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# An Analytic Operator-Valued Generalized Feynman Integral on Function Space

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**Abstract** In this paper, we use a generalized Brownian motion process to define an analytic operator-valued Feynman integral. We then establish the existence of the analytic operator-valued generalized Feynman integral. We next investigate a stability theorem for the analytic operator-valued generalized Feynman integral.

**Keywords** Analytic operator-valued function space integral · Analytic operator-valued generalized Feynman integral · Stability theorem

**Mathematics Subject Classification** Primary 60J25 · 28C20

## 1 Introduction

Cameron and Storvick [1] introduced an analytic operator-valued function space integral and showed that the integral satisfied an integral equation related to the Schrödinger equation. The existence of this integral was established as an operator from $L_2(\mathbb{R})$ to $L_2(\mathbb{R})$. Since then, Johnson and Lapidus [8] established the existence of the operator-valued function space integral as a bounded linear operator on $L_2(\mathbb{R}^n)$

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for certain functionals which only define finite Borel measures on the compact interval $[0, T]$ in $\mathbb{R}$. These integrals are based on the Wiener integral associated with the Wiener process.

On the other hand, Johnson [7] studied a bounded convergence theorem (stability theorem) for the operator-valued Feynman integral of functionals of the form $F(x) = \exp\{\int_0^T \theta(x(s))\mathrm{d}s\}$. Chang et al. [2] established a stability theorems for the operator-valued Feynman integral of certain functionals involving some Borel measures on an interval $(0, T)$ as a bounded linear operator from $L_1(\mathbb{R})$ to $C_0(\mathbb{R})$. Moreover, Chang and Lee [5] studied an analytic operator-valued generalized Feynman integral. The integral investigated in [5] is based on the function space integral associated with a generalized Brownian motion process.

The function space $C_{a,b}[0, T]$ induced by a generalized Brownian motion was introduced by Yeh in [10] and was studied extensively in [3,4,6]. In this paper, we define an analytic operator-valued generalized Feynman integral on the function space $C_{a,b}[0, T]$. We then establish the existence of the analytic operator-valued generalized Feynman integral and investigate a stability theorem for the analytic operator-valued generalized Feynman integral.

## 2 Definitions and Preliminaries

Let $D = [0, T]$ and let $(\Omega, \mathcal{B}, P)$ be a probability measure space. A real-valued stochastic process $Y$ on $(\Omega, \mathcal{B}, P)$ and $D$ is called *a generalized Brownian motion process* if $Y(0, \omega) = 0$ almost everywhere and for $0 = t_0 < t_1 < \cdots < t_n \leq T$, the $n$-dimensional random vector $(Y(t_1, \omega), \ldots, Y(t_n, \omega))$ is normally distributed with density function

$$K(\vec{t}, \vec{\eta}) = \left((2\pi)^n \prod_{j=1}^{n} \big(b(t_j) - b(t_{j-1})\big)\right)^{-1/2} \times \exp\left\{-\frac{1}{2}\sum_{j=1}^{n} \frac{\big(\big(\eta_j - a(t_j)\big) - \big(\eta_{j-1} - a(t_{j-1})\big)\big)^2}{b(t_j) - b(t_{j-1})}\right\}$$

where $\vec{\eta} = (\eta_1, \ldots, \eta_n)$, $\eta_0 = 0$, $\vec{t} = (t_1, \ldots, t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0, T]$ with $a(0) = 0$, $a'(t) \in L^2[0, T]$, and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0, T]$.

As explained in [11, pp. 18–20], $Y$ induces a probability measure $\mu$ on the measurable space $(\mathbb{R}^D, \mathcal{B}^D)$ where $\mathbb{R}^D$ is the space of all real-valued functions $x(t)$, $t \in D$, and $\mathcal{B}^D$ is the smallest $\sigma$-algebra of subsets of $\mathbb{R}^D$ with respect to which all the coordinate evaluation maps $e_t(x) = x(t)$ defined on $\mathbb{R}^D$ are measurable. The triple $(\mathbb{R}^D, \mathcal{B}^D, \mu)$ is a probability measure space. This measure space is called the function space induced by the generalized Brownian motion process $Y$ determined by $a(\cdot)$ and $b(\cdot)$.

We note that the generalized Brownian motion process $Y$ determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(s,t) = \min\{b(s), b(t)\}$. By [11, Theorem 14.2], the probability measure $\mu$ induced by $Y$, taking a separable version, is supported by $C_{a,b}[0,T]$ (which is equivalent to the Banach space of continuous functions $x$ on $[0,T]$ with $x(0) = 0$ under the sup norm). Hence, $(C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$ is the function space induced by $Y$ where $\mathcal{B}(C_{a,b}[0,T])$ is the Borel $\sigma$-algebra of $C_{a,b}[0,T]$. We then complete this function space to obtain $(C_{a,b}[0,T], \mathcal{W}(C_{a,b}[0,T]), \mu)$ where $\mathcal{W}(C_{a,b}[0,T])$ is the set of all $\mu$-Carathéodory measurable subsets of $C_{a,b}[0,T]$.

We note that the coordinate process defined by $e_t(x) = x(t)$ on $C_{a,b}[0,T] \times [0,T]$ is also the generalized Brownian motion process determined by $a(t)$ and $b(t)$. For more detailed studies about this function space $C_{a,b}[0,T]$, see [3,6,10].

Next, we state the definition of the analytic operator-valued generalized Feynman integral.

**Definition 2.1** Let $\mathbb{C}$ be the set of complex numbers, let $\mathbb{C}_+ = \{\lambda \in \mathbb{C}: Re(\lambda) > 0\}$ and let $\tilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C}: Re(\lambda) \geq 0, \lambda \neq 0\}$. Also, let $C[0,T]$ denote the space of real-valued continuous functions $x$ on $[0,T]$, and given a real number $\alpha$, let $\nu_\alpha$ be the measure on $\mathcal{B}(\mathbb{R})$ such that $\mathrm{d}\nu_\alpha = \exp\{\alpha\eta^2\}\mathrm{d}\eta$. Next let $F$ be a $\mathbb{C}$-valued functional on $C[0,T]$. For each $\lambda > 0$, $\psi \in L^2(\mathbb{R}, \nu_\alpha)$ and $\xi \in \mathbb{R}$, assume that the functional $F(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}x(T) + \xi)$ is $\mu$-integrable with respect to $x$ on $C_{a,b}[0,T]$, and let

$$(I_\lambda(F)\psi)(\xi) = \int_{C_{a,b}[0,T]} F\left(\lambda^{-1/2}x + \xi\right)\psi\left(\lambda^{-1/2}x(T) + \xi\right)\mathrm{d}\mu(x).$$

If $I_\lambda(F)\psi$ is in $L^2(\mathbb{R}, \nu_{-\alpha})$ as a function of $\xi$ and if the correspondence $\psi \to I_\lambda(F)\psi$ gives an element of $\mathcal{L} \equiv \mathcal{L}(L^2(\mathbb{R}, \nu_\alpha), L^2(\mathbb{R}, \nu_{-\alpha}))$, the space of continuous linear operators from $L^2(\mathbb{R}, \nu_\alpha)$ to $L^2(\mathbb{R}, \nu_{-\alpha})$, we say that the operator-valued function space integral $I_\lambda(F)$ exists. Next, suppose that there exists an $\mathcal{L}$-valued function which is analytic in $\mathbb{C}_+$ and agrees with $I_\lambda(F)$ on $(0, \infty)$, then this $\mathcal{L}$-valued function is denoted by $I_\lambda^{\mathrm{an}}(F)$ and is called the analytic operator-valued function space integral of $F$ associated with $\lambda$. Finally, suppose that there exists an operator $J_q^{\mathrm{an}}$ in $\mathcal{L}(L^2(\mathbb{R}, \nu_\alpha), L^2(\mathbb{R}, \nu_{-\alpha}))$ for some $\alpha > 0$ such that

$$\left\| I_\lambda^{\mathrm{an}}(F)\psi - J_q^{\mathrm{an}}(F)\psi \right\|_{L^2(\mathbb{R}, \nu_{-\alpha})} \to 0$$

as $\lambda \to -iq$ through $\mathbb{C}_+$, then $J_q^{\mathrm{an}}(F)$ is called the analytic operator-valued generalized Feynman integral of $F$ with parameter $q$.

## 3 An Analytic Operator-Valued Function Space Integral

Throughout the rest of this paper, we consider functionals of the form

$$F(x) = f\left(\int_0^T \theta(s, x(s))\mathrm{d}\eta(s)\right), \tag{3.1}$$

where $f$ is an analytic function on $\mathbb{C}$ and $\theta$ is an appropriate $\mathbb{C}$-valued function on $[0, T] \times \mathbb{R}$. $F(x)$ is a very important functional in quantum mechanics. We then establish the existence of the analytic operator-valued function space integral for functionals $F$ of the form (3.1).

Let $\mathcal{M}(0, T)$ denote the space of complex Borel measures $\eta$ on the open interval $(0, T)$. Then $\eta \in \mathcal{M}(0, T)$ has a unique decomposition $\eta = \beta + \beta_d$ into its continuous part $\beta$ and its discrete part $\beta_d$ [9]. Let $\delta_\tau$ denote the Dirac measure at $\tau \in (0, T)$. For convenience, we let

$$\eta = \beta + \omega\delta_\tau, \quad \omega \in \mathbb{C}. \tag{3.2}$$

Throughout the rest of this paper, we use the following notations:

(1) For $\lambda \in \tilde{\mathbb{C}}_+$ and $\psi \in L^2(\mathbb{R}, \nu_\alpha)$, let

$$\left(C_{(\lambda,K,L)}\psi\right)(\xi) \equiv \left(\frac{\lambda}{2\pi K}\right)^{1/2} \int_{\mathbb{R}} \psi(u) \exp\left\{-\frac{1}{2K}\left(\sqrt{\lambda}(u-\xi) - L\right)^2\right\} \mathrm{d}u \tag{3.3}$$

where $K$ and $L$ are real numbers with $K > 0$. Then $C_{(\lambda,K,L)}$ is in $\mathcal{L}(L^2(\mathbb{R}, \nu_\alpha), L^2(\mathbb{R}, \nu_{-\alpha}))$.

(2) For each $s \in (0, T)$, let $\theta(s)$ denote the operator of multiplication from $L^2(\mathbb{R}, \nu_{-\alpha})$ to $L^2(\mathbb{R}, \nu_\alpha)$ given by

$$(\theta(s)\psi)(\xi) = \theta(s, \xi)\psi(\xi), \quad \xi \in \mathbb{R}. \tag{3.4}$$

(3) Given a positive integer $l_1$, let

$$\Delta_{l_1;j}(T) \equiv \left\{\left(s_1, \ldots, s_{l_1}\right) | 0 < s_1 < \cdots < s_j < \tau < s_{j+1} < \cdots < s_{l_1} < T\right\}$$

and let

$$\Delta_{l_1}(T) \equiv \left\{\left(s_1, \ldots, s_{l_1}\right) | 0 < s_1 < \cdots < s_{l_1} < T\right\}.$$

Also, for $(s_1, \ldots, s_{l_1}) \in \Delta_{l_1;j}(T)$ and a positive integer $l_2$, let

$$\begin{aligned} \mathcal{L}^\lambda_{l_1;j} \equiv\ & C_{(\lambda,b(s_1),a(s_1))} \circ \theta(s_1) \circ \cdots \circ \theta(s_j) \circ C_{(\lambda,b(\tau)-b(s_j),a(\tau)-a(s_j))} \circ [\theta(\tau)]^{l_2} \\ & \circ C_{(\lambda,b(s_{j+1})-b(\tau),a(s_{j+1})-a(\tau))} \circ \theta(s_{j+1}) \circ \cdots \circ \theta(s_{l_1-1}) \\ & \circ C_{(\lambda,b(s_{l_1})-b(s_{l_1-1}),a(s_{l_1})-a(s_{l_1-1}))} \circ \theta(s_{l_1}) \circ C_{(\lambda,b(T)-b(s_{l_1}),a(T)-a(s_{l_1}))}. \end{aligned} \tag{3.5}$$

Finally, for $(s_1, \ldots, s_{l_1}) \in \Delta_{l_1}(T)$, let

$$\mathcal{L}^\lambda_{l_1} \equiv C_{(\lambda,b(s_1),a(s_1))} \circ \theta(s_1) \circ \cdots \circ \theta(s_{l_1}) \circ C_{(\lambda,b(T)-b(s_{l_1}),a(T)-a(s_{l_1}))}. \tag{3.6}$$

For example, we see that for $s_1 \in \Delta_{1;1}(T) = \{s_1 | 0 < s_1 < \tau < T\}$,

$$\begin{aligned} \mathcal{L}^\lambda_{1;1} =\ & C_{(\lambda,b(s_1),a(s_1))} \circ \theta(s_1) \circ C_{(\lambda,b(\tau)-b(s_1),a(\tau)-a(s_1))} \circ [\theta(\tau)]^{l_2} \\ & \circ C_{(\lambda,b(T)-b(\tau),a(T)-a(\tau))} \end{aligned}$$

and for $(s_1, s_2) \in \Delta_2(T)$,

$$\mathcal{L}_2^{\lambda} = C_{(\lambda, b(s_1), a(s_1))} \circ \theta(s_1) \circ C_{(\lambda, b(s_2)-b(s_1), a(s_2)-a(s_1))} \circ \theta(s_2) \circ C_{(\lambda, b(T)-b(s_2), a(T)-a(s_2))}.$$

Hence using Eqs. (3.3)–(3.6), we observe that for $\psi \in L^2(\mathbb{R}, \nu_\alpha)$,

$$\left(\mathcal{L}_{1;1}^{\lambda} \circ \psi\right)(\xi) = \left(\prod_{j=1}^{3} \frac{\lambda}{2\pi(b(s_j) - b(s_{j-1}))}\right)^{1/2} \int_{\mathbb{R}^3} \theta(s_1, u_1)\,[\theta(\tau, u_2)]^{l_2}\,\psi(u_3) \times \exp\left\{-\sum_{j=1}^{3} \frac{\left[\left(\sqrt{\lambda}u_j - a(s_j)\right) - \left(\sqrt{\lambda}u_{j-1} - a(s_{j-1})\right)\right]^2}{2\left(b(s_j) - b(s_{j-1})\right)}\right\} \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3,$$

and

$$\left(\mathcal{L}_2^{\lambda} \circ \psi\right)(\xi) = \left(\prod_{j=1}^{3} \frac{\lambda}{2\pi\left(b(s_j) - b(s_{j-1})\right)}\right)^{1/2} \int_{\mathbb{R}^3} \prod_{j=1}^{2} \theta(s_j, u_j)\psi(u_3) \times \exp\left\{-\sum_{j=1}^{3} \frac{\left[\left(\sqrt{\lambda}u_j - a(s_j)\right) - \left(\sqrt{\lambda}u_{j-1} - a(s_{j-1})\right)\right]^2}{2\left(b(s_j) - b(s_{j-1})\right)}\right\} \mathrm{d}u_1 \mathrm{d}u_2 \mathrm{d}u_3$$

where $s_0 = 0$, $a(s_0) = 0$, $u_0 = \xi$, $s_2 = \tau$ and $s_3 = T$.

Also we will use the following conventions: for all positive integer $l$ and $\lambda \in \tilde{\mathbb{C}}_+$, let

$$B_j^l\left(s_j; |\lambda|\right) \equiv \left(\frac{M_j|\lambda|}{2\pi}\right)^{1/2} \int_{\mathbb{R}} \left|\left[\theta(s_j, u_j)\right]^l\right| \exp\left\{M_j\,|\lambda|^{1/2}\left|u_j\right|\right\} \mathrm{d}u_j \tag{3.7}$$

for some $M_j > 0$, $j = 1, \ldots, l_1$. Furthermore, in order to ensure that analytic operator-valued generalized Feynman integral exists, we will assume that $B_j^l(s_j; |\lambda|)$, $a(\cdot)$ and $b(\cdot)$ satisfy the following conditions: for $j = 1, \ldots, l_1$ and $s_{l_1+1} = T$,

(1) $\int_0^T B_j^l\left(s_j; |\lambda|\right) \mathrm{d}|\eta|(s) < \infty$
(2) $\frac{1}{b(s_j) - b(s_{j-1})} \le L_{jn}$
(3) $\left|a'(s_j^*)\right| \le \left|b'(s_j^*)\right| M_{jn}$

for $s_j^* \in (s_{j-1}, s_j)$ and some positive real numbers $L_{jn}$ and $M_{jn}$.

The next lemma plays a key role in the proof of Theorem 3.2.

**Lemma 3.1** *Let $\mathcal{L}^{\lambda}_{l_1;j}$ be given by Eq. (3.5). Then for all $l_2 \in \mathbb{N}$, $\xi \in \mathbb{R}$, $\lambda \in \tilde{\mathbb{C}}_+$ and $\psi \in L^2(\mathbb{R}, \nu_\alpha)$,*

$$\left|\left(\mathcal{L}^{\lambda}_{l_1;j} \circ \psi\right)(\xi)\right| \le \left(\frac{L_{Tn}^2|\lambda|^2}{\pi\alpha}\right)^{\frac{1}{4}} \exp\left\{\frac{M_{Tn}^2}{2\alpha}|\lambda| + M_{1n}|\lambda|^{1/2}|\xi|\right\} \|\psi\|_{L^2(\mathbb{R},\nu_\alpha)} \times B_1(s_1;|\lambda|)\cdots B_\tau^{l_2}(\tau;|\lambda|)\cdots B_{s_{l_1}}(s_{l_1};|\lambda|) \tag{3.8}$$

*for some $\alpha > 0$.*

*Proof* Using Eq. (3.3)–(3.5), we have that for all $l \in \mathbb{N}$, $\xi \in \mathbb{R}$ and $\lambda \in \tilde{\mathbb{C}}_+$

$$\begin{aligned}
&\left|(\mathcal{L}^{\lambda}_{l_1;j} \circ \psi)(\xi)\right| \\
&= \Bigg|\left(\frac{\lambda}{2\pi b(s_1)}\right)^{1/2} \times\cdots\times \left(\frac{\lambda}{2\pi\left(b(\tau)-b(s_j)\right)}\right)^{1/2} \times\cdots\times \left(\frac{\lambda}{2\pi\left(b(T)-b(s_{l_1})\right)}\right)^{1/2} \\
&\quad\times \int_{\mathbb{R}^{l_1+2}} \theta(s_1,u_1)\cdots[\theta(\tau,u_\tau)]^{l_2}\cdots\theta(s_{l_1},u_{l_1})\psi(u_{l_1+1}) \\
&\quad\times \exp\left\{-\frac{1}{b(s_1)}\left(\sqrt{\lambda}(u_1-\xi)-a(s_1)\right)^2-\cdots\right. \\
&\quad -\frac{1}{2\left(b(\tau)-b(s_j)\right)}\left(\sqrt{\lambda}(u_\tau-u_j)-\left(a(\tau)-a(s_j)\right)\right)^2-\cdots \\
&\quad\left. -\frac{1}{2\left(b(T)-b(s_{l_1})\right)}\left(\sqrt{\lambda}(u_{l_1+1}-u_{l_1})-\left(a(T)-a(s_{l_1})\right)\right)^2\right\} du_1\cdots du_\tau\cdots du_{l_1+1}\Bigg| \\
&\le \left(\frac{L_{1n}|\lambda|}{2\pi}\right)^{1/2}\times\cdots\times\left(\frac{L_{\tau n}|\lambda|}{2\pi}\right)^{1/2}\times\cdots\times\left(\frac{L_{Tn}|\lambda|}{2\pi}\right)^{1/2} \\
&\quad\times \int_{\mathbb{R}^{l_1+2}} |\theta(s_1,u_1)|\cdots\left|[\theta(\tau,u_\tau)]^{l_2}\right|\cdots\left|\theta(s_{l_1},u_{l_1})\right|\left|\psi(u_{l_1+1})\right| \\
&\quad\times \exp\left\{M_{1n}|\lambda|^{1/2}(|u_1|+|\xi|)+M_{2n}|\lambda|^{1/2}(|u_2|+|u_1|)+\cdots\right. \\
&\quad + M_{\tau n}|\lambda|^{1/2}(|u_\tau|+|u_j|)+\cdots \\
&\quad\left. + M_{Tn}|\lambda|^{1/2}(|u_{l_1+1}|+|u_{l_1}|)\right\} du_1\cdots du_\tau\cdots du_{l_1+1} \\
&\le \exp\{M_{1n}|\lambda|^{\frac{1}{2}}|\xi|\}\left(\frac{L_{Tn}|\lambda|}{2\pi}\right)^{1/2}\int_{\mathbb{R}} |\psi(u_{l_1+1})|\exp\{M_{Tn}|\lambda|^{1/2}|u_{l_1+1}|\}du_{l_1+1} \\
&\quad\times\left(\frac{L_{1n}|\lambda|}{2\pi}\right)^{1/2}\int_{\mathbb{R}}|\theta(s_1,u_1)|\exp\{2M_{1n}|\lambda|^{1/2}|u_1|\}du_1 \\
&\qquad\vdots \\
&\quad\times\left(\frac{L_{\tau n}|\lambda|}{2\pi}\right)^{1/2}\int_{\mathbb{R}}|[\theta(\tau,u_\tau)]^{l_2}|\exp\{2M_{\tau n}|\lambda|^{1/2}|u_\tau|\}du_\tau \\
&\qquad\vdots
\end{aligned}$$

$$\times\left(\frac{L_{s_{l_1 n}}|\lambda|}{2\pi}\right)^{1/2}\int_{\mathbb{R}}|\theta(s_{l_1},u_{l_1})|\exp\{2M_{s_{l_1 n}}|\lambda|^{1/2}|u_{l_1}|\}\mathrm{d}u_{l_1}$$
$$\leq\left(\frac{L_{Tn}^2|\lambda|^2}{\pi\alpha}\right)^{\frac{1}{4}}\exp\left\{M_{1n}|\lambda|^{1/2}|\xi|+\frac{M_{Tn}^2|\lambda|}{2\alpha}\right\}\|\psi\|_{L^2(\mathbb{R},\nu_\alpha)}$$
$$\times B_1(s_1;|\lambda|)\cdots B_\tau^{l_2}(\tau;|\lambda|)\cdots B_{s_{l_1}}(s_{l_1};|\lambda|),$$

which completes the proof of Lemma 3.1. □

In our next theorem, we establish the existence of the analytic operator-valued function space integral for the functional $F$ given by (3.1) with $f(z)=z^n$.

**Theorem 3.2** *Let $\theta$ be a Borel measurable function on $[0,T]\times\mathbb{R}$. For $n=1,2,\ldots$, let*

$$F_n(x)=\left(\int_0^T\theta(s,x(s))d\eta(s)\right)^n. \tag{3.9}$$

*Let $\eta$ be given by (3.2). Then for all $\lambda\in\mathbb{C}_+$ and $\psi\in L^2(\mathbb{R},\nu_\alpha)$, the analytic operator-valued function space integral of $F_n$, $I_\lambda^{\mathrm{an}}(F_n)$, exists and is given by the formula*

$$\left(I_\lambda^{\mathrm{an}}(F_n)\psi\right)(\xi)=\sum_{\substack{l_1+l_2=n\\ l_2\neq 0}}\frac{n!\omega^{l_2}}{l_2!}\sum_{j=0}^{l_1}\int_{\Delta_{l_1;j}(T)}\left(\mathcal{L}_{l_1;j}^{\lambda}\circ\psi\right)(\xi)\mathrm{d}\prod_{l=1}^{l_1}\beta(s_l) \tag{3.10}$$

*where $\beta(s_0)=0$, $s_{l_1+1}=T$ and $\Delta_{0;j}(T)$ is an empty set.*

*Proof* Using Eq. (3.1) with $f(z)=z^n$, (3.3), (3.4) and the Fubini theorem, we first obtain that for all $\lambda>0$

$$\begin{aligned}(I_\lambda(F)\psi)(\xi)&=\int_{C_{a,b}[0,T]}F\left(\lambda^{-1/2}x+\xi\right)\psi\left(\lambda^{-1/2}x(T)+\xi\right)\mathrm{d}\mu(x)\\
&=\int_{C_{a,b}[0,T]}\left(\int_0^T\theta\left(s,\lambda^{-1/2}x(s)+\xi\right)\mathrm{d}\eta(s)\right)^n\psi\left(\lambda^{-1/2}x(T)+\xi\right)\mathrm{d}\mu(x)\\
&=\int_{C_{a,b}[0,T]}\left(\int_0^T\theta\left(s,\lambda^{-1/2}x(s)+\xi\right)\mathrm{d}\beta(s)+\omega\cdot\theta\left(\tau,\lambda^{-1/2}x(\tau)+\xi\right)\right)^n\\
&\quad\times\psi\left(\lambda^{-1/2}x(T)+\xi\right)\mathrm{d}\mu(x)\\
&=\int_{C_{a,b}[0,T]}\sum_{\substack{l_1+l_2=n\\ l_2\neq 0}}\frac{n!}{l_1!l_2!}\left(\int_0^T\theta\left(s,\lambda^{-1/2}x(s)+\xi\right)\mathrm{d}\beta(s)\right)^{l_1}\\
&\quad\times\left(\omega\cdot\theta\left(\tau,\lambda^{-1/2}x(\tau)+\xi\right)\right)^{l_2}\psi\left(\lambda^{-1/2}x(T)+\xi\right)\mathrm{d}\mu(x)\\
&=\sum_{\substack{l_1+l_2=n\\ l_2\neq 0}}\frac{n!\omega^{l_2}}{l_2!}\sum_{j=0}^{l_1}\int_{\Delta_{l_1;j}(T)}\left[\int_{C_{a,b}[0,T]}\theta\left(s_1,\lambda^{-1/2}x(s_1)+\xi\right)\times\cdots\right.\\
&\quad\times\theta\left(s_j,\lambda^{-1/2}x(s_j)+\xi\right)\left[\theta\left(\tau,\lambda^{-1/2}x(\tau)+\xi\right)\right]^{l_2}\end{aligned}$$

$$\times\ \theta\left(s_{j+1}, \lambda^{-1/2}x(s_{j+1})+\xi\right)\times\cdots$$
$$\times\,\theta\left(s_{l_1}, \lambda^{-1/2}x(s_{l_1})+\xi\right)\psi\left(\lambda^{-1/2}x(T)+\xi\right)\mathrm{d}\mu(x)\Big]\,\mathrm{d}\beta(s_1)\cdots\mathrm{d}\beta(s_{l_1})$$
$$=\sum_{\substack{l_1+l_2=n\\ l_2\neq 0}}\frac{n!\omega^{l_2}}{l_2!}\sum_{j=0}^{l_1}\int_{\Delta_{l_1;j}(T)}\left(\mathcal{L}^{\lambda}_{l_1;j}\circ\psi\right)(\xi)\mathrm{d}\prod_{l=1}^{l_1}\beta(s_l).$$

Next we will show that the existence of analytic operator-valued function space integral $I^{\text{an}}_{\lambda}(F_n)$ exists. Using Eq. (3.8), we obtain that for all $\lambda\in\mathbb{C}_+$

$$\sum_{\substack{l_1+l_2=n\\ l_2\neq 0}}\frac{n!\omega^{l_2}}{l_2!}\sum_{j=0}^{l_1}\int_{\Delta_{l_1;j}(T)}|(\mathcal{L}^{\lambda}_{l_1;j}\circ\psi)(\xi)|\mathrm{d}\prod_{l=1}^{l_1}|\beta|(s_l)$$
$$=\left(\frac{L^2_{Tn}|\lambda|^2}{\pi\alpha}\right)^{\frac{1}{4}}\exp\left\{M_{1n}|\lambda|^{1/2}|\xi|+\frac{M^2_{Tn}|\lambda|}{2\alpha}\right\}\|\psi\|_{L^2(\mathbb{R},\nu_\alpha)}$$
$$\times\sum_{\substack{l_1+l_2=n\\ l_2\neq 0}}\frac{n!\omega^{l_2}}{l_2!}\sum_{j=0}^{l_1}\int_{\Delta_{l_1;j}(T)}B_1(s_1;|\lambda|)\times\cdots\times B^{l_2}_{\tau}(\tau;|\lambda|)\times\cdots$$
$$\times\,B_{s_{l_1}}(s_{l_1};|\lambda|)\mathrm{d}|\beta|(s_1)\cdots\mathrm{d}|\beta|(s_{l_1})$$
$$=\left(\frac{L^2_{Tn}|\lambda|^2}{\pi\alpha}\right)^{\frac{1}{4}}\exp\left\{M_{1n}|\lambda|^{1/2}|\xi|+\frac{M^2_{Tn}|\lambda|}{2\alpha}\right\}\|\psi\|_{L^2(\mathbb{R},\nu_\alpha)}$$
$$\times\,n!\sum_{\substack{l_1+l_2=n\\ l_2\neq 0}}\frac{1}{l_1!l_2!}\left(\int_0^T B(s;|\lambda|)\mathrm{d}|\beta|(s)\right)^{l_1}(\omega B(\tau;|\lambda|))^{l_2}\Big]$$
$$=\left(\frac{L^2_{Tn}|\lambda|^2}{\pi\alpha}\right)^{\frac{1}{4}}\exp\left\{M_{1n}|\lambda|^{1/2}|\xi|+\frac{M^2_{Tn}|\lambda|}{2\alpha}\right\}\|\psi\|_{L^2(\mathbb{R},\nu_\alpha)}$$
$$\times\left(\int_0^T B(s;|\lambda|)\mathrm{d}|\beta|(s)+\omega B(\tau;|\lambda|)\right)^n$$
$$=\left(\frac{L^2_{Tn}|\lambda|^2}{\pi\alpha}\right)^{\frac{1}{4}}\exp\left\{M_{1n}|\lambda|^{1/2}|\xi|+\frac{M^2_{Tn}|\lambda|}{2\alpha}\right\}\|\psi\|_{L^2(\mathbb{R},\nu_\alpha)}$$
$$\times\left(\int_0^T B(s;|\lambda|)\mathrm{d}|\eta|(s)\right)^n<\infty. \tag{3.11}$$

Therefore, the analytic operator-valued function space integral $I^{\text{an}}_{\lambda}(F_n)$ exists and is given by Eq. (3.10).

Now we will show that $I^{\text{an}}_{\lambda}(F_n)$ is an element of $\mathcal{L}(L^2(\mathbb{R},\nu_\alpha), L^2(\mathbb{R},\nu_{-\alpha}))$. Using Eqs. (3.10) and (3.11), it follows that

$$\begin{aligned}
\left\| I_\lambda^{\mathrm{an}}(F_n)\psi \right\|^2_{L^2(\mathbb{R},\nu_{-\alpha})} &= \int_{\mathbb{R}} \left| \left( I_\lambda^{\mathrm{an}}(F_n)\psi \right)(\xi) \right|^2 \mathrm{d}\nu_{-\alpha}(\xi) \\
&= \left( \frac{L_{Tn}^2 |\lambda|^2}{\pi\alpha} \right)^{1/2} \|\psi\|^2_{L^2(\mathbb{R},\nu_\alpha)} \left( \int_0^T B(s;|\lambda|)\mathrm{d}|\eta|(s) \right)^{2n} \exp\left\{ \frac{M_{Tn}^2|\lambda|}{\alpha} \right\} \\
&\quad \times \int_{\mathbb{R}} \exp\left\{ M_{1n}|\lambda|^{1/2}|\xi| \right\} \mathrm{d}\nu_{-\alpha}(\xi) \\
&\le \left( \frac{4L_{Tn}^2 |\lambda|^2}{\alpha^2} \right)^{1/2} \|\psi\|^2_{L^2(\mathbb{R},\nu_\alpha)} \left( \int_0^T B(s;|\lambda|)\mathrm{d}|\eta|(s) \right)^{2n} \exp\left\{ \frac{|\lambda| \left( M_{1n}^2 + M_{Tn}^2 \right)}{\alpha} \right\}.
\end{aligned} \tag{3.12}$$

Hence, we obtain that for all $\lambda \in \mathbb{C}_+$,

$$\left\| I_\lambda^{\mathrm{an}}(F_n) \right\| \le \left( \frac{4L_{Tn}^2 |\lambda|^2}{\alpha^2} \right)^{\frac{1}{4}} \left( \int_0^T B(s;|\lambda|)\mathrm{d}|\eta|(s) \right)^{n} \exp\left\{ \frac{|\lambda|(M_{1n}^2 + M_{Tn}^2)}{\alpha} \right\}.$$

Thus, the theorem is proved. □

Let $f(z) = \sum_{n=1}^{\infty} \beta_n z^n$ be an analytic function on $\mathbb{C}$ such that

$$\sum_{n=1}^{\infty} |\beta_n| \Psi_n^k(|\lambda|) < \infty \tag{3.13}$$

for all $\lambda \in \tilde{\mathbb{C}}_+$, where

$$\Psi_n^k(|\lambda|) \equiv \left( \frac{4L_{Tn}^2 |\lambda|^2}{\alpha^2} \right)^{\frac{1}{4}} \left( \int_0^T B^k(s;|\lambda|)\mathrm{d}|\eta|(s) \right)^{n} \exp\left\{ \frac{|\lambda| \left( M_{1n}^2 + M_{Tn}^2 \right)}{\alpha} \right\} \tag{3.14}$$

for all positive integers $n$ and $k$. Let

$$F(x) = f\left( \int_0^T \theta(s, x(s))\mathrm{d}\eta(s) \right) \tag{3.15}$$

for $x \in C_{a,b}[0,T]$.

Our aim in this section is to establish the existence of the analytic operator-valued function space integral for the functionals $F$ given by (3.15).

**Theorem 3.3** *Let $F$ be given by Eq. (3.15). Then for all $\lambda \in \mathbb{C}_+$ and $\psi \in L^2(\mathbb{R}, \nu_\alpha)$, the analytic operator-valued function space integral of $F$, $I_\lambda^{\mathrm{an}}(F)$, exists and is given by the formula*

$$I_\lambda^{\mathrm{an}}(F)\psi = \sum_{n=1}^{\infty} \beta_n I_\lambda^{\mathrm{an}}(F_n)\psi$$

*where* $I_\lambda^{\text{an}}(F_n)$ *is given by Eq.* (3.10). *Furthermore,* $I_\lambda^{\text{an}}(F)$ *is an element of* $\mathcal{L}(L^2(\mathbb{R}, \nu_\alpha), L^2(\mathbb{R}, \nu_{-\alpha}))$.

*Proof* Since $F(x) = \sum_{n=1}^{\infty} \beta_n F_n(x)$, using (3.11) and (3.12) we have

$$I_\lambda^{\text{an}}(F)\psi = \sum_{n=1}^{\infty} \beta_n I_\lambda^{\text{an}}(F_n)\psi$$

and

$$\left\| I_\lambda^{\text{an}}(F)\psi \right\|_{L^2(\mathbb{R},\nu_{-\alpha})} \leq \sum_{n=1}^{\infty} |\beta_n| \Psi_n^1(|\lambda|) \|\psi\|_{L^2(\mathbb{R},\nu_\alpha)}$$

where $\Psi_n^1(|\lambda|)$ is given by Eq. (3.14) with $k = 1$. Next using the condition (3.13), the analytic operator-valued function space integral $I_\lambda^{\text{an}}(F)$ exists and $I_\lambda^{\text{an}}(F)$ is an element of $\mathcal{L}(L^2(\mathbb{R}, \nu_\alpha), L^2(\mathbb{R}, \nu_{-\alpha}))$. □

## 4 An Analytic Operator-Valued Generalized Feynman Integral

In Sect. 3, we established the existence of the analytic operator-valued function space integral for the functionals $F$ given by Eq. (3.15). In this section, we establish the existence of the analytic operator-valued generalized Feynman integral for the functionals $F$. To do this, in Theorem 4.1, we first obtain the analytic operator-valued generalized Feynman integral for the functionals $F_n$ given by (3.9).

**Theorem 4.1** *Let* $F_n$ *be given by Eq.* (3.9). *Then for all* $q \in \mathbb{R}\backslash\{0\}$, *the analytic operator-valued generalized Feynman integral of* $F_n$, $J_q^{\text{an}}(F_n)$, *exists and is given by the formula*

$$(J_q^{\text{an}}(F_n)\psi)(\xi) = \sum_{\substack{l_1+l_2=n \\ l_2\neq 0}} \frac{n!\omega^{l_2}}{l_2!} \sum_{j=0}^{l_1} \int_{\Delta_{l_1;j}(T)} \left(\mathcal{L}_{l_1;j}^{-iq} \circ \psi\right)(\xi) d\prod_{l=1}^{l_1} \beta(s_l) \tag{4.1}$$

*where* $\beta(s_0) = 0$, $s_{l_1+1} = T$ *and* $\Delta_{0;j}(T)$ *is an empty set.*

*Proof* In order to establish Eq. (4.1), it suffices to show that

$$\lim_{\lambda\to -iq} \int_{\mathbb{R}} \left| \left(I_\lambda^{\text{an}}(F_n)\right)(\psi) - \left(J_q^{\text{an}}(F_n)\right)(\psi) \right|^2 \mathrm{d}\nu_{-\alpha}(\xi) = 0.$$

But, for all $\lambda \in \mathbb{C}_+$, we have

$$\left| \left(I_\lambda^{\text{an}}(F_n)\right)(\psi) - \left(J_q^{\text{an}}(F_n)\right)(\psi) \right|^2 \leq 2\left|\left(I_\lambda^{\text{an}}(F_n)\right)(\psi)\right|^2 + 2\left|\left(J_q^{\text{an}}(F_n)\right)(\psi)\right|^2. \tag{4.2}$$

Using a similar method as those used in (3.12), we also see that $|(I_\lambda^{\text{an}}(F_n))(\psi)|^2$ and $|(J_q^{\text{an}}(F_n))(\psi)|^2$ are in $L^1(\mathbb{R}, \nu_{-\alpha})$. Hence, the second expression in Eq. (4.2) is in $L^1(\mathbb{R}, \nu_{-\alpha})$. Thus, using the dominated convergence theorem, we obtain the desired result. □

The next theorem is one of the main results in this paper.

**Theorem 4.2** *Let $F$ be given by Eq. (3.15). Then for all $q \in \mathbb{R}\backslash\{0\}$, the analytic operator-valued generalized Feynman integral of $F$, $J_q^{\text{an}}(F)$, exists and is given by the formula*

$$J_q^{\text{an}}(F)\psi = \sum_{n=1}^{\infty} \beta_n J_q^{\text{an}}(F_n)\psi \tag{4.3}$$

*where $J_q^{\text{an}}(F_n)$ is given by Eq. (4.1). Furthermore, $J_q^{\text{an}}(F)$ is an element of $\mathcal{L}(L^2(\mathbb{R}, \nu_\alpha), L^2(\mathbb{R}, \nu_{-\alpha}))$.*

*Proof* Using (3.11) and (3.12) with $\lambda$ replaced with $-iq$, we obtain

$$J_q^{\text{an}}(F)\psi = \sum_{n=1}^{\infty} \beta_n J_q^{\text{an}}(F_n)\psi$$

and

$$\left\| J_q^{\text{an}}(F)\psi \right\|_{L^2(\mathbb{R},\nu_{-\alpha})} \le \sum_{n=1}^{\infty} |\beta_n| \Psi_n^1(|-iq|) \, \|\psi\|_{L^2(\mathbb{R},\nu_\alpha)}$$

where $\Psi_n^1(|-iq|)$ is given by Eq. (3.14) with $k = 1$. Next using the condition (3.13), we conclude that the analytic operator-valued generalized Feynman integral $J_q^{\text{an}}(F)$ exists and $J_q^{\text{an}}(F)$ is an element of $\mathcal{L}(L^2(\mathbb{R}, \nu_\alpha), L^2(\mathbb{R}, \nu_{-\alpha}))$. □

The next two lemmas play key roles in the proof of Theorem 4.5.

**Lemma 4.3** *For each $k = 1, 2, \ldots$, let $F_n^{(k)}$ be given by (3.9) with $\theta$ replaced with $\theta^{(k)}$. Then for all $q \in \mathbb{R}\backslash\{0\}$, the analytic operator-valued generalized Feynman integral of $F_n^{(k)}$, $J_q^{\text{an}}(F_n^{(k)})$, exists and is given by the formula*

$$\left(J_q^{\text{an}}(F_n^{(k)})(\psi)\right) = \sum_{\substack{l_1+l_2=n \\ l_2\neq 0}} \frac{n!\omega^{l_2}}{l_2!} \sum_{j=0}^{l_1} \int_{\Delta_{l_1;j}(T)} \left(\mathcal{L}_{l_1;j;k}^{-iq} \circ \psi\right)(\xi) \mathrm{d} \prod_{l=1}^{l_1} \beta(s_l)$$

*where $\mathcal{L}_{l_1;j;k}^{-iq}$ is given by the right-hand side of Eq. (3.5) with $\theta$ replaced by $\theta^{(k)}$. Furthermore, we have*

$$J_q^{\text{an}}\left(F^{(k)}\right)\psi = \sum_{n=1}^{\infty} \beta_n J_q^{\text{an}}\left(F_n^{(k)}\right)\psi \tag{4.4}$$

*where* $F^{(k)} : C_{a,b}[0,T] \to \mathbb{C}$ *is given by*

$$F^{(k)}(x) = f\left(\int_0^T \theta^{(k)}(s, x(s))\mathrm{d}\eta(s)\right) \tag{4.5}$$

*for each* $k = 1, 2, \ldots$.

*Proof* The proof is straightforward by replacing $\theta$ with $\theta^{(k)}$ in Theorem 4.1. □

**Lemma 4.4** *Let* $F_n^{(k)}$ *be as in Lemma 4.3. Then for all* $q \in \mathbb{R}\backslash\{0\}$ *and* $\psi \in L^2(\mathbb{R}, \nu_\alpha)$*,*

$$\left\| J_q^{\mathrm{an}}(F_n^{(k)})\psi - J_q^{\mathrm{an}}(F_n)\psi \right\|_{L^2(\mathbb{R},\nu_{-\alpha})} \to 0 \quad as \quad k \to \infty. \tag{4.6}$$

*Proof* To establish Eq. (4.6) it will suffice to show that

$$\lim_{k\to\infty} \int_{\mathbb{R}} \left| \left(J_q^{\mathrm{an}}(F_n^{(k)})\psi\right)(\xi) - \left(J_q^{\mathrm{an}}(F_n)\psi\right)(\xi) \right|^2 \mathrm{d}\nu_{-\alpha}(\xi) = 0$$

for all $\psi \in L^2(\mathbb{R}, \nu_\alpha)$. But using similar methods as those used in (3.11), it follows that for each $n \in \mathbb{N}$,

$$\begin{aligned}
&\left| \left(J_q^{\mathrm{an}}(F_n^{(k)})\psi\right)(\xi) - \left(J_q^{\mathrm{an}}(F_n)\psi\right)(\xi) \right|^2 \\
&\quad \le 2\left| \left(J_q^{\mathrm{an}}(F_n^{(k)})\psi\right)(\xi) \right|^2 + 2\left| \left(J_q^{\mathrm{an}}(F_n)\psi\right)(\xi) \right|^2 \\
&\quad \le 2\left(\frac{L_{Tn}^2 q^2}{\pi\alpha}\right)^{1/2} \|\psi\|_{L^2(\mathbb{R},\nu_\alpha)}^2 \exp\left\{2M_{1n}\sqrt{|q|}|\xi| + \frac{M_{Tn}^2}{\alpha}|q|\right\} \\
&\qquad \times \left(\int_0^T B^{(k)}(s; |-iq|)\mathrm{d}|\eta|(s)\right)^{2n} \\
&\qquad + 2\left(\frac{L_{Tn}^2 q^2}{\pi\alpha}\right)^{1/2} \|\psi\|_{L^2(\mathbb{R},\nu_\alpha)}^2 \exp\left\{2M_{1n}\sqrt{|q|}|\xi| + \frac{M_{Tn}^2}{\alpha}|q|\right\} \\
&\qquad \times \left(\int_0^T B(s; |-iq|)\mathrm{d}|\eta|(s)\right)^{2n}
\end{aligned} \tag{4.7}$$

where $B^{(k)}$ is given by Eq. (3.7) with $\theta$ replaced with $\theta^{(k)}$. Also, the last expression of (4.7) is in $L^2(\mathbb{R}, \nu_{-\alpha})$ and it dominates the sequence of functions $|(J_q^{\mathrm{an}}(F_n^{(k)})\psi)(\xi) - (J_q^{\mathrm{an}}(F_n)\psi)(\xi)|^2$. Hence using the dominated convergence theorem, we obtain the desired result. Furthermore, using similar methods as those used in (3.12) we have

$$\left\| J_q^{\mathrm{an}}(F_n^{(k)})\psi \right\|_{L^2(\mathbb{R},\nu_{-\alpha})} \le \Psi_n^k(|-iq|)\|\psi\|_{L^2(\mathbb{R},\nu_\alpha)} \tag{4.8}$$

and

$$\left\| J_q^{\text{an}}(F_n)\psi \right\|_{L^2(\mathbb{R},\nu_{-\alpha})} \leq \Psi_n^1(|-iq|)\|\psi\|_{L^2(\mathbb{R},\nu_\alpha)}$$

where $\Psi_n^k(|-iq|)$ is given by Eq. (3.14). □

We are now ready to establish our main result, namely the stability theorem for the analytic operator-valued generalized Feynman integral.

**Theorem 4.5** *Let $\{\theta^{(k)}\}$ be a sequence of complex-valued functions such that $\theta^{(k)}(s,u) \to \theta(s,u)$, as $k \to \infty$, for $\eta \times m_L$-a.e. $(s,u)$. For $k = 1, 2, \ldots$, let the functional $F^{(k)}$ on $C_{a,b}[0,T]$ be given by Eq. (4.5). Then for all $q \in \mathbb{R}\backslash\{0\}$ and $\psi \in L^2(\mathbb{R},\nu_\alpha)$,*

$$\left\| J_q^{\text{an}}(F^{(k)})\psi - J_q^{\text{an}}(F)\psi \right\|_{L^2(\mathbb{R},\nu_{-\alpha})} \to 0 \quad as \quad k \to \infty$$

*where $J_q^{\text{an}}(F^{(k)})$ is given by Eq. (4.4).*

*Proof* Using Eqs. (4.3), (4.4) and (4.6) we have that

$$\begin{aligned}
\lim_{k\to\infty} J_q^{\text{an}}(F^{(k)})\psi &\overset{\text{(I)}}{=} \lim_{k\to\infty} \sum_{n=1}^{\infty} \beta_n J_q^{\text{an}}\left(F_n^{(k)}\right)\psi \\
&\overset{\text{(II)}}{=} \sum_{n=1}^{\infty} \lim_{k\to\infty} \beta_n J_q^{\text{an}}\left(F_n^{(k)}\right)\psi \\
&\overset{\text{(III)}}{=} \sum_{n=1}^{\infty} \beta_n J_q^{\text{an}}(F_n)\psi \\
&\overset{\text{(IV)}}{=} J_q^{\text{an}}(F)\psi
\end{aligned}$$

is in $L^2(\mathbb{R},\nu_{-\alpha})$. Step (I) follows from Lemma 4.3. From Eqs. (3.13) and (4.8), we have

$$\begin{aligned}
&\left\| \sum_{n=1}^{\infty} \beta_n J_q^{\text{an}}(F_n^{(k)})\psi \right\|_{L^2(\mathbb{R},\nu_{-\alpha})} \\
&\leq \sum_{n=1}^{\infty} |\beta_n| \left\| J_q^{\text{an}}(F_n^{(k)})\psi \right\|_{L^2(\mathbb{R},\nu_{-\alpha})} \\
&\leq \sum_{n=1}^{\infty} |\beta_n| \Psi_n^{(k)}(|-iq|)\|\psi\|_{L^2(\mathbb{R},\nu_\alpha)} < \infty.
\end{aligned}$$

Also, by using Eqs. (4.6) and (4.8), we can show that $J_q^{\text{an}}(F_n^{(k)})\psi \to J_q^{\text{an}}(F_n)\psi$ in $L^2(\mathbb{R},\nu_{-\alpha})$ as $k \to \infty$, and hence, $J_q^{\text{an}}(F_n)\psi$ exists. Hence, Step (II) now follows. From Lemma 4.4, we obtain Step (III). Step (IV) then follows from Theorem 4.2. □

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