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On Regular Modules over Commutative Rings

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Abstract In this paper, we investigate the class of von Neumann regular modules over commutative rings. More precisely, we introduce a characterization of regular modules, and then, we study some properties of these modules in viewpoint of this characterization. Among other things, we show that the Nakayama's Lemma and Krull's intersection theorem hold for this class of modules. Also, some explicit expressions for submodules of regular modules are introduced.

Keywords Von Neumann regular ring · Regular module · Semisimple module · Krull's intersection theorem · Prime submodule

Mathematics Subject Classification 16E50 · 13C13 · 13C05 · 16D40 · 16D60 · 13C11

1 Introduction

A (not necessary commutative) ring R is called *(von Neumann) regular* if for each element a of R there exists an element x of R such that $axa = a$. The notion of regularity has been extended to modules by D. Fieldhouse [5] and R. Ware [21]. The former author considered arbitrary modules over rings with identity element while the

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latter author dealt with projective modules only. Their definitions agree for projectives. Fieldhouse [5] called a module M over a (not necessary commutative) ring R *regular* if each submodule N of M is *pure* in M, i.e., the inclusion $0 \to N \to M$ remains exact upon tensoring by any (right) R-module. Regular modules have been studied under different definitions by Ware [21], Zelmanowitz [23], and Ramamurthi and Rangaswamy [20]. We follow the definition used in [5] and [8].

Definition 1.1 A left R-module is (von Neumann) *regular* if every submodule is pure.

Remark 1.2 Every regular ring R as R-module is regular. Over any ring R, a semisimple module is regular (by a *semisimple* module we mean one which is a direct sum of simple submodules); see [8]. It is known (see [4] or [6]) that over a (not necessarily commutative) local ring each regular module is semisimple. Cheatham [4] proves that over a Noetherian ring each regular module is semisimple.

In this paper, we are going to investigate regular modules over commutative rings. In Sect. 2, we introduce a characterization of regular modules (Theorem 2.3). Some properties of regular modules are explored in viewpoint of this characterization. In Sect. 3, among other things, we show that the Nakayama's Lemma and Krull's intersection theorem hold for regular modules. Section 4 is devoted to find some explicit expressions for submodules of regular modules.

2 A New Characterization of von Neumann Regular Modules

Unless otherwise stated, after this point, we assume that R is a commutative ring with nonzero identity and all R-modules are unitary. Also, M is an R-module and N is a submodule of M. The basic properties of commutative regular rings are collected together in the following lemma. For a treatment of more general case, we refer the reader to [7].

Lemma 2.1

(1) R is regular if and only if every R-module is flat.
(2) $R/\sqrt{0}$ is regular if and only if every prime ideal of R is maximal.
(3) R is regular if and only if $R_{\mathfrak{m}}$ is a field for each maximal ideal $\mathfrak{m}$.
(4) Every homomorphic image of a regular ring is regular.
(5) If a local ring is regular, then it is a field.
(6) Over a commutative regular ring, each module has a maximal submodule.

It is well known that a commutative ring S is a regular ring if and only if every ideal in S coincides with its radical (for example, see [16, Theorem 49]). This motivates us to introduce a characterization of regular modules (see Theorem 2.3).

We begin by recalling some definitions. A proper submodule L of M is said to be *prime* if $rm \in L$, where $r \in R$ and $m \in M \setminus L$, then $r \in (L :_R M)$ (see [12,18]). If L is prime, then ideal $\mathfrak{p} := (L : M)$ is a prime ideal of R. In this case, L is said to be $\mathfrak{p}$-*prime*. The set of all prime submodules of M is called the *prime spectrum* of M and is denoted by $\mathrm{Spec}(M)$. Similarly, the collection of all $\mathfrak{p}$-prime submodules of M for any $\mathfrak{p} \in \mathrm{Spec}(R)$ is designated by $\mathrm{Spec}_{\mathfrak{p}}(M)$. The set of all prime submodules of M containing N is denoted by $V(N)$. The *radical* of N, denoted by $\mathrm{rad}_M(N)$ or briefly $\mathrm{rad}(N)$, is defined to be the intersection of all prime submodules of M containing N. In the case where there are no such prime submodules, $\mathrm{rad}(N)$ is defined as M. If $\mathrm{rad}(N) = N$, we say that N is a *radical submodule* (see [13,17]). The *saturation* of N with respect to a prime ideal $\mathfrak{p}$ of R, denoted by $S_{\mathfrak{p}}(N)$, is the kernel of the composite homomorphism

$$M \to M/N \to M_{\mathfrak{p}}/N_{\mathfrak{p}}$$

where the first homomorphism is the canonical homomorphism (see [3, p.69]). More precisely,

$$S_{\mathfrak{p}}(N) = \left\{m \in M \,\middle|\, sm \in N \text{ for some } s \in R \setminus \mathfrak{p}\right\}.$$

The following lemma is proved in [15, Proposition 5.1] and is quite useful for our purpose.

Lemma 2.2 *Let Y be a set of prime ideals of a ring R which contains all the maximal ideals, M an R-module, and $N < M$. Then, $N = \bigcap_{\mathfrak{p} \in Y} S_{\mathfrak{p}}(N)$.*

Now, we can introduce one of the main results of this paper.

Theorem 2.3 *An R-module M is regular if and only if* $\mathrm{rad}(N) = N$ *for all proper submodule N of M.*

Proof Let M be regular and N be a proper submodule of M. Then, by Lemma 2.2 we have $N = \cap_{\mathfrak{p} \in \mathrm{Max}(R)} S_{\mathfrak{p}}(N)$. According to [15, Theorem 2.1], we can write

$$N = \bigcap_{\mathfrak{p} \in \mathrm{Supp}(M/N) \cap \mathrm{Max}(R)} S_{\mathfrak{p}}(N). \tag{2.1}$$

Let $\mathfrak{p} \in \mathrm{Supp}(M/N) \cap \mathrm{Max}(R)$. Then, $N_{\mathfrak{p}}$ is a proper submodule of the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$. By [5, Theorem 11.2], $M_{\mathfrak{p}}$ is regular, and so Remark 1.2 implies that $M_{\mathfrak{p}}$ is semisimple. Therefore, $N_{\mathfrak{p}}$ is intersection of some maximal (so prime) submodules. Hence, in the light of [14, Proposition 1], it is easy to see that $S_{\mathfrak{p}}(N)$ is a radical submodule of M. This fact together with Eq. (2.1) shows that $N = \mathrm{rad}(N)$.

Conversely, let N be a submodule of M and I be an ideal of R. Then, it is enough for us to show that $IM \cap N = IN$ (see [5, Proposition 8.1]). By definition of prime submodules, it is clear that $V(IM \cap N) = V(IN)$. Thus, by assumption we have

$$IM \cap N = \mathrm{rad}(IM \cap N) = \mathrm{rad}(IN) = IN.$$

This completes the proof. □

Example 2.4 By Theorem 2.3, every cosemisimple module is regular, since an R-module M is *cosemisimple* if and only if every proper submodule of M is an intersection of maximal (so prime) submodules (see [22, Proposition 23.1]).

The following theorem was proved in [5], but we provide a new proof by means of Theorem 2.3.

Theorem 2.5 *The following conditions are equivalent:*

(1) R is a regular ring.
(2) Every R-module is regular.

Proof (1) $\Rightarrow$ (2) Let R be a regular ring and M be an R-module. Suppose that N is a proper submodule of M. Then, by Lemma 2.2 we have $N = \cap_{\mathfrak{p}\in \mathrm{Max}(R)} S_{\mathfrak{p}}(N)$. According to [15, Theorem 2.1], we can write

$$N = \bigcap_{\mathfrak{p}\in \mathrm{Supp}(M/N)\cap \mathrm{Max}(R)} S_{\mathfrak{p}}(N).$$

Let $\mathfrak{p} \in \mathrm{Supp}(M/N) \cap \mathrm{Max}(R)$. Then, $N_{\mathfrak{p}}$ is a proper submodule of the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$. By Lemma 2.1(3), $R_{\mathfrak{p}}$ is a field, and so $N_{\mathfrak{p}}$ is a prime submodule. Now, [14, Proposition 1] shows that $S_{\mathfrak{p}}(N)$ is a prime submodule of M. This yields that $N = \mathrm{rad}(N)$. Now, the result follows from Theorem 2.3.

(2) $\Rightarrow$ (1) This is true by [16, Theorem 49], because every radical submodule of R-module R is a radical ideal of R. □

According to Theorem 2.3, we will provide a new proof for [5, Theorems 8.4 and 11.2].

Proposition 2.6 *Let M be a regular R-module and S be a multiplicatively closed subset of R. Then, the following statements hold.*

(1) $S^{-1}M$ is a regular $S^{-1}R$-module.
(2) Any submodule of M is regular.
(3) Any homomorphic image of M is regular.

Proof (1) Let G be a proper submodule of $S^{-1}M$. Obviously, $G \subseteq \mathrm{rad}_{S^{-1}M}(G)$. Let $m/s \in \mathrm{rad}_{S^{-1}M}(G)$, where $m \in M$ and $s \in S$. Then, for each prime submodule P minimal over G we have $m/1 \in P$. Now, suppose that Q is a prime submodule of M minimal over $G^c := f^{-1}(G)$, where f denotes the canonical map $M \to S^{-1}M$.

We claim that $S^{-1}Q$ is a prime submodule of $S^{-1}M$ minimal over G. Recall that by [14, Proposition 1], $S^{-1}Q$ is a prime submodule of $S^{-1}M$ and $Q = (S^{-1}Q)^c$. If there exists a prime submodule H of $S^{-1}M$ such that $G \subseteq H \subseteq S^{-1}Q$, then by [14, Proposition 1] we have $G^c \subseteq H^c \subseteq Q$. Hence, $H^c = Q$, by minimality of Q, and so $H = S^{-1}Q$.

Therefore, $m/1 \in S^{-1}Q$. Since Q is prime, we infer that $m \in Q$. This implies that $m \in \mathrm{rad}_M(G^c) = G^c$, by Theorem 2.3. Consequently, $m/s \in S^{-1}(G^c) = G$. This yields that $G = \mathrm{rad}_{S^{-1}M}(G)$. Now, the result follows from Theorem 2.3.

(2) Let M be a regular R-module and N be a submodule of M. If $N = M$, then we are done. So, we assume that N is proper. Let L be a proper submodule of N. Then, by assumption, there is a family $\{P_\lambda\}_{\lambda\in\Lambda}$ of prime submodules of M such that $L = \bigcap_{\lambda\in\Lambda} P_\lambda$. If $P_\lambda \cap N = N$ for each $\lambda \in \Lambda$, then

$$N = \mathrm{rad}_M(N) \subseteq \bigcap_{\lambda\in\Lambda} P_\lambda = L \subseteq N,$$

a contradiction. This yields that

$$\Lambda' := \left\{\lambda \in \Lambda \,\middle|\, P_\lambda \cap N \neq N\right\} \neq \emptyset.$$

It is easy to see that $\{P_\lambda \cap N \mid \lambda \in \Lambda'\} \subseteq \mathrm{Spec}(N)$. Now, one can easily show that

$$L = \bigcap_{\lambda\in\Lambda'} (P_\lambda \cap N).$$

Consequently, N is a regular R-module.

(3) It is enough for us to show that M/N is a regular R-module for each submodule N. For this aim, suppose that L/N is a proper submodule of M/N. Then, $\mathrm{rad}_M(L) = L$ and we have

$$\mathrm{rad}_{M/N}(L/N) = \mathrm{rad}_M(L)/N = L/N.$$

This completes the proof. □

In particular, every ideal of a regular ring is regular. This provides an ample source of regular modules.

Corollary 2.7 *The following statements are equivalent:*

(1) M is a regular R-module.
(2) M/N is a regular R-module for each submodule N of M.

Proof Use Proposition 2.6. □

Also by Proposition 2.6, we have the following corollary.

Corollary 2.8 *Let $\{M_i\}_{i\in I}$ be a collection of R-modules. If $\bigoplus_{i\in I} M_i$ is a regular module, then each M_i $(i \in I)$ is a regular module.*

If R is a ring with a free regular module M, then we deduce from Corollary 2.8 that R is a regular ring. Now, we show that if M is a flat R-module, then we can say more than Corollary 2.7, as the next proposition illustrates.

Proposition 2.9 *Let M be a flat R-module. Then, the following statements are equivalent:*

(1) M is a regular R-module.
(2) Every homomorphic image of M is flat.
(3) Every homomorphic image of M is regular.

Proof In [11, p.133], it is proved that for a flat R-module M and a submodule N of M, M/N is flat if and only if $IM \cap N = IN$ for every ideal I of R. Thus, by [5, Proposition 8.1] M/N is flat if and only if N is pure. Therefore, the results follows from Proposition 2.6 and Theorem 2.3. □

As we mentioned in Lemma 2.1, the dimension of any regular ring is zero. Here, we show that this is also true for regular modules.

Theorem 2.10 *Let M be a regular R-module such that* $\text{Ass}(M) \neq \emptyset$. *Then,* $\text{Ass}(M) \subseteq \text{Max}(R)$. *In particular, we have* $\dim(M) = 0$.

Proof Let $\mathfrak{p} \in \text{Ass}(M)$. Then, there is a nonzero element $m \in M$ such that $Rm \cong R/\mathfrak{p}$. By Proposition 2.6, $R/\mathfrak{p}$ is a regular R-module. Hence, [8, Lemma 1] implies that $R/\mathfrak{p}$ is a regular domain. Since every prime ideal of a regular ring is maximal, we conclude that $\mathfrak{p}$ is a maximal ideal of R. Thus, $\text{Ass}(M) \subseteq \text{Max}(R)$. Therefore, $\text{Ass}(M) = \text{Supp}(M)$ and so $\dim(M) = 0$. □

It would be desirable to show that *every nonzero regular R-module has at least one associated prime ideal*, but we have not been able to do this.

Theorem 2.11 *Let M be a nonzero regular R-module. Then, M is Noetherian if and only if it is Artinian.*

Proof Since M is regular, we have

$$(0) = \text{rad}(0) = \bigcap_{P \text{ is minimal in } \text{Spec}(M)} P.$$

Let M be Artinian and P be a minimal prime submodule of M over (0). By [1, Corollary 2.4], $\mathfrak{m} := (P : M)$ is a maximal ideal of R and $\mathfrak{m}M \subseteq P \neq M$. Thus, $\mathfrak{m}M$ is a prime submodule of M (see [12, Proposition 2]). Therefore, $P = \mathfrak{m}M$ by minimality of P. Hence, there is a collection $\{\mathfrak{m}_\lambda\}_{\lambda\in\Lambda}$ of maximal ideals of R such that $\bigcap_{\lambda\in\Lambda}(\mathfrak{m}_\lambda M) = 0$. Since M is Artinian, there is a finite subset $\Lambda' := \{\mathfrak{m}_1, \dots, \mathfrak{m}_n\}$ of Λ such that $\bigcap_{i=1}^{n}(\mathfrak{m}_i M) = 0$. Consequently,

$$M = M/\text{rad}(0) = M/\bigcap_{i=1}^{n}(\mathfrak{m}_i M)$$

is annihilated by $\mathfrak{m}_1\mathfrak{m}_2\cdots\mathfrak{m}_n$. This implies that M is a Noetherian R-module.

Conversely, suppose that M is Noetherian. Then, M has finitely many minimal prime submodules over (0) (see [19, Theorem 4.2]), say $P_1, \dots, P_t$. By [12, Theorem 1], Corollary 2.7, and Theorem 2.10, $\{(P_i : M)\} = \text{Ass}(M/P_i) \subseteq \text{Max}(R)$ for each $1 \leq i \leq t$. Therefore,

$$M = M/\text{rad}(0) = M/\bigcap_{i=1}^{t} P_i$$

is annihilated by $(P_1 : M)(P_2 : M) \cdots (P_t : M)$. This implies that M is an Artinian R-module. □

Proposition 2.12 *The following statements are equivalent:*

(1) R is a regular ring.
(2) R possesses a projective regular module M such that $\mathrm{Max}(R) \subseteq \mathrm{Supp}(M)$.

Proof (1) $\Rightarrow$ (2) This is clear. (2) $\Rightarrow$ (1) Recall from Lemma 2.1(3) that a ring R is regular if and only if $R_{\mathfrak{m}}$ is a field for each maximal ideal $\mathfrak{m} \in \mathrm{Max}(R)$. Let $\mathfrak{m}$ be a maximal ideal of R. In view of Proposition 2.6 and [9, Theorem 2], $M_{\mathfrak{m}}$ is a (nonzero) free regular $R_{\mathfrak{m}}$-module. By Corollary 2.8, we deduce that $R_{\mathfrak{m}}$ is a regular ring. According to Lemma 2.1(5), $R_{\mathfrak{m}}$ is a field, as desired. □

An example of a (commutative) ring R which is not regular but which possesses a projective regular module can be find in [21, p.242].

3 The Nakayama's Lemma

Nakayama's lemma, which we now prove it for regular modules, is one of the key tools in commutative algebra.

Lemma 3.1 *Let M be a regular R-module and $\mathfrak{p} \in$* $\mathrm{Supp}(M)$. *Then, there exists a prime submodule P of M such that $\mathfrak{p} \subseteq (P : M)$.*

Proof Let $\mathfrak{p} \in \mathrm{Supp}(M)$. Then, there is a nonzero element m in M such that $\mathrm{Ann}(m) \subseteq \mathfrak{p}$. Thus, $m \notin \mathfrak{p}m$. Hence, $\mathfrak{p}m$ is a proper submodule of M and by assumption and Theorem 2.3 there is a prime submodule P of M such that $\mathfrak{p}m \subseteq P$ and $m \notin P$. This yields that $\mathfrak{p} \subseteq (P : M)$. □

Note that Lemma 3.1 shows that if M is a nonzero regular R-module, then $\mathrm{Spec}_{\mathfrak{m}}(M)$ is non-empty for some maximal ideal $\mathfrak{m}$ of R. Any theorem concerning regular modules over an arbitrary ring is a generalization of a corresponding theorem about modules over a regular ring (see Theorem 2.5). The following is such a result. Recall that the intersection of all maximal ideals of R, the *Jacobson radical* of R, is designated by $\mathrm{Rad}(R)$.

Theorem 3.2 (Nakayama's Lemma) *Let M be a regular R-module and I be an ideal of R such that $I \subseteq$* $\mathrm{Rad}(R)$. *If $IM = M$, then $M = 0$.*

Proof Suppose that M is nonzero. Then, there is a maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{m} \in \mathrm{Supp}(M)$. By Lemma 3.1, there exists a prime submodule P of M such that $\mathfrak{m} = (P : M)$. Therefore, we have

$$M = IM \subseteq \mathrm{Rad}(R)M \subseteq \mathfrak{m}M = (P : M)M \subseteq P \neq M,$$

a contradiction. □

Let us mention some consequences of Theorem 3.2.

Corollary 3.3 *Let M be a regular R-module and I be an ideal of R such that $I \subseteq \mathrm{Rad}(R)$. If M/IM is a finitely generated R-module, then so is M.*

Proof Suppose that M/IM is generated by $\{g_i + IM\}_{i=1}^n$ and let $H := Rg_1 + \cdots + Rg_n$. Now, if $m \in M$, then there are elements $r_1, \ldots, r_n \in R$ such that $m + IM = \sum_{i=1}^n r_i(g_i + IM)$. This implies that $m \in H + IM$. Hence, $H + IM = M$. Therefore, Theorem 3.2 yields that $M = H$ and so M is finitely generated, as desired. □

Corollary 3.3 shows that if R is a local ring with the unique maximal ideal $\mathfrak{m}$ and M is a regular R-module such that $M/\mathfrak{m}M$ is finitely generated, then the number of any minimal generating set of M is equal to the vector dimension of vector space $M/\mathfrak{m}M$ over the field $R/\mathfrak{m}$.

Proposition 3.4 *Let R be a local ring with the unique maximal ideal $\mathfrak{m}$ and M be a regular R-module. If there exists an R-module N such that $M \otimes_R N \cong R$, then $M \cong R$. In particular, N must then also be isomorphic to R.*

Proof By assumption, we get

$$R/\mathfrak{m} \cong R/\mathfrak{m} \otimes_R (M \otimes_R N) \cong M/\mathfrak{m}M \otimes_{R/\mathfrak{m}} N/\mathfrak{m}N.$$

It follows from this that both $M/\mathfrak{m}M$ and $N/\mathfrak{m}N$ are one dimensional vector spaces over $R/\mathfrak{m}$. By Corollary 3.3, M is generated by a single element m. We have a surjective map $R \longrightarrow M$ via $1 \mapsto m$. The kernel of this map is $\mathrm{Ann}(m)$. Since $\mathrm{Ann}(m)$ annihilates $M \otimes N$, it must annihilates R. This implies that $\mathrm{Ann}(m) = (0)$, and $M \cong R$. □

Proposition 3.5 *Let M and L be two regular R-module. Then,*

$$\mathrm{Supp}(M \otimes L) = \mathrm{Supp}(M) \cap \mathrm{Supp}(L).$$

Proof Obviously, $\mathrm{Supp}(M \otimes L) \subseteq \mathrm{Supp}(M) \cap \mathrm{Supp}(L)$. Hence, let

$$\mathfrak{p} \in (\mathrm{Supp}(M) \cap \mathrm{Supp}(L)) \setminus \mathrm{Supp}(M \otimes L).$$

Then, $M_\mathfrak{p} \neq 0$, $L_\mathfrak{p} \neq 0$ and $(M \otimes L)_\mathfrak{p} = 0$. This implies that

$$0 = (M \otimes_R L)_\mathfrak{p} \otimes_{R_\mathfrak{p}} R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \cong M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \otimes_{R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}} L_\mathfrak{p}/\mathfrak{p}L_p.$$

Therefore, for these vector spaces we have $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} = 0$ or $L_\mathfrak{p}/\mathfrak{p}L_\mathfrak{p} = 0$. By Proposition 2.6, $M_\mathfrak{p}$ and $L_\mathfrak{p}$ are regular $R_\mathfrak{p}$-module. Hence, we deduce from Theorem 3.2 that $M_\mathfrak{p} = 0$ or $L_\mathfrak{p} = 0$, a contradiction. □

Note that according to the proof of Proposition 3.5, its assertion is still true if at least of M and L is finitely generated.

Theorem 3.6 *Let $f : R \longrightarrow S$ be a homomorphism of commutative rings and M a regular S-module. If $M \otimes_R (R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) = 0$ for every $\mathfrak{p} \in \mathrm{Spec}(R)$, then $M = 0$.*

Proof If $M \neq 0$, then there is a maximal ideal $\mathfrak{m}$ of S such that $M_{\mathfrak{m}} \neq 0$. So by Proposition 2.6 and Theorem 3.2, $M_{\mathfrak{m}}/\mathfrak{m}M_{\mathfrak{m}} \neq 0$. Let $\mathfrak{p} = f^{-1}(\mathfrak{m})$. Then, $M_{\mathfrak{m}}/\mathfrak{p}M_{\mathfrak{m}} \neq 0$, since $\mathfrak{p}M_{\mathfrak{m}} \subseteq \mathfrak{m}M_{\mathfrak{m}}$. Set $T := S\setminus\mathfrak{m}$ and $G := R\setminus\mathfrak{p}$. Then, localization $G^{-1}M = M_{\mathfrak{p}}$ of M as an R-module and the localization $(f(G))^{-1}M$ of M as a S-module coincide. Since $f(G) \subseteq T$, we have

$$M_{\mathfrak{m}} = T^{-1}M \cong T^{-1}((f(G))^{-1}M) = T^{-1}(M_{\mathfrak{p}}).$$

Thus,

$$M_{\mathfrak{m}}/\mathfrak{p}M_{\mathfrak{m}} = T^{-1}(M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}) = T^{-1}(M \otimes_R (R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})).$$

This implies that $M \otimes_R (R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}) \neq 0$. □

Proposition 3.7 *Let R be a local ring with the unique maximal ideal $\mathfrak{m}$ and let M be a regular R-module. If* $\mathrm{Hom}_R(M, R/\mathfrak{m}) = 0$*, then* $M = 0$.

Proof Assume that $M \neq 0$. Since M is regular, Theorem 3.2 implies that $M/\mathfrak{m}M \neq 0$ as a vector space over the field $R/\mathfrak{m}$. Let $f : M/\mathfrak{m}M \to R/\mathfrak{m}$ be a nonzero $R/\mathfrak{m}$-homomorphism. If we compose this with the canonical map $M \to M/\mathfrak{m}M$, we get a nonzero R-homomorphism $f : M \to R/\mathfrak{m}$, so we have $\mathrm{Hom}_R(M, R/\mathfrak{m}) \neq 0$, a contradiction. □

Proposition 3.8 *Let M be an R-module and N be a regular R-module and $f \in$* $\mathrm{Hom}_R(M, N)$*. Then, f is onto if and only if for each* $\mathfrak{m} \in \mathrm{Max}(R)$*, the induced map $\bar{f} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is onto.*

Proof Let C be the cokernel of f. Then, the exact sequence

$$M \xrightarrow{f} N \longrightarrow C \longrightarrow 0$$

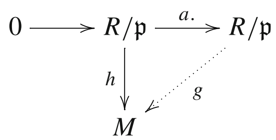
induces the sequence

$$M/\mathfrak{m}M \xrightarrow{\bar{f}} N/\mathfrak{m}N \longrightarrow C/\mathfrak{m}C \longrightarrow 0$$

which is exact. If f is onto, then $C = 0$, and so $\bar{f}$ is onto. Conversely, if $\bar{f}$ is onto for some maximal ideal $\mathfrak{m}$ of R, then $\mathfrak{m}C_{\mathfrak{m}} = C_{\mathfrak{m}}$. By Proposition 2.6, $C_{\mathfrak{m}}$ is a regular $R_{\mathfrak{m}}$-module. Thus, Theorem 3.2 implies that $C_{\mathfrak{m}} = 0$. Since this is true for all $\mathfrak{m} \in \mathrm{Max}(R)$, we infer that $C = 0$. □

Theorem 3.9 *Let R be a Noetherian local ring with the unique maximal ideal $\mathfrak{m}$. If M is a nonzero regular injective R-module, then* $\dim(R) = 0$.

Proof Suppose, contrary to our claim, that $\dim(R) \geq 1$. By assumption and Theorem 2.10, $\mathrm{Ass}(M) = \{\mathfrak{m}\}$. So, M has a submodule G isomorphic to $R/\mathfrak{m}$. Since $\dim(R) \geq 1$, there is a prime ideal $\mathfrak{p} \subset \mathfrak{m}$ and so $R/\mathfrak{p}$ admits a homomorphism onto $R/\mathfrak{m}$. Hence, $\mathrm{Hom}(R/\mathfrak{p}, M) \neq 0$. Let $a \in \mathfrak{m} \setminus \mathfrak{p}$. Then, a is a nonzero divisor on $R/\mathfrak{p}$. Since M is injective, for any homomorphism $h : R/\mathfrak{p} \longrightarrow M$ there is a homomorphism $g : R/\mathfrak{p} \longrightarrow M$ such that the following diagram is commutative.



Hence, $\mathrm{Hom}(R/\mathfrak{p}, M) = a\,\mathrm{Hom}(R/\mathfrak{p}, M)$. By Proposition 2.6, $\mathrm{Hom}(R/\mathfrak{p}, M)$ is a regular R-module. Now, Theorem 3.2 implies that $\mathrm{Hom}(R/\mathfrak{p}, M) = 0$, a contradiction. □

We say a subset $I \subseteq R$ *acts t-nilpotently on* M if, for every sequence $a_1, a_2, \ldots$ of elements in I and $m \in M$, we get $a_i a_{i-1} \cdots a_1 m = 0$ for some $i \in \mathbb{N}$ depending on m (see [22, p.257]).

Proposition 3.10 *Let a regular R-module M satisfy descending chain condition for cyclic submodules. Then,* $\mathrm{Rad}(R)$ *acts t-nilpotently on M.*

Proof Let $a_1, a_2, \ldots$ be a sequence of elements in $\mathrm{Rad}(R)$ and $m \in M$. Consider the descending chain of submodules $Ra_1m \supseteq Ra_2a_1m \supseteq Ra_3a_2a_1m \supseteq \cdots$. By assumption, there is $i \in \mathbb{N}$ such that

$$Ra_i a_{i-1} \cdots a_1 m = Ra_{i+1} a_i \cdots a_1 m \subseteq \mathrm{Rad}(R) a_i a_{i-1} \cdots a_1 m.$$

By Proposition 2.6 and Theorem 3.2, this means $a_i a_{i-1} \cdots a_1 m = 0$. □

Proposition 3.11 *Let R be a regular ring, $I \subseteq$* $\mathrm{Rad}(R)$ *be an ideal of R and M be an R-module. Then, $IM = 0$.*

Proof Consider the following short exact sequence.

$$0 \longrightarrow IM \longrightarrow M \longrightarrow M/IM \longrightarrow 0.$$

By Lemma 2.1, R/I is a flat R-module, and so we obtain the following short exact sequence:

$$0 \longrightarrow IM \otimes R/I \longrightarrow M \otimes R/I \longrightarrow M/IM \otimes R/I \longrightarrow 0.$$

Thus, the following sequence is exact.

$$0 \longrightarrow IM/I^2M \longrightarrow M/IM \longrightarrow M/IM \longrightarrow 0.$$

Hence, we deduce that $IM = I^2M$. Now, Theorem 3.2 implies that $IM = 0$, since IM is a regular R-module by Theorem 2.5. □

Let H be an R-module. Recall that an element $a \in R$ is said to be *H-regular* if $ax \neq 0$ for all $0 \neq x \in H$. Also, a sequence $a_1, \ldots, a_n$ of elements of R is an *H-sequence* (or an *H-regular sequence*) if the following two conditions hold:

(1) a_1 is H-regular, a_2 is (H/a_1H)-regular, ..., a_n is $(H/(a_1, \ldots, a_{n-1})H)$-regular;
(2) $H/(a_1, \ldots, a_n)H \neq 0$ (see [2, Definition 1.1.1]).

Let R be a Noetherian local ring and M be a finitely generated R-module. Then, every permutation of any M-sequence is an M-sequence (for example, see [2, Proposition 1.1.6]). As our next result shows, when M is a regular R-module (where R is an arbitrary ring), an M-sequence in the Jacobson radical is permutable. In the Noetherian case, regular sequences are finite. We do not know this for regular modules. Hence, the following proposition is to be interpreted as allowing infinite regular sequences if they exist.

Proposition 3.12 *Let M be a nonzero regular R-module and $\{x_\lambda\}_{\lambda \in \Lambda}$ be elements in* $\mathrm{Rad}(R)$ *constituting an M-sequence (finite or infinite). Then, any permutation of this M-sequence is also an M-sequence.*

Proof By [10, Theorem 118], it is enough for us to show that if a, b is an M-sequence in $\mathrm{Rad}(R)$, then b is not a zero-divisor on M. Let $C := (0 :_M b)$. It is easy to see that $aC = C$. Now, Theorem 3.2 implies that $C = 0$, as desired. □

Corollary 3.13 *Let R be a regular ring and M be an R-module. Then, any rearrangement of an M-sequence in Jacobson radical of R is again an M-sequence.*

Proof Use Theorem 2.5 and Proposition 3.12. □

The Krull's intersection theorem is one of the basic results in the theory of commutative Noetherian rings. The object of Corollary 3.16 is to prove this theorem and some of its consequence for the class of regular modules.

Proposition 3.14 *Let M be a nonzero regular R-module. Then, we have*

$$A := \bigcap \left\{ Q \in \mathrm{Spec}_{\mathfrak{p}}(M) \,\middle|\, \mathfrak{p} \in \mathrm{Max}(R) \right\} = 0.$$

Proof Note that by Lemma 3.1, $\mathrm{Spec}_{\mathfrak{m}}(M)$ is non-empty for some maximal ideal $\mathfrak{m}$ of R. Let x be a nonzero element in A. Then, there is a maximal ideal $\mathfrak{m}$ of R such that $\mathrm{Ann}(x) \subseteq \mathfrak{m}$. This implies that $x \notin \mathfrak{m}x$. So, $\mathfrak{m}x$ is a proper submodule of M. By Theorem 2.3, there is a prime submodule P of M such that $x \notin P$ and $\mathfrak{m}x \subseteq P$. Hence, we conclude that $P \in \mathrm{Spec}_{\mathfrak{m}}(M)$. This contradicts $x \in A$ and $x \notin P$. □

Corollary 3.15 *Let R be a local ring with the unique maximal ideal $\mathfrak{m}$ and M be a nonzero regular R-module. Then, every proper submodule of M is prime.*

Proof By Lemma 3.1, $\mathrm{Spec}_{\mathfrak{m}}(M)$ is non-empty. This implies that $\mathfrak{m}M$ is a prime submodule contained in every element of $\mathrm{Spec}_{\mathfrak{m}}(M)$ (see [12, Proposition 2]). We infer from Proposition 3.14 that $\mathfrak{m}M = 0$. Thus, $\mathrm{Ann}(M) = \mathfrak{m}$. Now, let N be a proper submodule of M. Then, $\mathfrak{m} = (N : M)$ and so N is a prime submodule by [12, Proposition 2]. □

Corollary 3.16 *Let M be a nonzero regular R-module. If r is in the Jacobson radical* $\mathrm{Rad}(R)$ *of R, then* $\bigcap_{n=1}^{\infty} r^n M = 0$.

Proof Let $r \in \mathrm{Rad}(R)$ and $P \in \mathrm{Spec}_{\mathfrak{m}}(M)$ for some maximal ideal $\mathfrak{m}$ of R. Then, for all $n \in \mathbb{N}$ we have

$$r^n M \subseteq \mathrm{Rad}(R)M \subseteq \mathfrak{m}M \subseteq P.$$

So, we infer from Proposition 3.14 that $\bigcap_{n=1}^{\infty} r^n M = 0$. □

Corollary 3.17 *Let M be a nonzero regular R-module, $\mathfrak{p}$ be a non-maximal prime ideal of R, and r be an element of* $\mathrm{Rad}(R) \setminus \mathfrak{p}$. *Then, the only possible $\mathfrak{p}$-prime submodule contained in rM is zero.*

Proof Let P be a $\mathfrak{p}$-prime submodule of M such that $P \subseteq rM$. We claim that $rP = P$. Obviously, $rP \subseteq P \subsetneq M$. Thus, rP is a proper submodule of the regular R-module M. By Theorem 2.3, there is a family $\{Q_\lambda\}_{\lambda \in \Lambda}$ of prime submodules of M such that

$$rP = \mathrm{rad}(rP) = \bigcap_{\lambda \in \Lambda} Q_\lambda.$$

So, it is enough for us to show that $P \subseteq Q_\lambda$ for each $\lambda \in \Lambda$. Suppose that, $P \nsubseteq Q_\alpha$ for some $\alpha \in \Lambda$. Hence, there exists $a \in P \setminus Q_\alpha$. This implies that

$$ar \in rP = \bigcap_{\lambda \in \Lambda} Q_\lambda \subseteq Q_\alpha.$$

Since Q_α is prime, we deduce that $r \in (Q_\alpha : M)$. Therefore, $P \subseteq rM \subseteq Q_\alpha$, a contradiction. This shows that $rP = P$. Therefore, $r^n P = P$ for all $n \in \mathbb{N}$. Now, by Corollary 3.16, we conclude that

$$P \subseteq \bigcap_{n=1}^{\infty} r^n M = (0).$$ □

4 An Explicit Expression for Submodules of Regular Modules

It is well known that every submodule of a semisimple module is an intersection of some maximal submodules. In Theorem 2.3, we established a similar result for regular modules where maximal submodules replaced with prime submodules. In this short section, we proceed further and we will introduce, as one of the main results of the paper, an explicit expression for submodules of regular modules.

Theorem 4.1 *Let M be a nonzero R-module. Then, M is regular if and only if*

$$N = \bigcap_{\mathfrak{p} \in \mathrm{Supp}(M/N)} S_{\mathfrak{p}}(N + \mathfrak{p}M) \tag{4.1}$$

for all proper submodules N of M.

Proof Suppose that M is regular. Let N be a proper submodule of M and $\mathrm{Min}(N)$ be the set of all minimal prime submodules of M over N. We claim that

$$\mathrm{Min}(N) = \left\{ S_{\mathfrak{p}}(N) \,\middle|\, \mathfrak{p} \in \mathrm{Supp}(M/N) \right\} = \left\{ S_{\mathfrak{p}}(N + \mathfrak{p}M) \,\middle|\, \mathfrak{p} \in \mathrm{Supp}(M/N) \right\}.$$

Suppose $\mathfrak{p} \in \mathrm{Supp}(M/N)$. Then, Proposition 2.6 implies that $M_{\mathfrak{p}}$ is a nonzero regular $R_{\mathfrak{p}}$-module. Moreover, Corollary 3.15 yields that the proper submodule $N_{\mathfrak{p}} = N_{\mathfrak{p}} + \mathfrak{p}M_{\mathfrak{p}}$ of $M_{\mathfrak{p}}$ is $\mathfrak{p}R_{\mathfrak{p}}$-prime. Now, by [14, Proposition 1], $S_{\mathfrak{p}}(N) = S_{\mathfrak{p}}(N + \mathfrak{p}M)$ is a $\mathfrak{p}$-prime submodule of M. According to [15, Result 3], $S_{\mathfrak{p}}(N) = S_{\mathfrak{p}}(N + \mathfrak{p}M)$ is a minimal prime submodule over N.

Now, let $Q \in \mathrm{Min}(N)$ be a $\mathfrak{q}$-prime submodule of M. Then, $N \subseteq S_{\mathfrak{q}}(N) \subseteq S_{\mathfrak{q}}(N + \mathfrak{q}M) \subseteq Q$. As we mentioned, $S_{\mathfrak{q}}(N)$ is a prime submodule of M. Hence, $Q = S_{\mathfrak{q}}(N) = S_{\mathfrak{q}}(N + \mathfrak{q}M)$ by minimality of Q. Consequently,

$$N = \mathrm{rad}(N) = \bigcap_{P \in \mathrm{Min}(N)} P = \bigcap_{\mathfrak{p} \in \mathrm{Supp}(M/N)} S_{\mathfrak{p}}(N) = \bigcap_{\mathfrak{p} \in \mathrm{Supp}(M/N)} S_{\mathfrak{p}}(N + \mathfrak{p}M).$$

Conversely, suppose that (4.1) holds for all proper submodules N of M. Let N be a proper submodule of M. If for all $\mathfrak{p} \in \mathrm{Supp}(M/N)$, we have $S_{\mathfrak{p}}(N + \mathfrak{p}M) = M$, then $N = M$, a contradiction. Suppose $\mathfrak{p} \in \mathrm{Supp}(M/N)$ such that $S_{\mathfrak{p}}(N + \mathfrak{p}M) \neq M$. This implies that

$$S_{\mathfrak{p}}(N + \mathfrak{p}M)/N = S_{\mathfrak{p}}(\mathfrak{p}(M/N)) \neq M/N.$$

In the light of [15, Corollary 3.7], we deduce that $S_{\mathfrak{p}}(\mathfrak{p}(M/N))$ is a prime submodule of M/N. So, $S_{\mathfrak{p}}(N + \mathfrak{p}M)$ is a prime submodule of M. Therefore, N is a radical submodule of M, i.e., $N = \mathrm{rad}(N)$. This completes the proof. □

Remark 4.2 Note that the proof of Theorem 4.1 shows that if M is a nonzero regular R-module, then

$$N = \bigcap_{\mathfrak{p} \in \mathrm{Supp}(M/N)} S_{\mathfrak{p}}(N)$$

for all proper submodules N of M.

Corollary 4.3 *Let M be a nonzero regular R-module. Then, the following statements hold.*

(1) For all proper submodules N of M, we have $N = \bigcap_{\mathfrak{p} \in V(N:M)} S_{\mathfrak{p}}(N)$.
(2) Let I be an ideal of R. Then, $IM = \bigcap_{\mathfrak{p} \in V(I)} S_{\mathfrak{p}}(IM)$. In particular, $\bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} S_{\mathfrak{p}}(0) = 0$.

Proof (1) Since $\mathrm{Supp}(M/N) \subseteq V(N:M)$, by Remark 4.2 and Theorem 4.1 we have

$$N \subseteq \bigcap_{\mathfrak{p}\in V(N:M)} S_{\mathfrak{p}}(N) \subseteq \bigcap_{\mathfrak{p}\in \mathrm{Supp}(M/N)} S_{\mathfrak{p}}(N) = N.$$

(2) Since $\mathrm{Supp}(M/IM) \subseteq V(IM:M) \subseteq V(I)$, the result follows from Theorem 4.1. □

When R is regular, we can say more than Theorem 4.1, as the next proposition illustrates.

Proposition 4.4 *Let R be a regular ring and N be a proper submodule of M. Then, $N = \bigcap_{\mathfrak{p}\in V(N:M)}(N+\mathfrak{p}M)$.*

Proof It follows from Theorem 4.1 that $N = \bigcap_{\mathfrak{p}\in \mathrm{Supp}(M/N)} S_{\mathfrak{p}}(N+\mathfrak{p}M)$. It is enough for us to show that for each $\mathfrak{p} \in \mathrm{Supp}(M/N)$, $N+\mathfrak{p}M$ is a $\mathfrak{p}$-prime submodule of M. By assumption and Theorem 2.5, M/N is a regular module, and so by Lemma 3.1 for each $\mathfrak{p} \in \mathrm{Supp}(M/N)$ there is a prime submodule P/N of M/N such that $\mathfrak{p} \subseteq (P:M)$. By Lemma 2.1, $\mathrm{Spec}(R) = \mathrm{Max}(R)$ and so $\mathfrak{p} = (P:M)$. This implies that

$$\mathfrak{p} \subseteq (\mathfrak{p}M:M) \subseteq (N+\mathfrak{p}M:M) \subseteq (P:M) = \mathfrak{p} \in \mathrm{Max}(R).$$

Hence, we can infer from [12, Proposition 2] that $N+\mathfrak{p}M$ is a $\mathfrak{p}$-prime submodule of M, and whence, [15, Result 2] implies that $S_{\mathfrak{p}}(N+\mathfrak{p}M) = N+\mathfrak{p}M$. Therefore,

$$\begin{aligned} N &\subseteq \bigcap_{\mathfrak{p}\in V(N:M)} S_{\mathfrak{p}}(N+\mathfrak{p}M) \subseteq \bigcap_{\mathfrak{p}\in \mathrm{Supp}(M/N)} S_{\mathfrak{p}}(N+\mathfrak{p}M) \\ &= \bigcap_{\mathfrak{p}\in \mathrm{Supp}(M/N)} (N+\mathfrak{p}M) = N. \end{aligned}$$

□

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