite graph with n edges having the vertex tripartition $\{A, B, C\}$. A *σ-tripartite labeling* of G is a one-to-one function $h\colon V(G) \to [0, 2n]$ that satisfies the following conditions:

(s1) h is a σ-labeling of G.
(s2) If $\{a, v\} \in E(G)$ with $a \in A$, then $h(a) < h(v)$.
(s3) If $e = \{b, c\} \in E(G)$ with $b \in B$ and $c \in C$, then there exists an edge $e' = \{b', c'\} \in E(G)$ with $b' \in B$ and $c' \in C$ such that $|h(c') - h(b')| + |h(c) - h(b)| = n$.
(s4) If $a \in A$ and $v \in B \cup C$, then $h(a) - h(v) \neq n$.
(s5) If $b \in B$ and $c \in C$, then $|h(b) - h(c)| \notin \{n, 2n\}$.

Note that e and e' in (s3) need not to be distinct. Also note that there need not be an edge $\{a, v\}$ in (s4) nor an edge $\{b, c\}$ in (s5). The following theorem from [3] shows that a σ-tripartite labeling yields results similar to those from α-labelings.

Theorem 7 *Let G be a tripartite graph with n edges. If G admits a σ-tripartite labeling, then there exists a cyclic G-decomposition of K_{2nt+1} for all positive integers t.*

Again, we illustrate how Theorem 7 works. Let G have n edges and let h be a σ-tripartite labeling of G with vertex tripartition $\{A, B, C\}$ as in the above definition. Let $B_1, B_2, \ldots, B_t$ be t vertex-disjoint copies of B, and let $C_1, C_2, \ldots, C_t$ be t vertex-disjoint copies of C. The vertex in B_i corresponding to $b \in B$ will be called b_i. Similarly, the vertex in C_i corresponding to $c \in C$ will be called c_i. Let $B^* = \bigcup_{i=1}^{t} B_i$ and $C^* = \bigcup_{i=1}^{t} C_i$. We define a new graph G^* with vertex set $A \bigcup B^* \bigcup C^*$ and edges $\{a, v_i\}$, $1 \le i \le t$, whenever $a \in A$ and $\{a, v\}$ is an edge of G, and $\{b_i, c_i\}$, $1 \le i \le t$, whenever $\{b, c\}$ is an edge of G with $b \in B$ and $c \in C$. Clearly G^* has nt edges and G divides G^*. Define a labeling h^* on G^* by

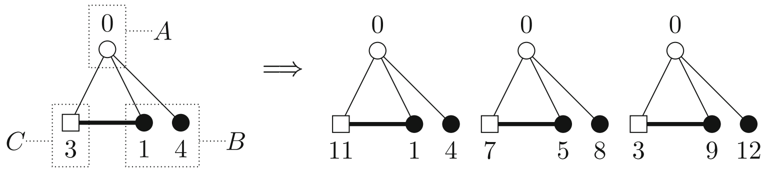


Fig. 1 A σ-tripartite labeling of a graph G with 4 edges and the 3 copies of G used to yield cyclic G-decompositions of K_{25}, of $K_{24}+I$, and of $K_{26}-I$

$$h^*(v) = \begin{cases} h(v) & v \in A, \\ h(b) + (i-1)n & v = b_i \in B_i, \\ h(c) + (t-i)n & v = c_i \in C_i. \end{cases}$$

The labeling h^* is a σ-labeling of G^* and the result follows by Theorem 1. Moreover, we can use Theorems 2 and 3 to obtain cyclic G-decompositions of $K_{2nt+2}-I$ and of $K_{2nt}+I$ as well. The $K_{2nt+2}-I$ result appears as Corollary 5 in [3].

Corollary 8 *Let G be a graph with n edges. If G admits a σ-tripartite labeling, then there exist cyclic G-decompositions of $K_{2nt+2}-I$ and of $K_{2nt}+I$ for every positive integer t.*

In Fig. 1, we demonstrate the use of Theorem 7 in the case $t = 3$ by showing a σ-tripartite labeling of the graph G consisting of a triangle with a pendent edge. The labeling of G on the left can be used to yield cyclic G-decompositions of K_9, of K_8+I, and of $K_{10}-I$. The labeling of the three copies of G on the right can be used to yield cyclic G-decompositions of K_{25}, of $K_{24}+I$, and of $K_{26}-I$.

Some Rosa-type labelings of various cubic graphs have been investigated. It is known that all bipartite prisms [6,7] and bipartite Möbius ladders [9] admit α-labelings. In [17], it is shown that if G is cubic and bipartite and if every component of G is either a prism, a Möbius ladder, or has order at most 14, then G admits an α-labeling. Hence, if such a bipartite G has n edges, then it cyclically decomposes K_{2nt+1}, $K_{2nt}+I$, and $K_{2nt+2}-I$ for every positive integer t. In [16], it is shown that if G is an odd prism, an even Möbius ladder, or a connected cubic tripartite graph of order at most 10, then G admits a ρ-tripartite labeling. Hence, such a G of size n would cyclically decompose K_{2nt+1} for every positive integer t. However, no G-decompositions of $K_{2nt}+I$ or $K_{2nt+2}-I$ can be obtained from this labeling. In [15], it is shown that every cubic graph of order at most 12, other than $2K_4$ and $3K_4$, admits a β-labeling. Vietri [13,14] has shown that certain classes of generalized Petersen graphs are graceful. It is also known that $2K_4$ does not admit a ρ-labeling, but $3K_4$ does.

In this article, we show that if G is an odd prism on 10 or more vertices or an even Möbius ladder, then G admits a σ-tripartite labeling, and hence, such a G of size n would cyclically decompose $K_{2nt}+I$ and $K_{2nt+2}-I$, in addition to K_{2nt+1}, for every positive integer t.

1.2 Additional Definitions and Notation

We denote the path with vertices $x_0, x_1, \ldots, x_k$, where x_i is adjacent to x_{i+1}, $0 \leq i \leq k-1$, by $(x_0, x_1, \ldots, x_k)$. In using this notation, we are thinking of traversing the path from x_0 to x_k so that x_0 is the first vertex, x_1 is the second vertex, and so on. Let $G_1 = (x_0, x_1, \ldots, x_j)$ and $G_2 = (y_0, y_1, \ldots, y_k)$. If G_1 and G_2 are vertex-disjoint except for $x_j = y_0$, then by G_1+G_2 we mean the path $(x_0, x_1, \ldots, x_j, y_1, y_2, \ldots, y_k)$. If the only vertices they have in common are $x_0 = y_k$ and $x_j = y_0$, then by $G_1 + G_2$ we mean the cycle $(x_0, x_1, \ldots, x_j, y_1, y_2, \ldots, y_{k-1}, x_0)$.

Let $P(2k)$ be the path with $2k$ edges and $2k+1$ vertices $0, 1, \ldots, 2k$ given by $(0, 2k, 1, 2k-1, 2, 2k-2, \ldots, k-1, k+1, k)$. Note that the set of vertices of this graph is $A \cup B$, where $A = [0, k]$, $B = [k+1, 2k]$, and every edge joins a vertex from A to one from B. Furthermore, the set of labels of the edges of $P(2k)$ is $[1, 2k]$.

Let a and b be nonnegative integers and k, d_1, and d_2 be positive integers such that $a+kd_1 < b$. Let $\hat{P}(2k, d_1, d_2, a, b)$ be the path with $2k$ edges and $2k+1$ vertices given by $(a, b+(k-1)d_2, a+d_1, b+(k-2)d_2, a+2d_1, \ldots, a+(k-1)d_1, b, a+kd_1)$. Note that $\hat{P}(2k, 1, 1, 0, k+1)$ is the graph $P(2k)$. Note that this graph $\hat{P}(2k, d_1, d_2, a, b)$ has the following properties:

- P1: $\hat{P}(2k, d_1, d_2, a, b)$ is a path with first vertex a, second vertex $b + (k-1)d_2$, and last vertex $a + kd_1$.
- P2: Each edge of $\hat{P}(2k, d_1, d_2, a, b)$ joins a vertex from $A = \{a+id_1 : 0 \leq i \leq k\}$ to a vertex with a larger label from $B = \{b + id_2 : 0 \leq i \leq k-1\}$.
- P3: The set of edge labels of $\hat{P}(2k, d_1, d_2, a, b)$ is $\{b - a - kd_1 + i(d_1 + d_2) : 0 \leq i \leq k-1\} \cup \{b - a - (k-1)d_1 + i(d_1 + d_2) : 0 \leq i \leq k-1\}$.

The path $\hat{P}(12, 2, 4, 10, 30)$ is shown in Fig. 2 below.

2 σ-Tripartite Labelings of Some Cubic Graphs

We will show that odd prisms and even Möbius ladders admit σ-tripartite labelings.

2.1 σ-Tripartite Labelings of Odd Prisms

By the *prism* D_n ($n \geq 3$) we mean the Cartesian product of a cycle with n vertices and a path with 2 vertices: $C_n \times P_2$. For convenience, we let $D_n = C_n \cup C'_n \cup F$, where

Fig. 2 Path $\hat{P}(12, 2, 4, 10, 30)$

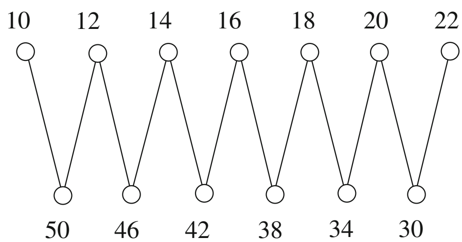
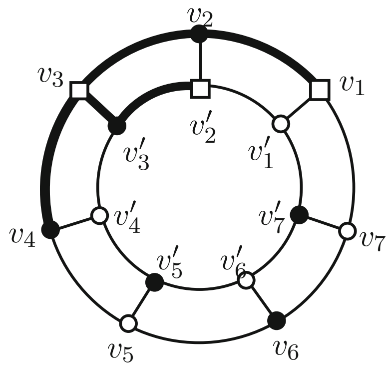


Fig. 3 Prism D_7



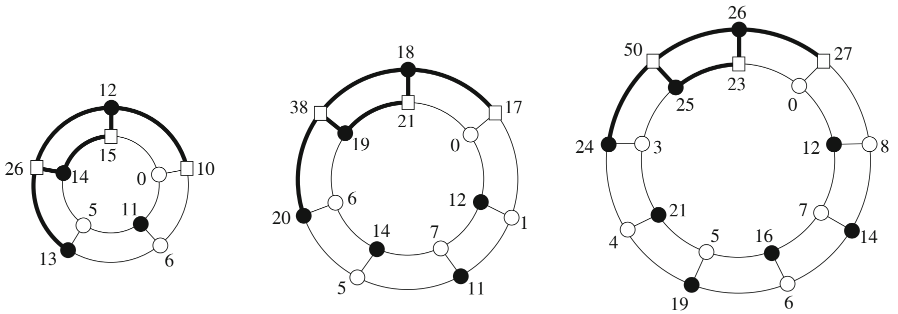


Fig. 4 A σ-tripartite labeling of D_5, D_7 and D_9

$C_n = (v_1, v_2, \ldots, v_n, v_1)$, $C'_n = (v'_1, v'_2, \ldots, v'_n, v'_1)$, and $F = \{\{v_i, v'_i\} : 1 \leq i \leq n\}$. We shall refer to C_n as the *outer cycle*, to C'_n as the *inner cycle*, and to F as the *spokes*. We note that D_{2n+1} (for $n > 1$) is necessarily tripartite with tripartition $\{A, B, C\}$ where $A = \{v'_1\} \cup \{v_{2i+1} : 2 \leq i \leq n\} \cup \{v'_{2i} : 2 \leq i \leq n\}$, $B = \{v_{2i} : 1 \leq i \leq n\} \cup \{v'_{2i+1} : 1 \leq i \leq n\}$, and $C = \{v_1, v_3, v'_2\}$. Figure 3 shows the prism D_7. In this figure, the vertices in A are shown with white circles while the vertices in B are shown with black circles and the vertices of C are shown with white squares. The edges between sets B and C are shown in thick lines. We will adopt this convention in all our figures. It is easy to see that D_3 cannot admit a σ-tripartite labeling. We will show that D_n admits a σ-tripartite labeling for all odd integers $n \geq 5$.

Lemma 9 *The prism D_3 does not admits a σ-tripartite labeling.*

Proof If $\{A, B, C\}$ is a vertex tripartition of D_3, then the number of edges between B and C is necessarily 3. Since the number of edges of D_3 is odd, it is impossible for D_3 to admit a σ-tripartite labeling.

Lemma 10 *The prism D_n admits a σ-tripartite labeling for all $n \in \{5, 7, 9\}$.*

Proof We give σ-tripartite labelings of D_5, D_7, and D_9 in Fig. 4. □

Theorem 11 *The prism D_n admits a σ-tripartite labeling for all odd $n \geq 5$.*

Proof The cases with $n \leq 9$ are covered in Lemma 10. We separate the rest of the proof into 3 cases.

Case 1 $n \equiv 1 \pmod 6$.
Let $n = 6t+1$ where $t \geq 2$. Thus, $|V(D_n)| = 12t+2$ and $|E(D_n)| = 18t+3$. Define a one-to-one function $f\colon V(D_{6t+1}) \to [0, 36t+6]$ as follows:

$$\begin{aligned}
f(v_1) &= 18t-1, \\
f(v_2) &= 18t, \\
f(v_3) &= 36t+2, \\
f(v_4) &= 18t+2, \\
f(v_i) &= i+2, && v_i \in A_1 = \{v_i : i \text{ odd},\ 5 \leq i \leq 6t-1\}, \\
f(v_i) &= 18t-2i+10, && v_i \in B_1 = \{v_i : i \text{ even},\ 6 \leq i \leq 2t+4\}, \\
f(v_i) &= 18t-2i+4, && v_i \in B_2 = \{v_i : i \text{ even},\ 2t+4 < i \leq 6t-2\}, \\
f(v_{6t}) &= 18t-5, \\
f(v_{6t+1}) &= 1, \\
f(v'_1) &= 0, \\
f(v'_2) &= 18t+3, \\
f(v'_3) &= 18t+1, \\
f(v'_4) &= 5, \\
f(v'_5) &= 12t-3, \\
f(v'_i) &= i+2, && v'_i \in A'_1 = \{v'_i : i \text{ even},\ 6 \leq i \leq 6t\}, \\
f(v'_i) &= 18t-2i+10, && v'_i \in B'_1 = \{v'_i : i \text{ odd},\ 7 \leq i \leq 2t+3\}, \\
f(v'_i) &= 18t-2i+4, && v'_i \in B'_2 = \{v'_i : i \text{ odd},\ 2t+3 < i \leq 6t-1\}, \\
f(v'_{6t+1}) &= 18t-7.
\end{aligned}$$

Note that $A = \{v_{6t+1}, v'_1, v'_4\} \cup A_1 \cup A'_1$, $B = \{v_2, v_4, v_{6t}, v'_3, v'_5, v'_{6t+1}\} \cup B_1 \cup B_2 \cup B'_1 \cup B'_2$ and $C = \{v_1, v_3, v'_2\}$. Thus, the domain of f is indeed $V(D_{6t+1})$. Next, we confirm that f is one-to-one. We compute

$$\begin{aligned}
f(A_1) &= \{7, 9, \ldots, 6t+1\}, \\
f(A'_1) &= \{8, 10, \ldots, 6t+2\}, \\
f(B_1) &= \{18t-2, 18t-6, \ldots, 14t+2\}, \\
f(B_2) &= \{14t-8, 14t-12, \ldots, 6t+8\}, \\
f(B'_1) &= \{18t-4, 18t-8, \ldots, 14t+4\}, \\
f(B'_2) &= \{14t-6, 14t-10, \ldots, 6t+6\}.
\end{aligned}$$

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 4 and that all labels are distinct. Thus, f is one-to-one. Moreover, $f(A) \subseteq [0, 6t+2]$ and $f(B \cup C) \subseteq [6t+6, 36t+2]$.

To help compute the edge labels, we will describe $f(V(D_{6t+1}))$ in terms of the $\hat{P}(2k, d_1, d_2, a, b)$ path notation. For convenience, we will identify the vertices of C_{6t+1} and C'_{6t+1} with their labels. We have $f(C_{6t+1}) = G_1 + G_2 + (6t + 1, 18t - 5, 1, 18t - 1, 18t, 36t + 2, 18t + 2, 7)$, where

$$G_1 = \hat{P}(2(t), 2, 4, 7, 14t + 2),$$
$$G_2 = \hat{P}(2(2t - 3), 2, 4, 2t + 7, 6t + 8).$$

By P3, the resulting edge label sets are:

$$\begin{aligned}\bar{f}(E(G_1)) &= \{12t - 5 + 6i : 0 \le i \le t - 1\} \cup \{12t - 3 + 6i : 0 \le i \le t - 1\}\\ &= \{\ell \equiv 1 \pmod 6 : 12t - 5 \le \ell \le 18t - 11\}\\ &\quad \cup \{\ell \equiv 3 \pmod 6 : 12t - 3 \le \ell \le 18t - 9\},\\ \bar{f}(E(G_2)) &= \{7 + 6i : 0 \le i \le 2t - 4\} \cup \{9 + 6i : 0 \le i \le 2t - 4\}\\ &= \{\ell \equiv 1 \pmod 6 : 7 \le \ell \le 12t - 17\}\\ &\quad \cup \{\ell \equiv 3 \pmod 6 : 9 \le \ell \le 12t - 15\}.\end{aligned}$$

Moreover, edge labels $12t - 6$, $18t - 6$, $18t - 2$, 1, $18t + 2$, $18t$, and $18t - 5$ occur on the path $(6t + 1, 18t - 5, 1, 18t - 1, 18t, 36t + 2, 18t + 2, 7)$.

Similarly, we have $f(C'_{6t+1}) = G'_1 + G'_2 + (6t + 2, 18t - 7, 0, 18t + 3, 18t + 1, 5, 12t - 3, 8)$, where

$$G'_1 = \hat{P}(2(t - 1), 2, 4, 8, 14t + 4),$$
$$G'_2 = \hat{P}(2(2t - 2), 2, 4, 2t + 6, 6t + 6).$$

By P3, the resulting edge label sets are:

$$\begin{aligned}\bar{f}(E(G'_1)) &= \{12t - 2 + 6i : 0 \le i \le t - 2\} \cup \{12t + 6i : 0 \le i \le t - 2\}\\ &= \{\ell \equiv 4 \pmod 6 : 12t - 2 \le \ell \le 18t - 14\}\\ &\quad \cup \{\ell \equiv 0 \pmod 6 : 12t \le \ell \le 18t - 12\},\\ \bar{f}(E(G'_2)) &= \{4 + 6i : 0 \le i \le 2t - 3\} \cup \{6 + 6i : 0 \le i \le 2t - 3\}\\ &= \{\ell \equiv 4 \pmod 6 : 4 \le \ell \le 12t - 14\}\\ &\quad \cup \{\ell \equiv 0 \pmod 6 : 6 \le \ell \le 12t - 12\}.\end{aligned}$$

Moreover, edge labels $12t - 9$, $18t - 7$, $18t + 3$, 2, $18t - 4$, $12t - 8$, and $12t - 11$ occur on the path $(6t + 2, 18t - 7, 0, 18t + 3, 18t + 1, 5, 12t - 3, 8)$.

For each spoke $\{v_i, v_i'\}$, the labels on the spokes are given by

$$\bar{f}(\{v_i, v_i'\}) = \begin{cases} 18t-1 & \text{for } i=1, \\ 3 & \text{for } i=2, \\ 18t+1 & \text{for } i=3, \\ 18t-3 & \text{for } i=4, \\ 12t-10 & \text{for } i=5, \\ 18t-3i+8 & \text{for } 6 \le i \le 2t+4, \\ 18t-3i+2 & \text{for } 2t+5 \le i \le 6t-1, \\ 12t-7 & \text{for } i=6t, \\ 18t-8 & \text{for } i=6t+1. \end{cases}$$

Thus, the set of edge labels on the spokes is

$$\begin{aligned} \bar{f}(E(F)) = & \{\ell \equiv 2 \pmod 3 : 12t-4 \le \ell \le 18t-10\} \\ & \cup \{\ell \equiv 2 \pmod 3 : 5 \le \ell \le 12t-13\} \\ & \cup \{18t-1, 3, 18t+1, 18t-3, 12t-10, 12t-7, 18t-8\}. \end{aligned}$$

It is easy to verify now that each $\ell \in [1, 18t+3]$ occurs on exactly one edge in D_{6t+1}. Hence, the defined labeling is a σ-labeling, and condition (s1) for a σ-tripartite labeling is satisfied. Condition (s2) also holds since $f(A) \subseteq [0, 6t+2]$ and $f(B \cup C) \subseteq [6t+6, 36t+2]$. Condition (s3) holds since $|f(v_1) - f(v_2)| + |f(v_2) - f(v_3)| = 18t+3$, $|f(v_3) - f(v_4)| + |f(v_2) - f(v_2')| = 18t+3$, and $|f(v_3) - f(v_3')| + |f(v_2') - f(v_3')| = 18t+3$, number of edges of D_{6t+1}. Condition (s4) clearly holds. Also $|f(b) - f(c)| \in \{18t+3, 36t+6\}$, where $b \in B$ and $c \in C$, is impossible since $|f(b) - f(c)| \in \{1, 2, 3, 18t, 18t+1, 18t+2\}$. Thus, condition (s5) holds, and we have a σ-tripartite labeling of D_{6t+1}. Figure 5 shows a σ-tripartite labeling of D_{13}.

Case 2 $n \equiv 3 \pmod 6$.
Let $n = 6t-3$ where $t \ge 3$. Thus, $|V(D_n)| = 12t-6$ and $|E(D_n)| = 18t-9$. Define a one-to-one function $f\colon V(D_{6t-3}) \to [0, 36t-18]$ as follows:

$$\begin{aligned} & f(v_1) = 18t-9, \\ & f(v_2) = 18t-10, \\ & f(v_3) = 36t-22, \\ & f(v_4) = 18t-12, \\ & f(v_i) = i-1, && v_i \in A_1 = \{v_i : i \text{ odd},\ 5 \le i \le 6t-3\}, \\ & f(v_i) = 18t-2i-4, && v_i \in B_1 = \{v_i : i \text{ even},\ 6 \le i \le 2t\}, \\ & f(v_i) = 18t-2i-5, && v_i \in B_2 = \{v_i : i \text{ even},\ 2t < i \le 4t-2\}, \\ & f(v_i) = 18t-2i-6, && v_i \in B_3 = \{v_i : i \text{ even},\ 4t-2 < i \le 6t-4\}, \\ & f(v_1') = 0, \end{aligned}$$

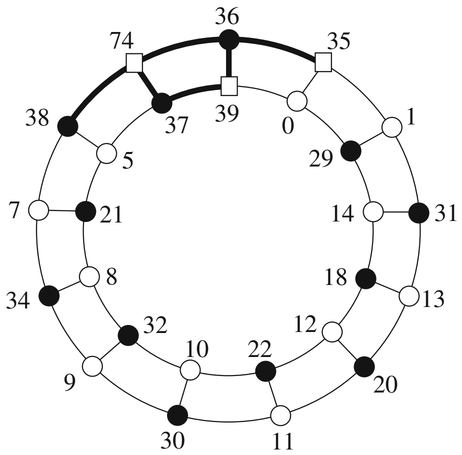


Fig. 5 A σ-tripartite labeling of D_{13}

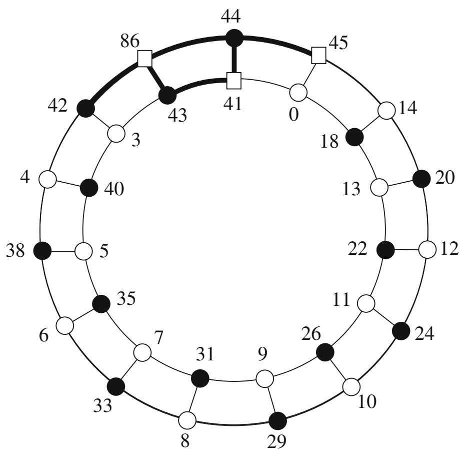


Fig. 6 A σ-tripartite labeling of D_{15}

$$
\begin{aligned}
&f(v_2') = 18t - 13, && \\
&f(v_3') = 18t - 11, && \\
&f(v_i') = i - 1, && v_i' \in A_1' = \{v_i' : i \text{ even},\ 4 \le i \le 6t - 4\}, \\
&f(v_i') = 18t - 2i - 4, && v_i' \in B_1' = \{v_i' : i \text{ odd},\ 5 \le i \le 2t - 1\}, \\
&f(v_i') = 18t - 2i - 5, && v_i' \in B_2' = \{v_i' : i \text{ odd},\ 2t - 1 < i \le 4t - 3\}, \\
&f(v_i') = 18t - 2i - 6, && v_i' \in B_3' = \{v_i' : i \text{ odd},\ 4t - 3 < i \le 6t - 3\}.
\end{aligned}
$$

Note that $A = \{v_1'\} \cup A_1 \cup A_1'$, $B = \{v_2, v_4, v_3'\} \cup B_1 \cup B_2 \cup B_3 \cup B_1' \cup B_2' \cup B_3'$ and $C = \{v_1, v_3, v_2'\}$. If we proceed as in Case 1, it is easy to verify that we have a σ-tripartite labeling of D_{6t-3}. Figure 6 shows a σ-tripartite labeling of D_{15}.

Case 3 $n \equiv 5 \pmod 6$.
Let $n = 6t-1$ where $t \geq 2$. Thus, $|V(D_n)| = 12t-2$ and $|E(D_n)| = 18t-3$. Define a one-to-one function $f : V(D_{6t-1}) \to [0, 36t-6]$ as follows:

$$
\begin{aligned}
f(v_1) &= 18t-7, \\
f(v_2) &= 18t-6, \\
f(v_3) &= 36t-10, \\
f(v_4) &= 18t-4, \\
f(v_i) &= i+2, & v_i \in A_1 = \{v_i : i \text{ odd}, \ 5 \leq i \leq 6t-3\}, \\
f(v_i) &= 18t-2i+1, & v_i \in B_1 = \{v_i : i \text{ even}, \ 6 \leq i \leq 2t+2\}, \\
f(v_i) &= 18t-2i-2, & v_i \in B_2 = \{v_i : i \text{ even}, \ 2t+2 < i \leq 6t-4\}, \\
f(v_{6t-2}) &= 18t-14, \\
f(v_{6t-1}) &= 1, \\
f(v'_1) &= 0, \\
f(v'_2) &= 18t-3, \\
f(v'_3) &= 18t-5, \\
f(v'_4) &= 5, \\
f(v'_i) &= i+2, & v'_i \in A'_1 = \{v'_i : i \text{ even}, \ 6 \leq i \leq 6t-2\}, \\
f(v'_i) &= 18t-2i+1, & v'_i \in B'_1 = \{v'_i : i \text{ odd}, 5 \leq i \leq 2t+3\}, \\
f(v'_i) &= 18t-2i-2, & v'_i \in B'_2 = \{v'_i : i \text{ even}, 2t+3 < i \leq 6t-3\}, \\
f(v'_{6t-1}) &= 18t-12.
\end{aligned}
$$

We have that $A = \{v_{6t-1}, v'_1, v'_4\} \cup A_1 \cup A'_1$, $B = \{v_2, v_4, v_{6t-2}, v'_3, v'_{6t-1}\} \cup B_1 \cup B_2 \cup B'_1 \cup B'_2$ and $C = \{v_1, v_3, v'_2\}$. If we proceed as in case 1, it is easy to verify that we have a σ-tripartite labeling of D_{6t-1}. Figure 7 shows a σ-tripartite labeling of D_{11}. □

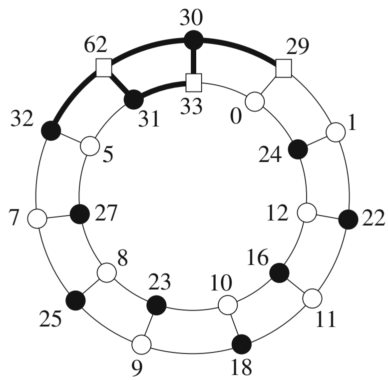
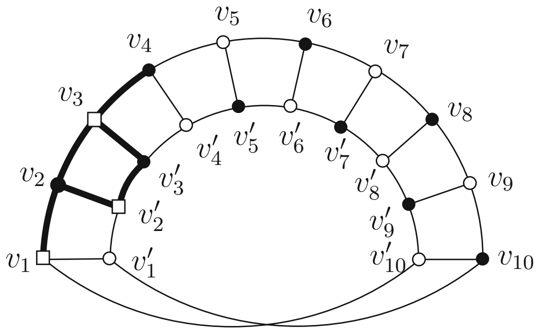


Fig. 7 A σ-tripartite labeling of D_{11}

Fig. 8 Möbius ladder M_{10}



2.2 σ-Tripartite Labelings of Even Möbius Ladders

For $n \geq 3$, let $v_1, v_2, \ldots, v_n$ and $v'_1, v'_2, \ldots, v'_n$ denote the consecutive vertices of two disjoint paths with n vertices. The *Möbius ladder M_n* is the graph obtained by joining v_i to v'_i for $i = 1, 2, \ldots, n$ and by joining v_1 to v'_n and v_n to v'_1. For convenience, we let $M_n = P_n \cup P'_n \cup F \cup H$, where $P_n = (v_1, v_2, \ldots, v_n)$, $P'_n = (v'_1, v'_2, \ldots, v'_n)$, $F = \{\{v_i, v'_i\} : 1 \leq i \leq n\}$ and $H = \{\{v_1, v'_n\}, \{v_n, v'_1\}\}$. We shall refer to P_n as the *outer path*, to P'_n as the *inner path*, and to F as the *spokes*. Figure 8 shows the Möbius ladder M_{10}. We note that M_{2n} (with $n \geq 2$) is necessarily tripartite with tripartition $\{A, B, C\}$, where $A = \{v'_1, v'_4\} \cup \{v_{2i-1}, v'_{2i} : 3 \leq i \leq n\}$, $B = \{v_2\} \cup \{v'_{2i-1}, v_{2i} : 2 \leq i \leq n\}$, and $C = \{v_1, v_3, v'_2\}$. We will show that M_n admits a σ-tripartite labeling for all even integers $n \geq 4$.

Lemma 12 *The Möbius ladder M_n admits a σ-tripartite labeling for all $n \in \{4, 6, 8, 10, 12\}$.*

Proof We give σ-tripartite labelings of M_4, M_6, M_8, M_{10}, and M_{12} in Fig. 9. □

Theorem 13 *The Möbius ladder M_n admits a σ-tripartite labeling for all even $n \geq 4$.*

Proof The cases with $n \leq 12$ are covered in Lemma 12. We separate the rest of the proof into 3 cases.
Case 1 $n \equiv 0 \pmod 6$.
Let $n = 6t$ where $t \geq 3$. Thus, $|V(M_n)| = 12t$ and $|E(M_n)| = 18t$. Define a one-to-one function $f : V(M_{6t}) \to [0, 36t]$ as follows:

$$\begin{aligned}
f(v_1) &= 18t, \\
f(v_2) &= 18t - 1, \\
f(v_3) &= 36t - 4, \\
f(v_i) &= i - 1, && v_i \in A_1 = \{v_i : i \text{ odd},\ 5 \leq i \leq 6t - 1\}, \\
f(v_i) &= 18t - 2i + 5, && v_i \in B_1 = \{v_i : i \text{ even},\ 4 \leq i \leq 2t\}, \\
f(v_i) &= 18t - 2i + 4, && v_i \in B_2 = \{v_i : i \text{ even},\ 2t < i \leq 4t\}, \\
f(v_i) &= 18t - 2i + 3, && v_i \in B_3 = \{v_i : i \text{ even},\ 4t < i \leq 6t\},
\end{aligned}$$

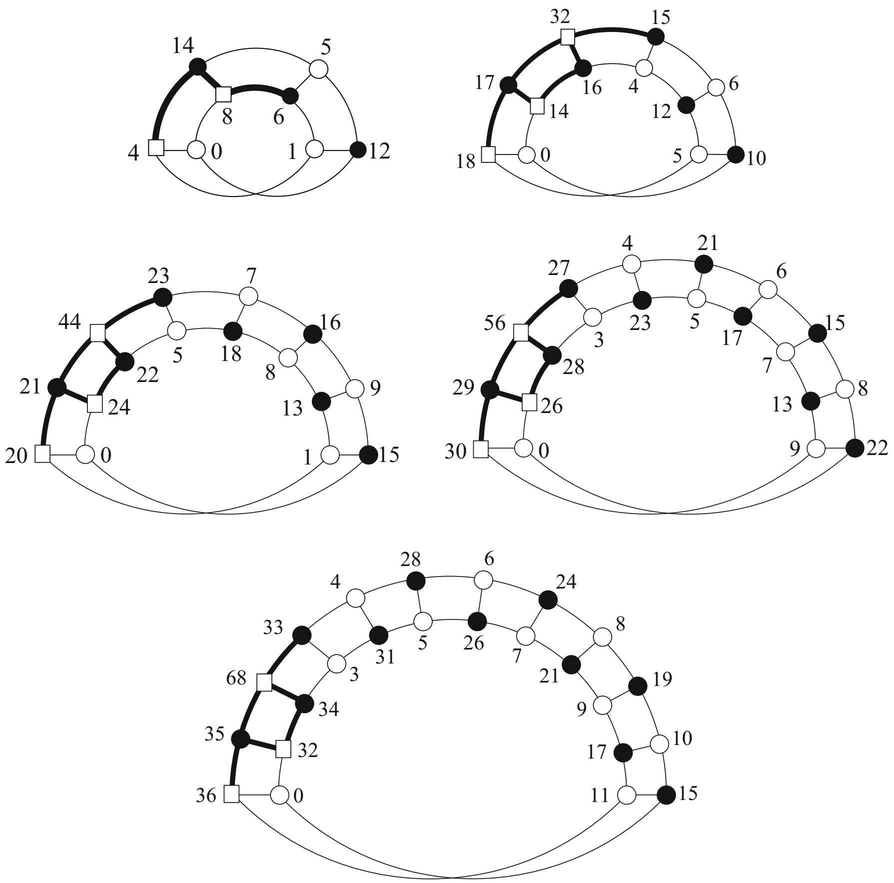


Fig. 9 σ-tripartite labelings of M_4, M_6, M_8, M_{10}, and M_{12}

$$
\begin{aligned}
&f(v_1') = 0, \\
&f(v_2') = 18t - 4, \\
&f(v_3') = 18t - 2, \\
&f(v_i') = i - 1, && v_i' \in A_1' = \{v_i' : i \text{ even},\ 4 \le i \le 6t\}, \\
&f(v_i') = 18t - 2i + 5, && v_i' \in B_1' = \{v_i' : i \text{ odd},\ 5 \le i \le 2t + 1\}, \\
&f(v_i') = 18t - 2i + 4, && v_i' \in B_2' = \{v_i' : i \text{ odd},\ 2t + 1 < i \le 4t - 1\}, \\
&f(v_i') = 18t - 2i + 3, && v_i' \in B_3' = \{v_i' : i \text{ odd},\ 4t - 1 < i \le 6t - 1\}.
\end{aligned}
$$

Note that $A = \{v_1'\} \cup A_1 \cup A_1'$, $B = \{v_2, v_3'\} \cup B_1 \cup B_2 \cup B_3 \cup B_1' \cup B_2' \cup B_3'$, and $C = \{v_1, v_3, v_2'\}$. Thus, the domain of f is indeed $V(M_{6t})$. Next, we confirm that f is one-to-one. We compute

$$
\begin{aligned}
f(A_1) &= \{4, 6, \ldots, 6t - 2\}, \\
f(A_1') &= \{3, 5, \ldots, 6t - 1\},
\end{aligned}
$$

$$
\begin{aligned}
f(B_1) &= \{18t-3, 18t-7, \ldots, 14t+5\},\\
f(B_2) &= \{14t, 14t-4, \ldots, 10t+4\},\\
f(B_3) &= \{10t-1, 10t-5, \ldots, 6t+3\},\\
f(B_1') &= \{18t-5, 18t-9, \ldots, 14t+3\},\\
f(B_2') &= \{14t-2, 14t-6, \ldots, 10t+6\},\\
f(B_3') &= \{10t+1, 10t-3, \ldots, 6t+5\}.
\end{aligned}
$$

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 4 and that all labels are distinct. Thus, f is one-to-one. Moreover, $f(A) \subseteq [0, 6t-1]$ and $f(B \cup C) \subseteq [6t+3, 36t-4]$.

To help compute the edge labels, we will describe $f(M_{6t})$ in terms of the $\hat{P}(2k, d_1, d_2, a, b)$ path notation. For convenience, we will identify the vertices of P_{6t} and P'_{6t} with their labels. We have $f(P_{6t}) = (18t, 18t-1, 36t-4, 18t-3, 4) + G_1 + G_2 + G_3 + (6t-2, 6t+3)$, where

$$
\begin{aligned}
G_1 &= \hat{P}(2(t-2), 2, 4, 4, 14t+5),\\
G_2 &= \hat{P}(2(t), 2, 4, 2t, 10t+4),\\
G_3 &= \hat{P}(2(t-1), 2, 4, 4t, 6t+7).
\end{aligned}
$$

By P3, the resulting edge label sets are:

$$
\begin{aligned}
\bar{f}(E(G_1)) &= \{12t+5+6i : 0 \le i \le t-3\} \cup \{12t+7+6i : 0 \le i \le t-3\}\\
&= \{\ell \equiv 5 \pmod 6 : 12t+5 \le \ell \le 18t-13\}\\
&\quad \cup \{\ell \equiv 1 \pmod 6 : 12t+7 \le \ell \le 18t-11\},\\
\bar{f}(E(G_2)) &= \{6t+4+6i : 0 \le i \le t-1\} \cup \{6t+6+6i : 0 \le i \le t-1\}\\
&= \{\ell \equiv 4 \pmod 6 : 6t+4 \le \ell \le 12t-2\}\\
&\quad \cup \{\ell \equiv 0 \pmod 6 : 6t+6 \le \ell \le 12t\},\\
\bar{f}(E(G_3)) &= \{9+6i : 0 \le i \le t-2\} \cup \{11+6i : 0 \le i \le t-2\}\\
&= \{\ell \equiv 3 \pmod 6 : 9 \le \ell \le 6t-3\}\\
&\quad \cup \{\ell \equiv 5 \pmod 6 : 11 \le \ell \le 6t-1\}.
\end{aligned}
$$

Moreover, edge labels 1, $18t-3$, $18t-1$, and $18t-7$ occur on the path $(18t, 18-1, 36t-4, 18t-3, 4)$ and the edge label 5 occurs on the edge $\{6t-2, 6t+3\}$.

Similarly, we have $f(P'_{6t}) = (0, 18t-4, 18t-2, 3) + G'_1 + G'_2 + G'_3$, where

$$
\begin{aligned}
G'_1 &= \hat{P}(2(t-1), 2, 4, 3, 14t+3),\\
G'_2 &= \hat{P}(2(t-1), 2, 4, 2t+1, 10t+6),\\
G'_3 &= \hat{P}(2(t), 2, 4, 4t-1, 6t+5).
\end{aligned}
$$

By P3, the resulting edge label sets are:

$$\begin{aligned}
\bar{f}(E(G_1')) &= \{12t+2+6i : 0 \le i \le t-2\} \cup \{12t+4+6i : 0 \le i \le t-2\} \\
&= \{\ell \equiv 2 \pmod 6) : 12t+2 \le \ell \le 18t-10\} \\
&\quad \cup \{\ell \equiv 4 \pmod 6) : 12t+4 \le \ell \le 18t-8\}, \\
\bar{f}(E(G_2')) &= \{6t+7+6i : 0 \le i \le t-2\} \cup \{6t+9+6i : 0 \le i \le t-2\} \\
&= \{\ell \equiv 1 \pmod 6 : 6t+7 \le \ell \le 12t-5\} \\
&\quad \cup \{\ell \equiv 3 \pmod 6 : 6t+9 \le \ell \le 12t-3\}, \\
\bar{f}(E(G_3')) &= \{6+6i : 0 \le i \le t-1\} \cup \{8+6i : 0 \le i \le t-1\} \\
&= \{\ell \equiv 0 \pmod 6 : 6 \le \ell \le 6t\} \\
&\quad \cup \{\ell \equiv 2 \pmod 6 : 8 \le \ell \le 6t+2\}.
\end{aligned}$$

Moreover, edge labels $18t-4$, 2, and $18t-5$ occur on the path $(0, 18t-4, 18t-2, 3)$.

For each spoke $\{v_i, v_i'\}$, the labels on the spokes are given by

$$\bar{f}(\{v_i, v_i'\}) = \begin{cases} 18t & \text{for } i = 1, \\ 3 & \text{for } i = 2, \\ 18t-2 & \text{for } i = 3, \\ 18t-3i+6 & \text{for } 4 \le i \le 2t+1, \\ 18t-3i+5 & \text{for } 2t+1 < i \le 4t, \\ 18t-3i+4 & \text{for } 4t < i \le 6t. \end{cases}$$

Thus, the set of edge labels on the spokes is

$$\begin{aligned}
\bar{f}(E(F)) &= \{\ell \equiv 0 \pmod 3 : 12t+3 \le \ell \le 18t-6\} \\
&\quad \cup \{\ell \equiv 2 \pmod 3 : 6t+5 \le \ell \le 12t-1\} \\
&\quad \cup \{\ell \equiv 1 \pmod 3 : 4 \le \ell \le 6t+1\} \cup \{18t, 3, 18t-2\}.
\end{aligned}$$

Moreover, edge labels $12t+1$ and $6t+3$ occur on the edges $\{v_1, v_{6t}'\}$ and $\{v_1', v_{6t}\}$.

It is easy to verify now that each $\ell \in [1, 18t]$ occurs on exactly one edge in M_{6t}. Hence, the defined labeling is a σ-labeling and condition (s1) for a σ-tripartite labeling is satisfied. Condition (s2) also holds since $f(A) \subseteq [0, 6t+2]$ and $f(B \cup C) \subseteq [6t+3, 36t]$. Condition (s3) holds since $|f(v_1) - f(v_2)| + |f(v_3) - f(v_4)| = 18t$, $|f(v_2)-f(v_3)|+|f(v_2)-f(v_2')| = 18t$, and $|f(v_2)-f(v_3')|+|f(v_3)-f(v_3')| = 18t$, the number of edges of M_{6t}. Condition (s4) clearly holds. Also $|f(b) - f(c)| \in \{18t, 36t\}$, where $b \in B$ and $c \in C$, is impossible since $|f(b)-f(c)| \in \{1, 2, 3, 18t-1, 18t-2, 18t-3\}$. Thus, condition (s5) holds, and we have a σ-tripartite labeling of M_{6t}. Figure 10 shows a σ-tripartite labeling of M_{18}.

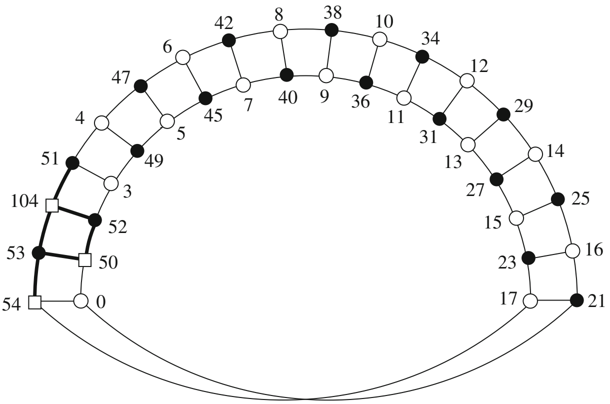


Fig. 10 A σ-tripartite labeling of M_{18}

Case 2 $n \equiv 2 \pmod 6$.
Let $n = 6t + 2$ where $t \geq 2$. Thus, $|V(M_n)| = 12t + 4$ and $|E(M_n)| = 18t + 6$. Define a one-to-one function $f : V(M_{6t+2}) \to [0, 36t + 12]$ as follows:

$$
\begin{aligned}
f(v_1) &= 18t + 2, \\
f(v_2) &= 18t + 3, \\
f(v_3) &= 36t + 8, \\
f(v_4) &= 18t + 5, \\
f(v_i) &= i + 2, && v_i \in A_1 = \{v_i : i \text{ odd},\ 5 \leq i \leq 6t + 1\}, \\
f(v_i) &= 18t - 2i + 10, && v_i \in B_1 = \{v_i : i \text{ even},\ 6 \leq i \leq 2t + 4\}, \\
f(v_i) &= 18t - 2i + 7, && v_i \in B_2 = \{v_i : i \text{ even},\ 2t + 4 < i \leq 6t\}, \\
f(v_{6t+2}) &= 18t - 3, \\
f(v'_1) &= 0, \\
f(v'_2) &= 18t + 6, \\
f(v'_3) &= 18t + 4, \\
f(v'_4) &= 5, \\
f(v'_i) &= i + 2, && v'_i \in A'_1 = \{v'_i : i \text{ even},\ 6 \leq i \leq 6t\}, \\
f(v'_i) &= 18t - 2i + 10, && v'_i \in B'_1 = \{v'_i : i \text{ odd},\ 5 \leq i \leq 2t + 3\}, \\
f(v'_i) &= 18t - 2i + 7, && v'_i \in B'_2 = \{v'_i : i \text{ odd},\ 2t + 3 < i \leq 6t - 1\}, \\
f(v'_{6t+1}) &= 18t - 5, \\
f(v'_{6t+2}) &= 1.
\end{aligned}
$$

Note that $A = \{v'_1, v'_4, v'_{6t+2}\} \cup A_1 \cup A'_1$, $B = \{v_2, v_4, v_{6t+2}, v'_3, v'_{6t+1}\} \cup B_1 \cup B_2 \cup B'_1 \cup B'_2$, and $C = \{v_1, v_3, v'_2\}$. If we proceed as in Case 1, it is easy to verify that we have a σ-tripartite labeling of M_{6t+2}. Figure 11 shows a σ-tripartite labeling of M_{14}.

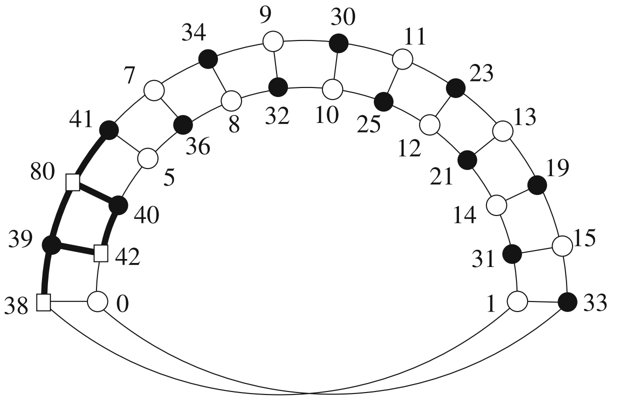


Fig. 11 A σ-tripartite labeling of M_{14}

Case 3 $n \equiv 4 \pmod 6$.
Let $n = 6t - 2$, where $t \geq 3$. Thus, $|V(M_n)| = 12t - 4$ and $|E(M_n)| = 18t - 6$. Define a one-to-one function $f : V(M_{6t-2}) \to [0, 36t - 12]$ as follows:

$$
\begin{aligned}
f(v_1) &= 18t - 6, \\
f(v_2) &= 18t - 7, \\
f(v_3) &= 36t - 16, \\
f(v_i) &= i - 1, && v_i \in A_1 = \{v_i : i \text{ odd},\ 5 \leq i \leq 6t - 3\}, \\
f(v_i) &= 18t - 2i - 1, && v_i \in B_1 = \{v_i : i \text{ even},\ 4 \leq i \leq 2t\}, \\
f(v_i) &= 18t - 2i - 3, && v_i \in B_2 = \{v_i : i \text{ even},\ 2t < i \leq 4t - 2\}, \\
f(v_i) &= 18t - 2i - 5, && v_i \in B_3 = \{v_i : i \text{ even},\ 4t - 2 < i \leq 6t - 4\}, \\
f(v_{6t-2}) &= 12t - 2, \\
f(v'_1) &= 0, \\
f(v'_2) &= 18t - 10, \\
f(v'_3) &= 18t - 8, \\
f(v'_i) &= i - 1, && v'_i \in A'_1 = \{v'_i : i \text{ even},\ 4 \leq i \leq 6t - 2\}, \\
f(v'_i) &= 18t - 2i - 1, && v'_i \in B'_1 = \{v'_i : i \text{ odd},\ 5 \leq i \leq 2t - 1\}, \\
f(v'_i) &= 18t - 2i - 3, && v'_i \in B'_2 = \{v'_i : i \text{ odd},\ 2t - 1 < i \leq 4t - 3\}, \\
f(v'_i) &= 18t - 2i - 5, && v'_i \in B'_3 = \{v'_i : i \text{ odd},\ 4t - 3 < i \leq 6t - 3\}.
\end{aligned}
$$

Note that $A = \{v'_1\} \cup A_1 \cup A'_1$, $B = \{v_2, v_{6t-2}, v'_3\} \cup B_1 \cup B_2 \cup B_3 \cup B'_1 \cup B'_2 \cup B'_3$, and $C = \{v_1, v_3, v'_2\}$. If we proceed as in Case 1, it is easy to verify that we have a σ-tripartite labeling of M_{6t-2}. Figure 12 shows a σ-tripartite labeling of M_{16}. □

Because it is known that bipartite prisms and bipartite Möbius ladders admit α-labelings and in light of our results here, we have the following.

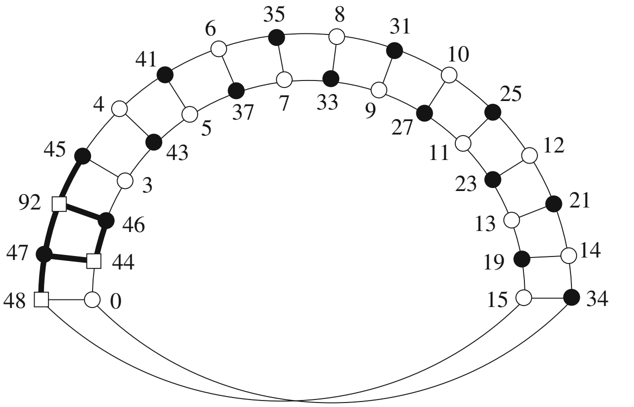


Fig. 12 A σ-tripartite labeling of M_{16}

Corollary 14 *If G of size n is a prism (other than D_3) or a Möbius ladder, then there exists a cyclic G-decomposition of K_{2nt+1}, of $K_{2nt} + I$, and of $K_{2nt+2} - I$ for all positive integers t.*

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