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A Fixed-Component Point Theorem and Applications

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Abstract We prove a topologically based characterization of the existence of fixed-component points for an arbitrary family of set-valued maps defined on a product set by using topologically based structures, without linear or convexity structures. Then, applying this general result, we derive sufficient conditions for the existence of coincidence-component points of families of set-valued maps and intersection points of families of sets, as examples for many other important points in nonlinear analysis. Applications to systems of variational relations and abstract economies are provided as examples for other optimization-related problems.

Keywords Fixed-component points · Coincidence-component points · Intersection points · Maximal elements · KKM-structures · Optimization-related problems

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1 Introduction

Numerous fixed point theorems have been developed and used for centuries as the most useful tool in dealing with the existence of solutions in various fields of science. Relatively recently, motivated by problems involving functions or sets defined on product spaces, e.g., in game theory, problems on sets with convex sections, systems of variational inequalities, etc., several authors (see [1,21]) established fixed point theorems for a family of mappings defined in product spaces. These points are called fixed-component points or collectively fixed points. Such results were developed, e.g., in [1,6,7,9,19,24,28,31,34], with successful applications in existence studies for optimization-related problems.

For considerations of the existence of fixed points in particular and of many other important points in nonlinear analysis in general, it was believed for a long period that one needed both topological and algebraic machineries. Wu [33] and Horvath [12] started two directions of dealing with existence issues in pure topological settings. Wu's approach is based on replacing convexity assumptions by connectedness conditions, and Horvath's one on replacing a convex hull by an image of a simplex through a continuous map. Recently, an attempt to unify these two directions in obtaining topological full (two-way) characterizations of the existence of various important points like intersection points, maximal elements, coincidence points, sectional points, etc., was carried out in [18,20], based on the so-called KKM-structures and connectedness structures. Realizing the basic role of fixed points in existence studies, in [19] we developed a new characterization of the existence of such points in topologically based settings. An extension to fixed-component points was also included, but only for a family of finite number of mappings. We cannot employ the same proof technique to extend this result for infinitely many mappings. It is worth noting that among the above-encountered references, only [18–20] dealt with necessary and sufficient conditions for existence, the others included only sufficient ones.

The above discussions inspire us to find another way in this paper to consider full characterizations of the existence of fixed-component points of general infinite families of mappings. Namely, based on our KKM-structures and using continuous partitions of unity and the classical Tikhonov fixed point theorem, in this paper we extend Theorem 2.5 of [19], which is a necessary and sufficient condition for the existence of fixed-component points, to the case of an arbitrary family of mappings defined on product sets. Applications of this result to aforementioned important points in nonlinear analysis and to optimization-related problems are also included.

The layout of the paper is simple. In the rest of this section, we recall some needed definitions. Section 2 contains full characterizations of the existence of fixed-component points together with discussions on imposed assumptions as well as consequences and relations to previous results in the literature. Section 3 is devoted to applications, including studies of the existence of some important points in nonlinear analysis and for solutions of optimization-related problems.

Throughout this paper, for a nonempty set X, $\langle X\rangle$ stands for the set of all finite subsets of X. For $N = \{x_0, x_1, \ldots, x_n\} \in \langle X\rangle$ and $M = \{x_{i_0}, x_{i_1}, \ldots, x_{i_k}\} \subset N$, $\Delta_{|N|} := \Delta_n$ stands for the standard n-simplex of Euclidean space $\mathbb{R}^{n+1}$ with vertices being unit vectors $e_0 = (1, 0, \ldots, 0)$, $e_1 = (0, 1, 0, \ldots, 0)$,..., $e_n = (0, 0, \ldots, 1)$,

Δ_M denotes the face of $\Delta_{|N|}$ with vertices $e_{i_0}, e_{i_1}, \ldots, e_{i_k}$. Let $H : X \rightrightarrows Y$ be a set-valued map between nonempty sets X and Y. For $x \in X$ and $y \in Y$, an image and a fiber (or inverse image) of H is the set $H(x)$ and $H^{-1}(y) = \{x \in X \mid y \in H(x)\}$, respectively (resp).

Definition 1 ([19,20]) For nonempty sets X and Y, a pair $\mathscr{F} := (\Phi_X, \Im_Y)$ is called a KKM-structure of the pair (X, Y) if $\Im_Y$ is a topology on Y and $\Phi_X = \{\varphi_N : \Delta_{|N|} \to Y \mid N \in \langle X \rangle\}$ is a family of maps with all $\varphi_N \in \Phi_X$ being $\Im_Y$-continuous. In the special case where $X = Y$, such a $\mathscr{F}$ is termed a KKM-structure of X. If $\Im_Y$ is compact, i.e., Y is $\Im_Y$-compact, $(\Phi_X, \Im_Y)$ is called a compact KKM-structure.

If $X = Y$ is a convex subset of a topological vector space E, $\Im_X$ is the topology on X induced by that of E, and

$$\Phi_X = \Big\{\varphi_N : \Delta_{|N|} \to X \mid \varphi_N(e) = \sum_{x_i \in N} \lambda_i x_i \ \text{ for } \ e = \sum \lambda_i e_i \in \Delta_{|N|},\ N \in \langle X \rangle\Big\},$$

then $\mathscr{F} := (\Phi_X, \Im_X)$ is called the natural KKM-structure of X.

Let I be any index set, X_i a nonempty set and $\mathscr{F}_i := (\Phi_{X_i}, \Im_{X_i})$ a KKM-structure on X_i for each $i \in I$. We define a KKM-structure of $X := \prod_{i \in I} X_i$ as follows. Let $\Im_X$ be the Tikhonov product topology on X of the topologies $\Im_{X_i}$. For $N = \{(x_i^1)_{i \in I}, (x_i^2)_{i \in I}, \ldots (x_i^n)_{i \in I}\} \in \langle X \rangle$, we define $N_i = \{x_i^1, x_i^2, \ldots, x_i^n\} \in \langle X_i \rangle$ and denote $N := \otimes_{i \in I} N_i$. Each N_i is called the "ith component" of N. We also denote $x := \otimes_{i \in I} x_i$ for each element $x = (x_i)_{i \in I} \in X$. Let

$$\Phi_X = \Big\{\varphi_N : \Delta_{|N|} \to X \mid \varphi_N(e) = \otimes_{i \in I} \varphi_{N_i}(e) \text{ for } e \in \Delta_{|N|},\ N = \otimes_{i \in I} N_i \in \langle X \rangle\Big\}.$$

Then, $\mathscr{F} := (\Phi_X, \Im_X)$ is a KKM-structure on X, called the product KKM-structure of the KKM-structures $\mathscr{F}_i := (\Phi_{X_i}, \Im_{X_i})$, and denoted by $\mathscr{F} = \prod_{i \in I} \mathscr{F}_i$.

Definition 2 ([19,20]) Let X be a nonempty set and $(\Phi_X, \Im_X)$ a KKM-structure of X. We say that a subset B of X is Φ_X-convex if, for all $\varphi_N \in \Phi_X$ and $M \subset N \cap B$, $\varphi_N(\Delta_M) \subset B$. For $C \subset X$, the smallest Φ_X-convex set containing C, denoted by Φ_X-co$C$, is called Φ_X-convex hull of C. It is not hard to check that Φ_X-co$C = \bigcup_{N \in \langle C \rangle} \Phi_X$-co$N$.

When $\mathscr{F} := (\Phi_X, \Im_X)$ is the natural KKM-structure of X, the notions of a Φ_X-convex set and a Φ_X-convex hull collapse to the usual notions of a convex set and a convex hull, resp. Note that notions of generalized convex sets and convex hulls in some spaces, previously introduced by many authors, such as a convex space [22], H-space [12], G-convex space [26], FC-space [8], GFC-space [17], etc., are particular cases of the notions in Definition 2 because in each of these spaces there is a KKM-structure implicitly. However, there are several convex structures for which notions of convex sets and convex hulls do seemingly not naturally match with the notions in Definition 2. We discuss first the Takahashi-convex structure. Recall that a Takahashi-convex structure on a metric space (X, d) is a function $h : X \times X \times [0, 1] \longrightarrow X$

satisfying $d(s, h(u, v, t)) \leq td(s, u)+(1-t)d(s, v)$ for all $(s, u, v, t) \in X \times X \times X \times [0, 1]$ (see [29]). A subset B of X is said to be convex if $h(u, v, t) \in B$ for any $u, v \in B$ and $t \in [0, 1]$. We construct a KKM-structure on X as follows. For each $N \in \langle X \rangle$ and $e = \sum_{i=0}^{|N|} \lambda_i e_i \in \Delta_{|N|}$, let $i_e = \min\{i \mid \lambda_i \neq 0\}$ and $i^e = \max\{i \mid \lambda_i \neq 0\}$. Let Φ_X includes maps $\varphi_N : \Delta_{|N|} \longrightarrow X$ defined by $\varphi_N(e) = h(x_{i_e}, x_{i^e}, \lambda_{i_e} + \lambda_{i^e})$ for all $e = \sum_{i=0}^{|N|} \lambda_i e_i \in \Delta_{|N|}$. Let

$$\Im_X = \bigcap_{N \in \langle X \rangle} \left\{ U \subset X \mid \varphi_N^{-1}(U) \text{ is open in } \Delta_{|N|} \right\}.$$

Then, $(\Phi_X, \Im_X)$ is a KKM-structure of X, and any convex subset in Takahashi-convex metric space (X, d, h) is also Φ_X-convex. However, if we fix the topology $\Im$ induced by the metric d of X (and do not consider the above topology $\Im_X$, then we still do not know if there is or not a family Φ_X such that $(\Phi_X, \Im)$ is a KKM-structure of X with each convex subset in (X, d, h) being also Φ_X-convex. Another approach to obtaining a notion of convex hull was proposed in [14], which was interesting, without any convex structure. But the notions in Section 4 of [14] and in Definition 2 are not comparable.

2 Fixed-Component Point Theorems

Definition 3 Let I be any index set. For each $i \in I$, let X_i be a nonempty set, $\mathscr{F}_i := (\Phi_{X_i}, \Im_{X_i})$ be a KKM-structure of X_i, and $P_i, Q_i : X := \prod_{i \in I} X_i \rightrightarrows X_i$ be set-valued maps. $\{Q_i\}_{i \in I}$ is called $\{\Phi_{X_i}\}$-weak-convex with respect to (w.r.t.) $\{P_i\}_{i \in I}$ if whenever $x = (x_i)_{i \in I} \in X$, $\varphi_{N_i} \in \Phi_{X_i}$, and $M_i \subset N_i \cap P_i(x)$ satisfying $x_i \in \varphi_{N_i}(\Delta_{M_i})$ simultaneously for all $i \in I$, one has $\varphi_{N_i}(\Delta_{M_i}) \subset Q_i(x)$ for each $i \in I$.

If Definition 3 holds with the natural KKM-structures $\mathscr{F}_i$, we say that $\{Q_i\}_{i \in I}$ is weak-convex w.r.t. $\{P_i\}_{i \in I}$.

Remark 1 For each $i \in I$, let us consider the following conditions for the pair (P_i, Q_i).

(h1) Whenever $x = (x_i)_{i \in I} \in X$, $\varphi_{N_i} \in \Phi_{X_i}$, and $M_i \subset N_i \cap P_i(x)$ satisfying $x_i \in \varphi_{N_i}(\Delta_{M_i})$, one has $\varphi_{N_i}(\Delta_{M_i}) \subset Q_i(x)$. In this case, we say that Q_i is Φ_{X_i}-weak-convex w.r.t. P_i.
(h2) For all $x \in X$, $\varphi_{N_i} \in \Phi_{X_i}$ and $M_i \subset N_i \cap P_i(x)$, one has $\varphi_{N_i}(\Delta_{M_i}) \subset Q_i(x)$. If this condition holds, Q_i is called Φ_{X_i}-convex w.r.t. P_i.
(h3) Φ_{X_i}-co$P_i(x) \subset Q_i(x)$ for each $x \in X$.
(h4) $P_i(x) \subset Q_i(x)$ and $Q_i(x)$ is Φ_{X_i}-convex for each $x \in X$.

It is not hard to see that, if the KKM-structures $\mathscr{F}_i := (\Phi_{X_i}, \Im_{X_i})$ are given, then (h4) $\Rightarrow$ (h3) $\Rightarrow$ (h2) $\Rightarrow$ (h1). Example 1 below shows that, in general, the reverse implications are not true. However, in the case $\mathscr{F}_i$ is the natural KKM-structure of X_i, (h2) coincides with (h3). From Definition 3, we see that "*if for each* $i \in I$, *one of conditions* (h1)-(h4) *holds for* (P_i, Q_i), *then* $\{Q_i\}_{i \in I}$ *is* $\{\Phi_{X_i}\}$*-weak-convex w.r.t.* $\{P_i\}_{i \in I}$". *The converse does not hold as shown by Example* 2 *below.*

Example 1 Let $X_1 = X_2 = X_3 = [0, 2]$, $\mathscr{F}_1 := (\Phi_{X_1}, \Im_{X_1})$ be the natural KKM-structure of X_1, $\mathscr{F}_2 := (\Phi_{X_2}, \Im_{X_2})$ and $\mathscr{F}_3 := (\Phi_{X_3}, \Im_{X_3})$ be KKM-structures of X_2 and X_3, resp., defined by: $\Im_{X_2} = \Im_{X_3}$ being the usual topology on $[0, 2]$, $\Phi_{X_2} = \{\varphi_{N_2} : \Delta_{|N_2|} \to X_2 \mid \varphi_{N_2}(e) = 0$ for all $e \in \Delta_{|N_2|},\ N_2 \in \langle X_2\rangle\}$, and $\Phi_{X_3} = \{\varphi_{N_3} : \Delta_{|N_3|} \to X_3 \mid \varphi_{N_3}(e) = \frac{\min N_3 + \max N_3}{2}$ for all $e \in \Delta_{|N_3|},\ N_3 \in \langle X_3\rangle\}$. Let, for all $x = (x_1, x_2, x_3) \in X := X_1 \times X_2 \times X_3$,

$$P_1(x) = \begin{cases} [0, \frac{x_1}{3}) & \text{if } x_1 \in (0, 2], \\ 0 & \text{if } x_1 = 0, \end{cases} \qquad Q_1(x) = \{0, 1\},$$

$$P_2(x) = \begin{cases} 0 & \text{if } x_2 = 0, \\ [0, x_2) & \text{if } x_2 \in (0, 2], \end{cases} \qquad Q_2(x) = \{0, x_2\},$$

$$P_3(x) = \begin{cases} 0 & \text{if } x_3 = 0, \\ [0, \frac{x_3}{3}) & \text{if } x_3 \in (0, 2], \end{cases} \qquad Q_3(x) = \left[0, \frac{x_3}{3}\right] \cup \left[\frac{2x_3}{3}, 2\right].$$

For $x = (x_1, x_2, x_3) \in X$, $\varphi_{N_1} \in \Phi_{X_1}$ and $M_1 \subset N_1$, we see the following string of equivalent statements

$$\begin{cases} M_1 \subset P_1(x), \\ x_1 \in \varphi_{N_1}(\Delta_{M_1}) \end{cases} \iff \begin{cases} M_1 \subset P_1(x), \\ x_1 \in \text{co}M_1 \end{cases}$$
$$\iff \begin{cases} M_1 \subset P_1(x) = \{0\}, \\ x_1 = 0, \\ x_1 \in [\min M_1, \max M_1] \end{cases} \text{or} \begin{cases} x_1 \in (0, 2], \\ x_1 \in [\min M_1, \max M_1], \\ M_1 \subset P_1(x) = [0, \frac{x_1}{3}) \end{cases}$$
$$\iff \begin{cases} x_1 = 0, \\ M_1 = \{0\}. \end{cases}$$

Then, $\varphi_{N_1}(\Delta_{M_1}) = \text{co}\{0\} \subset Q_1(x)$. Thus, (P_1, Q_1) satisfies condition (h1). (P_1, Q_1) does not fulfill condition (h2) because, for $x = (1, 1, 1)$ and $N_1 = M_1 = \{0, \frac{1}{4}\}$ satisfying $M_1 \subset P_1(x)$, but $\varphi_{N_1}(\Delta_{M_1}) = \text{co}\{0, \frac{1}{4}\} \not\subset Q_1(x)$.

We easily see that condition (h2) holds for (P_2, Q_2), but condition (h3) does not, because Φ_{X_2}-co$P_2(x) = [0, x_2) \not\subset \{0, x_2\} = Q_2(x)$ for all $x = (x_1, x_2, x_3) \in X_1 \times (0, 2] \times X_3$.

For (P_3, Q_3), we see that the values of P_3 are Φ_{X_3}-convex and $P_3(x) \subset Q_3(x)$ for all $x \in X$, i.e., condition (h3) holds, while condition (h4) does not because the values of Q_3 are not Φ_{X_3}-convex.

Finally, since (P_i, Q_i) $(i = 1, 2, 3)$ satisfy at least one of conditions (h1)-(h4), $\{Q_1, Q_2, Q_3\}$ is $\{\Phi_{X_1}, \Phi_{X_2}, \Phi_{X_3}\}$- weak-convex w.r.t. $\{P_1, P_2, P_3\}$.

Example 2 Let $X_1 = X_2 = [0, 1]$, $\mathscr{F}_1 := (\Phi_{X_1}, \Im_{X_1}) = \mathscr{F}_2 := (\Phi_{X_2}, \Im_{X_2})$ be the natural KKM-structure of $[0, 1]$. Let $P_1, Q_1 : X := X_1 \times X_2 \rightrightarrows X_1$ and $P_2, Q_2 : X \rightrightarrows X_2$ defined by, for all $x = (x_1, x_2) \in X$,

$$P_1(x) = Q_2(x) = [0, x_2], \quad Q_1(x) = P_2(x) = [0, x_1].$$

For all $x = (x_1, x_2) \in X$, $\varphi_{N_1} \in \Phi_{X_1}$, $\varphi_{N_2} \in \Phi_{X_2}$, $M_1 \subset N_1$ and $M_2 \subset N_2$, we have the equivalent assertions:

$$\begin{cases} M_1 \subset P_1(x), \\ x_1 \in \varphi_{N_1}(\Delta_{M_1}), \\ M_2 \subset P_2(x), \\ x_2 \in \varphi_{N_2}(\Delta_{M_2}) \end{cases} \Longleftrightarrow \begin{cases} M_1 \subset [0, x_2], \\ x_1 \in [\min M_1, \max M_1], \\ M_2 \subset [0, x_1], \\ x_2 \in [\min M_2, \max M_2] \end{cases}$$
$$\Longleftrightarrow \begin{cases} \max M_1 \le x_2, \\ x_1 \in [\min M_1, \max M_1], \\ \max M_2 \le x_1, \\ x_2 \in [\min M_2, \max M_2] \end{cases}$$
$$\Longleftrightarrow x_1 = x_2 = \max M_1 = \max M_2.$$

Then, $\varphi_{N_1}(\Delta_{M_1}) = [\min M_1, \max M_1] \subset [0, x_1] = Q_1(x)$ and $\varphi_{N_2}(\Delta_{M_2}) = [\min M_2, \max M_2] \subset [0, x_2] = Q_2(x)$. Thus, $\{Q_1, Q_2\}$ is $\{\Phi_{X_1}, \Phi_{X_2}\}$-weak-convex w.r.t. $\{P_1, P_2\}$. For $N_1 = \{0, \frac{1}{2}, 1\}$, $M_1 = \{0, 1\}$ and $x = (0, 1)$, we see that $x_1 = 0 \in [0, 1] = \varphi_{N_1}(\Delta_{M_1})$ and $M_1 \subset [0, 1] = P_1(x)$, but $\varphi_{N_1}(\Delta_{M_1}) = [0, 1] \not\subset \{0\} = Q_1(x)$. Thus, condition (h1) does not hold for (P_1, Q_1) (hence conditions (h2)-(h4) not either). We also check easily that condition (h1) does not hold for (P_2, Q_2).

From now on, for an index set I, nonempty sets X_i $(i \in I)$, $X := \prod_{i\in I} X_i$, and KKM-structures $\mathscr{F}_i := (\Phi_{X_i}, \Im_{X_i})$ of X_i, if not otherwise stated, $\Im_X$ denotes the Tikhonov product topology of topologies $\Im_{X_i}$ on X.

Coercivity condition Let $\{X_i\}_{i\in I}$ be a family of nonempty sets, $X := \prod_{i\in I} X_i$, $\mathscr{F}_i = (\Phi_{X_i}, \Im_{X_i})$ a KKM-structure of X_i, and $P_i : X \rightrightarrows X_i$. The following condition is a coercivity condition for the family $\{P_i\}_{i\in I}$

(C) there exists a nonempty $\Im_X$-compact subset K of X and, for each $N_i \in \langle X_i\rangle$, there is a $\Im_{X_i}$-compact and Φ_{X_i}-convex subset $L_{N_i} \subset X_i$ containing N_i such that, for each $x \in (\prod_{i\in I} L_{N_i})\backslash K$ and $i \in I$, there exists $x_i' \in L_{N_i}$ with $x \in \operatorname{int}_{\Im_X} P_i^{-1}(x_i')$.

Let I, X_i, X and Q_i be as in Definition 3. A point $\bar{x} := (\bar{x}_i)_{i\in I} \in X$ is called a fixed-component point of the family $\{Q_i\}_{i\in I}$ if $\bar{x}_i \in Q_i(\bar{x})$ for all $i \in I$.

Theorem 1 *For an arbitrary index set I and $i \in I$, let X_i be a nonempty set and $Q_i : X := \prod_{i\in I} X_i \rightrightarrows X_i$. Then, the family $\{Q_i\}_{i\in I}$ has a fixed-component point if and only if, for each $i \in I$, there exist a KKM-structure $\mathscr{F}_i = (\Phi_{X_i}, \Im_{X_i})$ of X_i and a set-valued mapping $P_i : X \rightrightarrows X_i$ such that, for each $i \in I$, $X = \bigcup_{x_i\in X_i} \operatorname{int}_{\Im_X} P_i^{-1}(x_i)$, $\{Q_i\}_{i\in I}$ is $\{\Phi_{X_i}\}_{i\in I}$-weak-convex w.r.t. $\{P_i\}_{i\in I}$, and condition* (C) *holds for $\{P_i\}_{i\in I}$.*

Proof Necessity. Suppose that $\bar{x} := (\bar{x}_i)_{i\in I} \in X$ is a fixed-component point of $\{Q_i\}_{i\in I}$. For $i \in I$, let $P_i : X \rightrightarrows X_i$ be defined by $P_i(x) = \{\bar{x}_i\}$ for all $x \in X$, and a KKM-structure $\mathscr{F}_i$ of X_i defined as follows: $\Im_{X_i} = \{U \subset X_i \mid \bar{x}_i \notin U\} \cup \{X_i\}$ and

$$\Phi_{X_i} = \left\{\varphi_{N_i} : \Delta_{|N_i|} \to X_i \mid \varphi_{N_i}(e) = \bar{x}_i \text{ for all } e \in \Delta_{|N_i|}, N_i \in \langle X_i\rangle\right\}.$$

For each $i \in I$, the condition $X = \bigcup_{x_i \in X_i} \text{int}_{\Im_X} P_i^{-1}(x_i)$ is obviously satisfied. Since each $\Im_{X_i}$ is clearly a compact topology, $\Im_X$ is compact too. Therefore, condition (C) is satisfied with $K = X$ and $L_{N_i} = X_i$ for all $N_i \in \langle X_i \rangle$. For all $x := (x_i)_{i \in I} \in X$, $\varphi_{N_i} \in \Phi_{X_i}$ and $M_i \subset N_i$, we obtain the equivalent statements:

$$\begin{cases} M_i \subset P_i(x), \\ x_i \in \varphi_{N_i}(\Delta_{M_i}), \\ i \in I \end{cases} \Longleftrightarrow \begin{cases} M_i \subset \{\bar{x}_i\}, \\ x_i \in \{\bar{x}_i\}, \\ i \in I \end{cases}$$
$$\Longleftrightarrow x = \bar{x} = (\bar{x}_i)_{i \in I} \text{ and } M_i = \{\bar{x}_i\} \text{ for all } i \in I.$$

Then, $\varphi_{N_i}(\Delta_{M_i}) = \{\bar{x}_i\} \subset Q_i(x)$ for all $i \in I$. Hence, $\{Q_i\}_{i \in I}$ is $\{\Phi_{X_i}\}_{i \in I}$-convex w.r.t. $\{P_i\}_{i \in I}$. Thus, the proof of the necessity is complete.

Sufficiency. Assume that, for each $i \in I$, there exist a KKM-structure $\mathscr{F}_i$ of X_i and a set-valued mapping $P_i : X \rightrightarrows X_i$ such that the conditions mentioned in Theorem 1 hold. For each $i \in I$, since $X = \bigcup_{x_i \in X_i} \text{int}_{\Im_X} P_i^{-1}(x_i)$, there exists $N_i \in \langle X_i \rangle$ such that $K \subset \bigcup_{x_i \in N_i} \text{int}_{\Im_X} P_i^{-1}(x_i)$ (K is given by condition (C)). For each $i \in I$, by (C) there is a $\Im_{X_i}$-compact and Φ_{X_i}-convex subset $L_{N_i} \subset X_i$ containing N_i such that $L_N \backslash K \subset \bigcup_{x_i \in L_{N_i}} \text{int}_{\Im_X} P_i^{-1}(x_i)$, where $L_N := \prod_{i \in I} L_{N_i}$. Hence, $L_N \subset (L_N \backslash K) \cup K \subset \bigcup_{x_i \in L_{N_i}} \text{int}_{\Im_X} P_i^{-1}(x_i)$. Since L_N is compact, there exists $\overline{N}_i := \{x_i^0, x_i^1, \ldots, x_i^{n_i}\} \in \langle L_{N_i} \rangle$ such that $L_N = \bigcup_{j=0}^{n_i} \text{int}_{\Im_X} P_i^{-1}(x_i^j)$. For each $i \in I$, let $\{\psi_j\}_{j=0}^{n_i}$ be a continuous partition of unity of L_N associated with the finite open covering $\{\text{int}_{\Im_X} P_i^{-1}(x_i^j)\}_{j=0}^{n_i}$. Then, for each $x \in L_N$ and $k \in J_i(x) := \{j \in \{0, 1, \ldots, n_i\} | \psi_j(x) \neq 0\}$, one has $x \in \text{int}_{\Im_X} P_i^{-1}(x_i^k) \subset P_i^{-1}(x_i^k)$. Therefore,

$$\overline{M}_i(x) := \left\{x_i^k \,|\, k \in J_i(x)\right\} \subset \overline{N}_i \cap P_i(x) \text{ for all } x \in L_N. \tag{1}$$

Now, for each $i \in I$, we define a map $\gamma_i : L_N \to \Delta_{|\overline{N}_i|}$ by $\gamma_i(x) = \sum_{j=0}^{n_i} \psi_j(x) e_j$ for all $x \in L_N$. We have, for all $x \in L_N$,

$$\varphi_{\overline{N}_i}(\gamma_i(x)) = \varphi_{\overline{N}_i}\left(\sum_{j=0}^{n_i} \psi_j(x) e_j\right) = \varphi_{\overline{N}_i}\left(\sum_{k \in J_i(x)} \psi_k(x) e_k\right) \in \varphi_{\overline{N}_i}\left(\Delta_{\overline{M}_i(x)}\right). \tag{2}$$

Let $\Omega = \prod_{i \in I} \Delta_{|\overline{N}_i|}$. Then, Ω is a compact convex subset of the locally convex space $\mathbb{R}^I = \prod_{i \in I} \mathbb{R}^{n_i+1}$. Let $\Gamma : L_N \to \Omega$ and $\Psi : \Omega \to L_N$ be defined by

$\Gamma(x) = (\gamma_i(x))_{i \in I}$ for all $x \in L_N$, and
$\Psi(t) = (\varphi_{\overline{N}_i}(p_i(t)))_{i \in I}$ for all $t \in \Omega$,

where $p_i(t)$ is the projection of t on $\Delta_{|\overline{N}_i|}$. Then, Γ and Ψ are continuous and so is $\Gamma \circ \Psi : \Omega \to \Omega$. By the Tikhonov fixed point theorem (see [32]), a $\bar{t} \in \Omega$ exists such that $\bar{t} = (\Gamma \circ \Psi)(\bar{t})$. Setting $\bar{x} = (\bar{x}_i)_{i \in I} = \Psi(\bar{t}) = (\varphi_{\overline{N}_i}(p_i(\bar{t})))_{i \in I}$, we have $\bar{t} = (p_i(\bar{t}))_{i \in I} = \Gamma(\bar{x}) = (\gamma_i(\bar{x}))_{i \in I}$. Then, $\bar{x}_i = \varphi_{\overline{N}_i}(\gamma_i(\bar{x}))$ for all $i \in I$, and by (2),

$$\bar{x}_i = \varphi_{\overline{N}_i}(\gamma_i(\bar{x})) \in \varphi_{\overline{N}_i}\left(\Delta_{\overline{M}_i(\bar{x})}\right) \text{ for all } i \in I. \tag{3}$$

Then, (1) gives

$$\overline{M}_i(\bar{x}) \subset \overline{N}_i \cap P_i(\bar{x}) \text{ for all } i \in I. \tag{4}$$

Since $\{Q_i\}_{i\in I}$ is $\{\Phi_{X_i}\}_{i\in I}$-convex w.r.t. $\{P_i\}_{i\in I}$, (3) and (4) imply that $\bar{x}_i \in \varphi_{\overline{N}_i}(\Delta_{\overline{M}_i(\bar{x})}) \subset Q_i(\bar{x})$ for all $i \in I$. The proof is complete. □

Remark 2 (a) For each $i \in I$, the condition $X = \bigcup_{x_i\in X_i} \mathrm{int}_{\Im_X} P_i^{-1}(x_i)$ in Theorem 1 is satisfied if P_i has the nonempty values and $\Im_X$-open inverse images.

(b) For I, X_i, X and Q_i as in Theorem 1, applying this theorem, condition (h4) and the statement at the end of Remark 1 with P_i and Q_i replaced by Q_i and Φ_{X_i}-co$Q_i(\cdot)$, resp, we obtain

Corollary 1 *Let I, X_i, X, and Q_i be as in Theorem 1. If, for each $i \in I$, there exists a KKM-structure $\mathscr{F}_i := (\Phi_{X_i}, \Im_{X_i})$ of X_i such that $X = \bigcup_{x_i\in X_i} \mathrm{int}_{\Im_X} Q_i^{-1}(x_i)$ and condition* (C) *holds for $\{Q_i\}_{i\in I}$. Then, the family $\{\Phi_{X_i}\text{-co}Q_i(\cdot)\}_{i\in I}$ has a fixed-component point.*

(c) If each X_i has the natural KKM-structure, we easily check the necessary condition of Theorem 1 with maps P_i taken in the proof of the necessity part of Theorem 1. Therefore, we have a particular case of Theorem 1 as follows.

Theorem 2 *For an arbitrary index set I and $i \in I$, let X_i be a nonempty convex set of a topological vector space and $Q_i : X := \prod_{i\in I} X_i \rightrightarrows X_i$. Then, the family $\{Q_i\}_{i\in I}$ has a fixed-component point if and only if, for each $i \in I$, there exists $P_i : X \rightrightarrows X_i$ such that $X = \bigcup_{x_i\in X_i} \mathrm{int}\, P_i^{-1}(x_i)$, $\{Q_i\}_{i\in I}$ is weak-convex w.r.t. $\{P_i\}_{i\in I}$, and the following condition holds for $\{P_i\}_{i\in I}$:*

(C′) *there exists a nonempty compact subset K of X and, for each $N_i \in \langle X_i\rangle$, there is a compact and convex subset $L_{N_i} \subset X_i$ containing N_i such that, for each $x \in \prod_{i\in I} L_{N_i}\setminus K$ and $i \in I$, there exists $x_i' \in L_{N_i}$ with $x \in \mathrm{int}\, P_i^{-1}(x_i')$.*

(d) Let I, X_i, X be as in Theorem 1, $X^i := \prod_{j\in I, j\neq i} X_j$, and x^i be the canonical projection of $x \in X$ on X^i. For each $i \in I$, let $\widetilde{P}_i, \widetilde{Q}_i : X^i \rightrightarrows X_i$ be set-valued maps. We can state Definition 3 for $\{\widetilde{P}_i\}_{i\in I}$ and $\{\widetilde{Q}_i\}_{i\in I}$ and change conditions (h1)-(h4) for the pair $(\widetilde{P}_i, \widetilde{Q}_i)$ in the manner that the phrases "$P_i(x)$," "$Q_i(x)$," "for all $x \in X$" and "for each $x \in X$" in Definition 3 are replaced by "$\widetilde{P}_i(x^i)$," "$\widetilde{Q}_i(x^i)$," "for all $x^i \in X^i$" and "for each $x^i \in X^i$," resp. A point $\bar{x} := (\bar{x}_i)_{i\in I} \in X$ satisfying $\bar{x}_i \in \widetilde{Q}_i(\bar{x}^i)$ for all $i \in I$ is also called a fixed-component point of the family $\{\widetilde{Q}_i\}_{i\in I}$.

The following consequence of Theorem 1 is formulated in terms of the family $\{\widetilde{Q}_i\}_{i\in I}$.

Theorem 3 *Let I, X_i, X be as in Theorem 1, $X^i := \prod_{j\in I, j\neq i} X_j$ and $\widetilde{Q}_i : X^i \rightrightarrows X_i$. Then, the family $\{\widetilde{Q}_i\}_{i\in I}$ has a fixed-component point if and only if, for each $i \in I$, there exist a KKM-structure $\mathscr{F}_i = (\Phi_{X_i}, \Im_{X_i})$ of X_i and a set-valued mapping $\widetilde{P}_i : X^i \rightrightarrows X_i$ such that $X^i = \bigcup_{x_i\in X_i} \mathrm{int}_{\Im_{X^i}} \widetilde{P}_i^{-1}(x_i)$ for each $i \in I$, $\{\widetilde{Q}_i\}_{i\in I}$ is $\{\Phi_{X_i}\}_{i\in I}$-weak-convex w.r.t. $\{\widetilde{P}_i\}_{i\in I}$, and the following condition holds:*

($\widetilde{\mathrm{C}}$) *there exists a nonempty $\Im_X$-compact subset K of X, and, for each $N_i \in \langle X_i \rangle$, there is a $\Im_{X_i}$-compact and Φ_{X_i}-convex subset $L_{N_i} \subset X_i$ containing N_i such that, for each $x \in (\prod_{i\in I} L_{N_i})\setminus K$ and $i \in I$, there exists $x_i' \in L_{N_i}$ with $x^i \in \operatorname{int}_{\Im_{X^i}} \widetilde{P}_i^{-1}(x_i')$.*

We derive Theorem 3 from Theorem 1. The necessary condition is easily checked with the KKM-structures $\mathscr{F}_i := (\Phi_{X_i}, \Im_{X_i})$ taken from the proof of the necessity part of Theorem 1 and maps $\widetilde{P}_i : X^i \rightrightarrows X_i$ defined by $\widetilde{P}_i(x^i) = \{\bar{x}_i\}$ for all $x^i \in X^i$, where $\bar{x} := (\bar{x}_i)_{i\in I}$ is the fixed-component point of $\{\widetilde{Q}_i\}_{i\in I}$. For the sufficiency, we define maps $P_i, Q_i : X \rightrightarrows X_i$ by $P_i(x) = \widetilde{P}_i(x^i)$ and $Q_i(x) = \widetilde{Q}_i(x^i)$ for all $x \in X$. Then, for each $i \in I$ and $x_i \in X_i$, $P_i^{-1}(x_i) = X_i \times \widetilde{P}_i^{-1}(x_i)$. Hence,

$$\bigcup_{x_i\in X_i} \operatorname{int}_{\Im_X} P_i^{-1}(x_i) = \bigcup_{x_i\in X_i} \operatorname{int}_{\Im_X} \left(X_i \times \widetilde{P}_i^{-1}(x_i)\right) \supset \bigcup_{x_i\in X_i} \left(X_i \times \operatorname{int}_{\Im_{X^i}} \widetilde{P}_i^{-1}(x_i)\right)$$

$$\supset X_i \times \left(\bigcup_{x_i\in X_i} \operatorname{int}_{\Im_{X^i}} \widetilde{P}_i^{-1}(x_i)\right) = X_i \times X^i = X.$$

Moreover, it is clear that the $\{\Phi_{X_i}\}_{i\in I}$-weak-convexity w.r.t. $\{\widetilde{P}_i\}_{i\in I}$ of $\{\widetilde{Q}_i\}_{i\in I}$ implies the $\{\Phi_{X_i}\}_{i\in I}$-weak-convexity w.r.t. $\{P_i\}_{i\in I}$ of $\{Q_i\}_{i\in I}$ and the condition ($\widetilde{\mathrm{C}}$) for $\{\widetilde{P}_i\}_{i\in I}$ implies the condition (C) for $\{P_i\}_{i\in I}$. Thus, by Theorem 1, $\{Q_i\}_{i\in I}$ has a fixed-component point $\bar{x}$. This $\bar{x}$ is also a fixed-component point of $\{\widetilde{Q}_i\}_{i\in I}$.

Conversely, Theorem 3 implies Theorem 1 or not is still an open question for us, though this looks likely.

We can suitably modify the proof of Theorem 1 to obtain Theorem 3 with the condition ($\widetilde{\mathrm{C}}$) replaced by the following weaker condition:

($\widehat{\mathrm{C}}$) for each $i \in I$, there exists a nonempty $\Im_{X^i}$-compact subset K^i of X^i and, for each $N_i \in \langle X_i \rangle$, there is a $\Im_{X_i}$-compact and Φ_{X_i}-convex subset $L_{N_i} \subset X_i$ containing N_i such that there exists $x_i' \in L_{N_i}$ with $x^i \in \operatorname{int}_{\Im_{X^i}} \widetilde{P}_i^{-1}(x_i')$ for each $x^i \in (\prod_{j\in I, j\neq i} L_{N_i})\setminus K^i$.

(e) The sufficiency part of Theorem 1 implies Theorems 3.1–3.4 of [9]. Hence, it also implies Theorem 3.2 of [7], Theorem 7 of [34], Theorems 2.3 and 3.1 of [31], Theorem 2.1 of [28], and Theorem 2.2 of [4]. We also deduce Theorem 3.1 of [17] and thus Theorem 1 of [1] and Theorem 2.1 of [21] from this part of Theorem 1. When I is a singleton, Theorem 1 becomes Theorem 2.5 of [19] and hence its sufficiency part generalizes many fixed point theorems, including the classical Browder fixed point theorem in [4] (which implies the seminal Kakutani fixed point theorem in [13]) and Tarafdar's fixed point theorem in [30], etc. (cf. [19]).

(f) To illustrate Theorem 1, we revisit Examples 1 and 2.

For Example 1, we see that

$$P_1^{-1}(x_1) = \begin{cases} X_1 \times X_2 \times X_3 & \text{if } x_1 = 0, \\ (3x_1, 2] \times X_2 \times X_3 & \text{if } 0 < x_1 \le \frac{2}{3}, \\ \emptyset & \text{if } \frac{2}{3} < x_1 \le 2, \end{cases} \quad \text{for all } x_1 \in X_1,$$

$$P_2^{-1}(x_2) = \begin{cases} X_1 \times X_2 \times X_3 & \text{if } x_2 = 0, \\ X_1 \times (x_2, 2] \times X_3 & \text{if } x_2 \in (0, 2], \end{cases} \quad \text{for all } x_2 \in X_2,$$

$$P_3^{-1}(x_3) = \begin{cases} X_1 \times X_2 \times X_3 & \text{if } x_3 = 0, \\ X_1 \times X_2 \times (3x_3, 2] & \text{if } 0 < x_3 < \frac{2}{3}, \\ \emptyset & \text{if } \frac{2}{3} \le x_3 \le 2, \end{cases} \quad \text{for all } x_3 \in X_3.$$

Clearly, the maps P_1, P_2, P_3 have the nonempty values and open fibers. Moreover, $X = X_1 \times X_2 \times X_3$ is compact. Thus, by Theorem 1, $\{Q_1, Q_2, Q_3\}$ has a fixed-component point.

For Example 2, we have $P_1^{-1}(x_1) = X_1 \times [x_1, 1]$ for all $x_1 \in X_1$, and $P_2^{-1}(x_2) = [x_2, 1] \times X_2$ for all $x_2 \in X_2$. It is clear that X is compact, and $\bigcup_{x_i \in X_i} \text{int}_{\Im_X} P_i^{-1}(x_i) = X$ for $i \in \{1, 2\}$. Thus, $\{Q_1, Q_2\}$ has a fixed-component point.

Remark 3 Using Theorem 1, we deduce a result on the existence of common fixed points. Common fixed point theorems were studied by many authors (e.g., [2,3,5]). Let I be any index set, A be a nonempty set and $\{T_i : A \rightrightarrows A,\ i \in I\}$ be a family of set-valued maps. A point $\bar{a} \in A$ is called a common fixed point of the family $\{T_i\}_{i \in I}$ if $\bar{a} \in T_i(\bar{a})$ for all $i \in I$.

Let $\mathscr{F} = (\Phi_A, \Im_A)$ be a KKM-structure of A and $P : A \rightrightarrows A$ be a set-valued map. We say that the family $\{T_i\}_{i \in I}$ is Φ_A-weak-convex with respect to (w.r.t.) P if whenever $a \in A$, $\varphi_N \in \Phi_A$, and $M \subset N \cap P(a)$ satisfying $a \in \varphi_N(\Delta_M)$, one has $\varphi_N(\Delta_M) \subset T_i(a)$ for each $i \in I$. When this notion holds with the natural KKM-structure, we say that $\{T_i\}_{i \in I}$ is weak-convex w.r.t. P.

Theorem 4 *Let A be a nonempty set and I be an arbitrary index set. For each $i \in I$, let $T_i : A \rightrightarrows A$. Then, the family $\{T_i\}_{i \in I}$ has a common fixed point if and only if there exist a KKM-structure $\mathscr{F} = (\Phi_A, \Im_A)$ of A and a set-valued mapping $P : A \rightrightarrows A$ such that $A = \bigcup_{a \in A} \text{int}_{\Im_A} P^{-1}(a)$, $\{T_i\}_{i \in I}$ is Φ_A-weak-convex w.r.t. P, and the following condition holds:*

(D) *there exists a nonempty $\Im_A$-compact subset K of A and, for each $N \in \langle A \rangle$, there is a $\Im_A$-compact and Φ_A-convex subset $L_N \subset A$ containing N such that, for each $a \in L_N \backslash K$, there exists $a' \in L_N$ with $a \in \text{int}_{\Im_A} P^{-1}(a')$.*

This theorem is deduced from Theorem 1 by setting $X_i = A$, $X = \prod_{i \in I} X_i = A^I$, $P_i(x) = P(x_i)$, and $Q_i(x) = \bigcap_{j \in I} T_j(x_i)$.

Example 3 Let $A = [0, 1]$, $\mathscr{F}$ be the natural KKM-structure of $[0, 1]$. Let $T_1, T_2, T_3 : A \rightrightarrows A$ be defined by, for all $a \in A$,

$$T_1(a) = \left[0, \frac{a}{3}\right], \quad T_2(a) = \left[\frac{2a}{3}, 1\right], \quad T_3(a) = \left[\frac{a}{3}, \frac{2a}{3}\right].$$

For $P : A \rightrightarrows A$ defined by $P(0) = \{0\}$ and $P(a) = [0, \frac{a}{4})$ if $a \neq 0$, we have: for all $a \in A$, $\varphi_N \in \Phi_A$ and $M \subset N$,

$$\begin{cases} M \subset P(a), \\ a \in \varphi_N(\Delta_M) \end{cases} \Longleftrightarrow \begin{cases} M \subset P(0) = \{0\}, \\ a = 0 \end{cases} \text{or} \begin{cases} M \subset [0, \frac{a}{4}], \\ a \in [\min M, \max M] \end{cases}$$

$$\iff \begin{cases} M = \{0\}, \\ a = 0. \end{cases}$$

Then, $\varphi_N(\Delta_M) = \{0\} \subset T_i(0)$ $(i = 1, 2, 3)$. Thus, $\{T_1, T_2, T_3\}$ is weak-convex w.r.t. P. Moreover, P has the nonempty values and fibers, for $a \in A$,

$$P^{-1}(a) = \begin{cases} A & \text{if } a = 0, \\ (4y, 1] & \text{if } 0 < a \leq \frac{1}{4}, \\ \emptyset & \text{if } \frac{1}{4} < a \leq 1 \end{cases}$$

are open in A. Hence, $A = \bigcup_{a \in A} \operatorname{int} P^{-1}(a)$. Since A is compact, the condition (D) of Theorem 4 holds. Thus, by Theorem 4, there exists $\bar{a} \in A$ such that $\bar{a} \in T_i(\bar{a})$ $(i = 1, 2, 3)$.

3 Applications

Since fixed point theorems have a wide range of applications and fixed-component point results imply corresponding ones for fixed points by considering a family consisting of one element, we can deduce many applications from the results obtained in Sect. 2. Here, we focus on two kinds of applications only. First, we study the existence of some important points which were mentioned in Sect. 1 and seemingly more general than fixed points, to emphasize the generality of our results. Then, we show that from these results the existence of solutions to numerous optimization-related problems can be obtained.

3.1 Coincidence-Component Point Theorems

Theorem 5 *For each $i \in I$, let X_i, Y_i be nonempty sets, $X := \prod_{i \in I} X_i$, $Y := \prod_{i \in I} Y_i$, and $F_i : X \rightrightarrows Y_i$ and $G_i : Y \rightrightarrows X_i$ be nonempty-valued. Assume that, for each $i \in I$, there exist KKM-structures $\mathscr{F}_i := (\Phi_{X_i}, \Im_{X_i})$ of X_i and $\mathscr{G}_i := (\Phi_{Y_i}, \Im_{Y_i})$ of Y_i such that*

- (i) *for each $(x_i, y_i) \in X_i \times Y_i$, $F_i^{-1}(y_i)$ and $G_i^{-1}(x_i)$ are $\Im_X$-open and $\Im_Y$-open, resp;*
- (ii) *for each $(x, y) \in X \times Y$, $F_i(x)$ and $G_i(y)$ are Φ_{Y_i}-convex and Φ_{X_i}-convex, resp;*
- (iii) *there exists a nonempty $\Im_{X \times Y}$-compact subset K of $X \times Y$ and, for each $N_i \in \langle X_i \times Y_i \rangle$, there is a $\Im_{X_i \times Y_i}$-compact and $\Phi_{X_i \times Y_i}$-convex subset $L_{N_i} \subset X_i \times Y_i$ containing N_i such that, for each $(x, y) \in (\prod_{i \in I} L_{N_i}) \backslash K$ and $i \in I$, there exists $(x_i', y_i') \in L_{N_i}$ such that $(x, y) \in F_i^{-1}(y_i') \times G_i^{-1}(x_i')$.*

Then, there exists $\bar{x} := (\bar{x}_i)_{i \in I} \in X$ and $\bar{y} := (\bar{y}_i)_{i \in I} \in Y$ such that $\bar{y}_i \in F_i(\bar{x})$ and $\bar{x}_i \in G_i(\bar{y})$ for all $i \in I$.

Proof For each $i \in I$, let $\mathscr{H}_i := (\Phi_{X_i \times Y_i}, \Im_{X_i \times Y_i})$ be the product KKM-structure of $\mathscr{F}_i$ and $\mathscr{G}_i$ on $X_i \times Y_i$. Let $D_i : X \times Y \rightrightarrows X_i \times Y_i$ be defined by $D_i(x, y) := G_i(y) \times F_i(x)$ for all $(x, y) \in X \times Y$. Then, D_i has the nonempty values, and

$D_i^{-1}(x_i, y_i) = F_i^{-1}(y_i) \times G_i^{-1}(x_i)$ is $\Im_{X\times Y}$-open for all $(x_i, y_i) \in X_i \times Y_i$. Hence, $X \times Y = \bigcup_{(x_i,y_i)\in X_i\times Y_i} \mathrm{int}_{\Im_{X\times Y}} D_i^{-1}(x_i, y_i)$. Assumption (ii) implies that $D_i(x, y)$ is $\Phi_{X_i\times Y_i}$-convex for each $(x, y) \in X \times Y$. Assumption (iii) shows that condition (C) holds for $\{D_i\}_{i\in I}$. Thus, applying Theorem 1 together with condition (h4) and the statement at the end of Remark 1 with $X \times Y$, $X_i \times Y_i$ and D_i replacing X, X_i, $P_i \equiv Q_i$, resp, we have a $(\bar{x}, \bar{y}) \in X \times Y$ such that $(x_i, y_i) \in D_i(x, y)$ for all $i \in I$, i.e., $\bar{y}_i \in F_i(\bar{x})$ and $\bar{x}_i \in G_i(\bar{y})$ for all $i \in I$. □

3.2 Maximal Element Theorems

Theorem 6 *For an arbitrary index set I and $i \in I$, let X_i be a nonempty set and $S_i, T_i : X := \prod_{i\in I} X_i \rightrightarrows X_i$. Assume that, for each $i \in I$, there exists a KKM-structure $\mathscr{F}_i := (\Phi_{X_i}, \Im_{X_i})$ of X_i such that*

(i) *for each $x_i \in X_i$, $S_i^{-1}(x_i)$ is $\Im_X$-open;*
(ii) *for each $x \in X$, Φ_{X_i}-co$S_i(x) \subset T_i(x)$;*
(iii) *there exists $i \in I$ such that $x_i \notin T_i(x)$ for all $x = (x_i)_{i\in I} \in X$;*
(iv) *there exists a nonempty $\Im_X$-compact subset K of X, and, for each $N_i \in \langle X_i\rangle$, there is a $\Im_{X_i}$-compact and Φ_{X_i}-convex subset $L_{N_i} \subset X_i$ containing N_i such that, for each $x \in (\prod_{i\in I} L_{N_i})\backslash K$ and $i \in I$, $S_i(x) \cap L_{N_i} \neq \emptyset$.*

Then, there exist $\bar{x} \in X$ and $i_0 \in I$ such that $S_{i_0}(\bar{x}) = \emptyset$.

Proof Suppose to the contrary that, for all $x \in X$ and $i \in I$, $S_i(x) \neq \emptyset$. Then, this and (i) imply that $X = \bigcup_{x_i\in X_i} S_i^{-1}(x_i) = \bigcup_{x_i\in X_i} \mathrm{int}_{\Im_X} S_i^{-1}(x_i)$ for each $i \in I$. (iv) together with (i) ensures condition (C) for $\{S_i\}_{i\in I}$. Applying Theorem 1 for $\{P_i \equiv S_i\}_{i\in I}$ and $\{Q_i \equiv T_i\}_{i\in I}$ via condition (h3) of Remark 1 and the statement at the end of this remark, we have $\bar{x} := (\bar{x}_i)_{i\in I} \in X$ such that $\bar{x}_i \in T_i(\bar{x})$ for all $i \in I$, contradicting (iii). □

Corollary 2 *Let X be a nonempty set and $S, T : X \rightrightarrows X$. Assume that there exists a KKM-structure $\mathscr{F} := (\Phi_X, \Im_X)$ of X such that*

(i) *for each $x \in X$, $S^{-1}(x)$ is $\Im_X$-open, Φ_X-co$S(x) \subset T(x)$, and $x \notin T(x)$;*
(ii) *there exists a nonempty $\Im_X$-compact subset K of X and, for each $N \in \langle X\rangle$, there is a $\Im_X$-compact and Φ_X-convex subset $L_N \subset X$ containing N such that $S(x) \cap L_N \neq \emptyset$ for each $x \in L_N\backslash K$.*

Then, there exists $\bar{x} \in X$ such that $S(\bar{x}) = \emptyset$.

Proof This result is a special case of Theorem 6 with the index set I being a singleton. □

3.3 Intersection Point Theorems

Theorem 7 *Let $\{X_i\}_{i\in I}$ be a family of nonempty sets, $X := \prod_{i\in I} X_i$, $\{A_i\}_{i\in I}$, $\{B_i\}_{i\in I}$ two families of nonempty subsets of X, and x^i the canonical projection of x on $X^i := \prod_{j\in I, j\neq i} X_j$. Assume that, for each $i \in I$, there exists a KKM-structure $\mathscr{F}_i := (\Phi_{X_i}, \Im_{X_i})$ of X_i such that*

(i) *for each* $x_i \in X_i$, $\{x' \in X | (x_i, x'^i) \in B_i\}$ *is* $\Im_X$*-open;*
(ii) *for each* $x \in X$, $\{x'_i \in X_i | (x'_i, x^i) \in B_i\}$ *is nonempty, and* Φ_{X_i}-co$\{x'_i \in X_i | (x'_i, x^i) \in B_i\} \subset \{x'_i \in X_i | (x'_i, x^i) \in A_i\}$*;*
(iii) *there exists a nonempty* $\Im_X$*-compact subset* K *of* X *and, for each* $N_i \in \langle X_i \rangle$, *there is a* $\Im_{X_i}$*-compact and* Φ_{X_i}*-convex subset* $L_{N_i} \subset X_i$ *containing* N_i *such that, for each* $x \in (\prod_{i\in I} L_{N_i})\backslash K$ *and* $i \in I$, *there exists* $x'_i \in L_{N_i}$ *such that* $(x'_i, x^i) \in B_i$.

Then, $\bigcap_{i\in I} A_i \neq \emptyset$

Proof For each $i \in I$, we define $P_i, Q_i : X \rightrightarrows X_i$ by $P_i(x) := \{x'_i \in X_i | (x'_i, x^i) \in B_i\}$ and $Q_i(x) := \{x'_i \in X_i | (x'_i, x^i) \in A_i\}$ for all $x \in X$. It is clear that the assumptions of Theorem 1 (under the condition (h3)) are satisfied for $\{P_i\}_{i\in I}$ and $\{Q_i\}_{i\in I}$. Applying this theorem, we have $\bar{x} := (\bar{x}_i)_{i\in I} \in X$ such that $\bar{x}_i \in \{x'_i \in X_i | (x'_i, \bar{x}^i) \in A_i\}$ for each $i \in I$, i.e., $\bar{x} = (\bar{x}_i, \bar{x}^i) \in A_i$ for all $i \in I$. □

Corollary 3 *Let* $\{X_i\}_{i\in I}$ *be a family of nonempty sets,* $X := \prod_{i\in I} X_i$, $\{A_i\}_{i\in I}$ *a family of nonempty subsets of* X, *and* x^i *the canonical projection of* x *on* $X^i := \prod_{j\in I, j\neq i} X_j$. *Assume that, for each* $i \in I$, *there exists a KKM-structure* $\mathscr{F}_i = (\Phi_{X_i}, \Im_{X_i})$ *of* X_i *such that*

(i) *for each* $x_i \in X_i$, $\{x' \in X | (x_i, x'^i) \in A_i\}$ *is* $\Im_X$*-open;*
(ii) *for each* $x \in X$, $\{x'_i \in X_i | (x'_i, x^i) \in A_i\}$ *is nonempty and* Φ_{X_i}*-convex;*
(iii) *there exists a nonempty* $\Im_X$*-compact subset* K *of* X *and, for each* $N_i \in \langle X_i \rangle$, *there is a* $\Im_{X_i}$*-compact and* Φ_{X_i}*-convex subset* $L_{N_i} \subset X_i$ *containing* N_i *such that, for each* $x \in (\prod_{i\in I} L_{N_i})\backslash K$ *and* $i \in I$, *there exists* $x'_i \in L_{N_i}$ *such that* $(x'_i, x^i) \in A_i$.

Then, $\bigcap_{i\in I} A_i \neq \emptyset$.

Proof This is a particular case of Theorem 7 with $A_i \equiv B_i$ for all $i \in I$. □

An equivalent formulation of Corollary 3 is the following.

Corollary 4 *For* $i \in I$, *let* X_i *be a nonempty set,* $X := \prod_{i\in I} X_i$, $X^i := \prod_{j\in I, j\neq i} X_j$, x^i *the canonical projection of* x *on* X^i, $f_i : X \to \mathbb{R}$ *and* $\delta_i \in \mathbb{R}$. *Assume that, for each* $i \in I$, *there exists a KKM-structure* $\mathscr{F}_i := (\Phi_{X_i}, \Im_{X_i})$ *of* X_i *such that*

(i) *for each* $x_i \in X_i$, $\{x' \in X | f_i(x_i, x'^i) > \delta_i\}$ *is* $\Im_X$*-open;*
(ii) *for each* $x \in X$, $\{x'_i \in X_i | f_i(x'_i, x^i) > \delta_i\}$ *is nonempty and* Φ_{X_i}*-convex;*
(iii) *there exists a* $\Im_X$*-compact subset* K *of* X, *and, for each* $N_i \in \langle X_i \rangle$, *there is a* $\Im_{X_i}$*-compact and* Φ_{X_i}*-convex subset* $L_{N_i} \subset X_i$ *containing* N_i *such that* $(\prod_{i\in I} L_{N_i})\backslash K \subset \bigcup_{x_i\in L_{N_i}} \{x' \in X | f_i(x_i, x'^i) > \delta_i\}$.

Then, there exists $\bar{x} \in X$ *such that* $f_i(\bar{x}) > \delta_i$ *for all* $i \in I$.

Proof Corollary 3 implies Corollary 4. It is clear that the assumptions of Corollary 4 imply the assumptions of Corollary 3 with $A_i := \{x \in X | f_i(x) > \delta_i\}$ for $i \in I$. Applying Corollary 3, we have $\bar{x} \in X$ such that $\bar{x} \in A_i$ for all $i \in I$, which means that $f_i(\bar{x}) > \delta_i$ for all $i \in I$.

Corollary 4 implies Corollary 3. We define real functions $f_i : X \to \mathbb{R}$ by, for $x \in X$,

$$f_i(x) = \begin{cases} 1 & \text{if } x \in A_i, \\ 0 & \text{if } x \notin A_i. \end{cases}$$

Then, for each $i \in I$, we have $\{x' \in X|\, f_i(x_i, x'^i) > 0\} = \{x' \in X|\, (x_i, x'^i) \in A_i\}$ for each $x_i \in X_i$, and $\{x_i' \in X_i|\, f_i(x_i', x^i) > 0\} = \{x_i' \in X_i|\, (x_i', x^i) \in A_i\}$ for each $x \in X$. Hence, the assumptions of Corollary 3 imply those of Corollary 4 for $\{f_i\}_{i\in I}$ and $\{\delta_i = 0\}_{i\in I}$. Corollary 4 gives $\bar{x} \in X$ satisfying $f_i(\bar{x}) = 1$ for all $i \in I$, i.e., $\bar{x} \in A_i$ for all $i \in I$. □

Remark 4 (a) We can prove that Theorem 6 is still true if we replace x, x', X and $\Im_X$ by x^i, x'^i, X^i and $\Im_{X^i}$, resp, in assumptions (i) and (ii). A similar replacement can be made for Corollaries 3 and 4. In this case, Theorem 6 and Corollary 3 extend some results on sets with convex sections in [11] (Theorems 14–16) and [21] (Theorems 2.3 and 2.4).

(b) Assumption (i) of Corollary 4 is satisfied if, for each $x_i \in X_i$, $f_i(x_i, \cdot)$ is $\Im_{X^i}$-lower semicontinuous on X^i. Corollary 4 generalizes Theorem 3 of [10] and Theorem 2.5 of [21].

Corollary 5 *For $i \in I$, let X_i be a nonempty set, $X := \prod_{i\in I} X_i$, $X^i := \prod_{j\in I, j\neq i} X_j$, x^i the canonical projection of x on X^i, and $f_i : X \to \mathbb{R}$. Assume that there exists a compact KKM-structure $\mathscr{F}_i = (\Phi_{X_i}, \Im_{X_i})$ of X_i for each $i \in I$ such that*

(i) *f_i is $\Im_X$-continuous;*

(ii) *for any $\epsilon > 0$ sufficiently close to 0 and $x \in X$, the set $\{a_i \in X_i \mid f_i(x^i, a_i) > \max_{a_i' \in X_i} f_i(x^i, a_i') - \epsilon\}$ is Φ_{X_i}-convex.*

Then, there exists $\bar{x} \in X$ such that $f_i(\bar{x}) = \max_{a_i\in X_i} f_i(\bar{x}^i, a_i)$ for all $i \in I$.

Proof Applying Corollary 4 for functions $h_i : X \longrightarrow \mathbb{R}$ defined by $h_i(x) = f_i(x) - \max_{a_i\in X_i} f_i(a_i, x^i)$ and $\delta_i = -\epsilon$, we have $\bar{x} \in X$ such that $h_i(\bar{x}) > -\epsilon$. Since ϵ is arbitrary, the proof is complete. □

Note that results similar to Corollaries 4 and 5 were proved in [19] for the case with a finite index set I.

3.4 Systems of Variational Relations

In this subsection, we discuss applications to a general model of systems of variational relations since it encompasses most problems related to optimization. Variational relations were first studied in [16,25] and then extended to a case of a system of relations in [23]. Let I be an index set, $\{X_i\}_{i\in I}$ be a family of nonempty sets and $X := \prod_{i\in I} X_i$. Let $\Lambda_i, \Omega_i : X \rightrightarrows X_i$ be nonempty-valued and $R_i(x, a_i)$ be a relation linking $x \in X$ and $a_i \in X_i$. We consider the following system of variational relations

$$\text{(SVR)} \quad \begin{cases} \text{find } \bar{x} \in X \text{ such that, for all } i \in I, \ \bar{x}_i \in \Lambda_i(\bar{x}) \text{ and } a_i \in \Omega_i(\bar{x}), \\ R_i(\bar{x}, a_i) \text{ holds.} \end{cases}$$

Problem (SVR) was studied in [15] for the case where $\{X_i\}_{i\in I}$ is a family of nonempty convex subsets of topological vector spaces. In this section, using Theorem 1 we will establish an existence result for the general case.

For $i \in I$, we set $\Theta_i(x) := \{a_i \in X_i \mid R_i(x, a_i)$ does not hold$\}$ for all $x \in X$, and $W_i := \{x \in X \mid$ for all $a_i \in \Omega_i(x)$, $R_i(x, a_i)$ holds$\}$.

Theorem 8 *For problem* (SVR)*, assume that there exists a compact KKM-structure* $\mathscr{F}_i := (\Phi_{X_i}, \Im_{X_i})$ *of* X_i *for each* $i \in I$ *such that*

(i) $X = \bigcup_{a_i\in X_i} \mathrm{int}_{\Im_X}\big((\Theta_i^{-1}(a_i) \cup W_i) \cap \Omega_i^{-1}(a_i)\big)$;
(ii) Λ_i *is* Φ_{X_i}*-weak-convex w.r.t.* Ω_i*;*
(iii) *for each* $x := (x_i)_{i\in I} \in X$*,* $x_i \notin \Phi_{X_i}$-co$\Theta_i(x)$.

Then, (SVR) *has a solution.*

Proof For each $i \in I$, we define set-valued maps $P_i, Q_i : X \rightrightarrows X_i$ by

$$P_i(x) = \begin{cases} \Theta_i(x) \cap \Omega_i(x) & \text{if } x \notin W_i, \\ \Omega_i(x) & \text{if } x \in W_i, \end{cases}$$

$$Q_i(x) = \begin{cases} \Phi_{X_i}-\mathrm{co}\Theta_i(x) & \text{if } x \notin W_i, \\ \Lambda_i(x) & \text{if } x \in W_i. \end{cases}$$

For each $a_i \in X_i$ we have

$$\begin{aligned} P_i^{-1}(a_i) &= \Big(\Theta_i^{-1}(a_i) \cap \Omega_i^{-1}(a_i) \cap (X \setminus W_i)\Big) \cup \Big(\Omega_i^{-1}(a_i) \cap W_i\Big) \\ &= \Big([\Theta_i^{-1}(a_i) \cap (X \setminus W_i)] \cup W_i\Big) \cap \Omega_i^{-1}(a_i) \\ &= (\Theta_i^{-1}(a_i) \cup W_i) \cap \Omega_i^{-1}(a_i). \end{aligned} \tag{5}$$

The equality (5) and assumption (i) imply that $X = \bigcup_{a_i\in X_i} \mathrm{int}_{\Im_X} P_i^{-1}(a_i)$.

Take any $\varphi_{N_i} \in \Phi_{X_i}$, $M_i \subset N_i$, $x = (x_i)_{i\in I} \in X$ satisfying $x_i \in \varphi_{N_i}(\Delta_{M_i})$, and $M_i \subset P_i(x)$. As $x \notin W_i$, $M_i \subset \Theta_i(x) \cap \Omega_i(x) \subset \Theta_i(x)$. Then, $\varphi_{N_i}(\Delta_{M_i}) \subset \Phi_{X_i}$-co$\Theta_i(x) = Q_i(x)$. If $x \in W_i$, then $M_i \subset \Omega_i(x)$. By (ii) we obtain $\varphi_{N_i}(\Delta_{M_i}) \subset \Lambda_i(x) = Q_i(x)$. Thus, Q_i is Φ_{X_i}-weak-convex w.r.t. P_i.

Theorem 1 under condition (h1) of Remark 1 implies that an $\bar{x} := (\bar{x}_i)_{i\in I} \in X$ exists such that $\bar{x}_i \in Q_i(\bar{x})$ for each $i \in I$. If $\bar{x} \notin W_i$, then $\bar{x}_i \in \Phi_{X_i}$-co$\Theta_i(\bar{x})$, contradicting (iii). Hence, $\bar{x} \in W_i$, and therefore $\bar{x}_i \in \Lambda_i(\bar{x})$ and $R_i(\bar{x}, a_i)$ holds for all $a_i \in \Omega_i(\bar{x})$. □

Remark 5 (a) Because the map P_i in the proof of Theorem 8 has the nonempty values, assumption (i) can be replaced by the following

(i′) *for each* $a_i \in X_i$, $(\Theta_i^{-1}(a_i) \cup W_i) \cap \Omega_i^{-1}(a_i)$ *is* $\Im_X$*-open.*

Furthermore, (i′) is satisfied if

(i″) W_i *is* $\Im_X$*-open, and, for each* $a_i \in X_i$, $\Theta_i^{-1}(a_i) \cap (X \setminus W_i)$ *is relatively* $\Im_X$*-open in* $X \setminus W_i$ *and* $\Omega_i^{-1}(a_i)$ *is* $\Im_X$*-open.*

Indeed, as $\Theta_i^{-1}(a_i) \cap (X \setminus W_i)$ is relatively $\Im_X$-open in $X \setminus W_i$, a $\Im_X$-open set $U_i \subset X$ exists such that $\Theta_i^{-1}(a_i) \cap (X \setminus W_i) = U_i \cap (X \setminus W_i)$. Then, $(\Theta_i^{-1}(a_i) \cup W_i) \cap \Omega_i^{-1}(a_i)$ = $([U_i \cap (X \setminus W_i)] \cup W_i) \cap \Omega_i^{-1}(a_i)$ = $(U_i \cup W_i) \cap \Omega_i^{-1}(a_i)$ is $\Im_X$-open.

(b) It is clear that assumption (iii) of Theorem 8 is fulfilled if "*for each* $x := (x_i)_{i \in I} \in X$, $R_i(x, x_i)$ *holds, and* $\Theta_i(x)$ *is* Φ_{X_i}*-convex*."

3.5 Abstract Economies

Finally, we discuss a practical model. Let I be any set of agents. For each $i \in I$, let X_i be a nonempty set of actions available for the agent i and $X := \prod_{i \in I} X_i$. An abstract economy (see [27]) is a family of ordered triples $\mathscr{E} := (X_i, A_i, B_i)_{i \in I}$, where $A_i : X \rightrightarrows X_i$ is a constraint correspondence such that $A_i(x)$ is the state attainable for the agent i at x, and $B_i : X \rightrightarrows X_i$ is a preference correspondence such that $B_i(x)$ is the state preference by the agent i at x. An equilibrium point of $\mathscr{E}$ is a point $\bar{x} \in X$ such that $\bar{x}_i \in A_i(\bar{x})$ and $B_i(\bar{x}) \cap A_i(\bar{x}) = \emptyset$ for each $i \in I$.

Of course, we can deduce the following existence result from our results in Sect. 2. But, to explain the generality of variational relations studied in the preceding subsection we will apply Theorem 7.

Theorem 9 *For the abstract economy* $\mathscr{E} := (X_i, A_i, B_i)_{i \in I}$, *let* $W_i := \{x \in X \,|\, B_i(x) \cap A_i(x) = \emptyset\}$ *for each* $i \in I$. *Assume that, for each* $i \in I$, *there exists a compact KKM-structure* $\mathscr{F}_i = (\Phi_{X_i}, \Im_{X_i})$ *of* X_i *such that*

(i) $X = \bigcup_{a_i \in X_i} \mathrm{int}_{\Im_X}\big((B_i^{-1}(a_i) \cup W_i) \cap A_i^{-1}(a_i)\big)$;
(ii) *for each* $x \in X$, $A_i(x)$ *is* Φ_{X_i}*-convex;*
(iii) *for each* $x = (x_i)_{i \in I} \in X$, $x_i \notin \Phi_{X_i}\text{-co}B_i(x)$;

Then, $\mathscr{E}$ *has an equilibrium point.*

Proof We see that $\bar{x} \in X$ is an equilibrium point of $\mathscr{E}$ if and only if, for each $i \in I$, $\bar{x}_i \in A_i(\bar{x})$, and for all $a_i \in A_i(\bar{x})$, $a_i \notin B_i(\bar{x})$. This means that $\bar{x}$ is a solution of (SVR) with $\Lambda_i = A_i$, $\Omega_i = A_i$ and the relation R_i defined by: $R_i(x, a_i)$ holds if and only if $a_i \notin B_i(x)$. Applying Theorem 7, we complete the proof. □

Remark 6 The observations in Remark 5 are valid also for Theorem 9. Furthermore, if A_i and B_i are $(\Im_X, \Im_{X_i})$-closed (i.e., their graphs are $\Im_X \times \Im_{X_i}$-closed in $X \times X_i$), then W_i is $\Im_X$-closed. Hence, assumption (i) of Theorem 9 is satisfied if

(i') A_i *and* B_i *are* $(\Im_X, \Im_{X_i})$*-closed, and their fibers are* $\Im_X$*-open.*

In some cases condition (*i*') is restrictive, e.g., if the topologies $\Im_{X_i}$ are connected (in particular, if KKM-structures $\mathscr{F}_i = (\Phi_{X_i}, \Im_{X_i})$ are the natural KKM-structures), then (i') is fulfilled if and only if A_i and B_i are constant maps. However, there are instances where this condition is relatively easy to check.

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