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Weighted Composition Operators from Analytic Besov Spaces into the Bloch Space

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Abstract In this paper, we give some new essential norm estimates of weighted composition operators uC_φ from analytic Besov spaces into the Bloch space, where u is a function analytic on the unit disk $\mathbb{D}$ and φ is an analytic self-map of $\mathbb{D}$. Moreover, new characterizations for the boundedness, compactness and essential norm of weighted composition operators uC_φ are obtained by the nth power of the symbol φ and the Volterra operators I_u and J_u.

Keywords Besov space · Bloch space · Essential norm · Weighted composition operator

Mathematics Subject Classification 30H30 · 47B33

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1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. Let $S(\mathbb{D})$ denote the set of all analytic self-maps of $\mathbb{D}$. The Bloch space, denoted by $\mathcal{B} = \mathcal{B}(\mathbb{D})$, is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_\beta = \sup_{z\in\mathbb{D}} \left(1 - |z|^2\right) \left|f'(z)\right| < \infty.$$

Under the norm $\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_\beta$, the Bloch space is a Banach space.

For $p \in (1, \infty)$, the analytic Besov space B_p is the set of all $f \in H(\mathbb{D})$ for which

$$b_p(f)^p := \int_{\mathbb{D}} \left|f'(z)\right|^p \left(1 - |z|^2\right)^{p-2} dA(z) < \infty,$$

where dA is the normalized area measure on $\mathbb{D}$. The quantity b_p is a seminorm and the Besov norm is defined by $\|f\|_{B_p} = |f(0)| + b_p(f)$. In particular, B_2 is the classical Dirichlet space with an equivalent norm. See [18] for more results of the analytic Besov space.

Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. For $f \in H(\mathbb{D})$, the composition operator C_φ and the multiplication operator M_u are defined by

$$(C_\varphi f)(z) = f(\varphi(z)) \quad \text{and} \quad (M_u f)(z) = u(z) f(z),$$

respectively. The weighted composition operator uC_φ is defined by

$$(uC_\varphi f)(z) = u(z) \cdot f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

It is clear that the weighted composition operator uC_φ is the generalization of C_φ and M_u. A main problem concerning concrete operators (such as composition operator, multiplication operator, weighted composition operator, Toeplitz operator and Hankel operator) is to relate operator theoretic properties to their function theoretic properties of their symbols.

It is well known that C_φ is bounded on $\mathcal{B}$ by the Schwarz-Pick lemma for any $\varphi \in S(\mathbb{D})$. The compactness of C_φ on $\mathcal{B}$ was studied in [10,14,16]. Wulan et al. [16] proved that $C_\varphi : \mathcal{B} \to \mathcal{B}$ is compact if and only if $\lim_{n\to\infty} \|\varphi^n\|_{\mathcal{B}} = 0$. Zhao [17] showed that $\|C_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} = \frac{e}{2} \limsup_{n\to\infty} \|\varphi^n\|_{\mathcal{B}}$. Ohno and Zhao [13] studied the boundedness and compactness of the weighted composition operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$. The essential norm of the operator $uC_\varphi : \mathcal{B} \to \mathcal{B}$ was studied in [5,9,11]. For more results on composition operator and weighted composition operators mapping into the Bloch space, see [1–3,5–11,13,15–17] and the related references therein.

In [2], the authors characterized the boundedness and compactness of weighted composition operator $uC_\varphi : B_p \to \mathcal{B}$. Among others, they proved that, under the assumption that $uC_\varphi : B_p \to \mathcal{B}$ is bounded, $uC_\varphi : B_p \to \mathcal{B}$ is compact if and only if $\lim_{|\varphi(z)|\to 1} \|uC_\varphi f_{\varphi(z)}\|_{\mathcal{B}} = 0$ and $\lim_{n\to\infty} \|u\varphi^n\|_{\mathcal{B}} = 0$, as well as if and only if

$$\lim_{|\varphi(z)|\to 1}\left(1-|z|^2\right)\left|u'(z)\right|\left(\log\frac{2}{1-|\varphi(z)|^2}\right)^{1-\frac{1}{p}}=0$$

and

$$\lim_{|\varphi(z)|\to 1}\frac{\left(1-|z|^2\right)\left|u(z)\varphi'(z)\right|}{1-|\varphi(z)|^2}=0.$$

Here

$$f_a(z)=\left(\log\frac{e}{1-|a|^2}\right)^{-\frac{1}{p}}\log\frac{e}{1-\bar{a}z}.$$

Motivated by the above result, in this paper, we give the corresponding estimates for the essential norm of the operator $uC_\varphi : B_p \to \mathcal{B}$. Moreover, we give a new characterization for the boundedness, compactness and essential norm for the operator $uC_\varphi : B_p \to \mathcal{B}$.

Recall that the essential norm $\|T\|_{e,X\to Y}$ of a bounded linear operator $T : X \to Y$ is defined as the distance from T to the set of compact operators K mapping X into Y, that is, $\|T\|_{e,X\to Y} = \inf\{\|T - K\|_{X\to Y} : K \text{ is compact }\}$, where $\|\cdot\|_{X\to Y}$ is the operator norm.

Throughout this paper, we say that $A \lesssim B$ if there exists a constant C such that $A \le CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2 Essential Norm of $uC_\varphi : B_p \to \mathcal{B}$

In this section, we give some estimates for the essential norm of the operator $uC_\varphi : B_p \to \mathcal{B}$. For this purpose, we need some lemmas which will be used in the proofs of the main results in this paper.

Lemma 2.1 [14] *Let X, Y be two Banach spaces of analytic functions on $\mathbb{D}$. Suppose that*

(1) *The point evaluation functionals on Y are continuous.*
(2) *The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.*
(3) *$T : X \to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.*

Then, T is a compact operator if and only if given a bounded sequence $\{f_n\}$ in X such that $f_n \to 0$ uniformly on compact sets, then the sequence $\{Tf_n\}$ converges to zero in the norm of Y.

Lemma 2.2 [2] *Let $1 < p < \infty$. If $f \in B_p$, then*

(i) $|f(z)| \lesssim \|f\|_{B_p}\left(\log\frac{2}{1-|z|^2}\right)^{1-\frac{1}{p}}$, *for every $z \in \mathbb{D}$;*

(ii) $|f'(z)| \lesssim \frac{1}{1-|z|^2}\|f\|_{B_p}$, *for every* $z \in \mathbb{D}$.

Let $a \in \mathbb{D}$. We define

$$g_a(z) = \frac{\left(\log \frac{e}{1-\bar{a}z}\right)^2}{\left(\log \frac{e}{1-|a|^2}\right)^{1+\frac{1}{p}}}, \qquad h_a(z) = \frac{(a-z)\left(1-|a|^2\right)}{(1-\bar{a}z)^2}, \quad z \in \mathbb{D}.$$

We state and prove the first result in this section.

Theorem 2.1 *Let* $1 < p < \infty$, $u \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$ *such that* $uC_\varphi : B_p \to \mathcal{B}$ *is bounded. Then*

$$\|uC_\varphi\|_{e,B_p\to\mathcal{B}} \approx \max\left\{A, B, C\right\} \approx \max\left\{E, F\right\},$$

where

$$A := \limsup_{|a|\to 1} \|uC_\varphi f_a\|_{\mathcal{B}}, \quad B := \limsup_{|a|\to 1} \|uC_\varphi g_a\|_{\mathcal{B}}, C := \limsup_{|a|\to 1} \|uC_\varphi h_a\|_{\mathcal{B}},$$

$$E := \limsup_{|\varphi(z)|\to 1} \left(1-|z|^2\right)\left|u'(z)\right|\left(\log \frac{e}{1-|\varphi(z)|^2}\right)^{1-\frac{1}{p}}$$

and

$$F := \limsup_{|\varphi(z)|\to 1} \frac{\left(1-|z|^2\right)\left|u(z)\varphi'(z)\right|}{1-|\varphi(z)|^2}.$$

Proof Without loss of generality, we assume that $\|\varphi\|_\infty = 1$. We first prove that

$$\max\left\{A, B, C\right\} \lesssim \|uC_\varphi\|_{e,B_p\to\mathcal{B}}. \tag{2.1}$$

As shown in [2], $f_a, g_a, h_a \in B_p$, $\|f_a\|_{B_p}$, $\|g_a\|_{B_p}$, $\|h_a\|_{B_p}$ are bounded by a constant independent of a, and the all f_a, g_a and h_a converge to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. Thus, for any compact operator $K : B_p \to \mathcal{B}$, by Lemma 2.1 we have

$$\lim_{|a|\to 1} \|Kf_a\|_{\mathcal{B}} = 0, \lim_{|a|\to 1} \|Kg_a\|_{\mathcal{B}} = 0, \lim_{|a|\to 1} \|Kh_a\|_{\mathcal{B}} = 0.$$

Since $uC_\varphi : B_p \to \mathcal{B}$ is bounded, we have

$$\|uC_\varphi - K\|_{B_p\to\mathcal{B}} \gtrsim \|(uC_\varphi - K)f_a\|_{\mathcal{B}} \geq \|uC_\varphi(f_a)\|_{\mathcal{B}} - \|Kf_a\|_{\mathcal{B}},$$
$$\|uC_\varphi - K\|_{B_p\to\mathcal{B}} \gtrsim \|(uC_\varphi - K)g_a\|_{\mathcal{B}} \geq \|uC_\varphi(g_a)\|_{\mathcal{B}} - \|Kg_a\|_{\mathcal{B}}$$

and

$$\|uC_\varphi - K\|_{B_p\to\mathcal{B}} \gtrsim \|(uC_\varphi - K)h_a\|_{\mathcal{B}} \geq \|uC_\varphi(h_a)\|_{\mathcal{B}} - \|Kh_a\|_{\mathcal{B}}.$$

Thus,

$$\|uC_\varphi - K\|_{B_p\to\mathcal{B}} \gtrsim A, \qquad \|uC_\varphi - K\|_{B_p\to\mathcal{B}} \gtrsim B, \qquad \|uC_\varphi - K\|_{B_p\to\mathcal{B}} \gtrsim C.$$

Therefore,

$$\|uC_\varphi\|_{e,B_p\to\mathcal{B}} = \inf_K \|uC_\varphi - K\|_{B_p\to\mathcal{B}} \gtrsim \max\Big\{A, B, C\Big\}.$$

Next, we prove that

$$\max\Big\{E, F\Big\} \lesssim \|uC_\varphi\|_{e,B_p\to\mathcal{B}}. \tag{2.2}$$

Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. Define

$$k_j(z) = \frac{\log \frac{e}{1-\overline{\varphi(z_j)}z}}{\left(\log \frac{e}{1-|\varphi(z_j)|^2}\right)^{\frac{1}{p}}} - \frac{1}{2}\frac{\left(\log \frac{e}{1-\overline{\varphi(z_j)}z}\right)^2}{\left(\log \frac{e}{1-|\varphi(z_j)|^2}\right)^{1+\frac{1}{p}}}$$

and

$$l_j(z) = \frac{\big(\varphi(z_j) - z\big)\left(1 - |\varphi(z_j)|^2\right)}{\left(1 - \overline{\varphi(z_j)}z\right)^2}.$$

We know that the both k_j and l_j belong to B_p and converge to zero uniformly on compact subsets of $\mathbb{D}$. Moreover,

$$\left|k_j\left(\varphi(z_j)\right)\right| = \frac{1}{2}\left(\log \frac{e}{1 - |\varphi(z_j)|^2}\right)^{1-\frac{1}{p}}, \qquad k_j'\left(\varphi(z_j)\right) = 0,$$

and

$$l_j\left(\varphi(z_j)\right) = 0, \qquad \left|l_j'\left(\varphi(z_j)\right)\right| = \frac{1}{1 - \left|\varphi(z_j)\right|^2}.$$

Then, for any compact operator $K : B_p \to \mathcal{B}$, we obtain

$$\|uC_\varphi - K\|_{B_p\to\mathcal{B}} \gtrsim \|(uC_\varphi - K)k_j\|_{\mathcal{B}} \gtrsim \|uC_\varphi(k_j)\|_{\mathcal{B}} - \|Kk_j\|_{\mathcal{B}}$$

and

$$\|uC_\varphi - K\|_{B_p\to\mathcal{B}} \gtrsim \|(uC_\varphi - K)l_j\|_{\mathcal{B}} \gtrsim \|uC_\varphi(l_j)\|_{\mathcal{B}} - \|Kl_j\|_{\mathcal{B}}.$$

Taking $j \to \infty$, we get

$$\begin{aligned}\|uC_\varphi - K\|_{B_p\to\mathcal{B}} &\gtrsim \limsup_{j\to\infty} \|uC_\varphi(k_j)\|_{\mathcal{B}} \\ &\gtrsim \limsup_{j\to\infty} \left(1-|z_j|^2\right)\left|u'(z_j)\right|\left(\log\frac{e}{1-|\varphi(z_j)|^2}\right)^{1-\frac{1}{p}} \\ &= \limsup_{|\varphi(z)|\to 1} \left(1-|z|^2\right)\left|u'(z)\right|\left(\log\frac{e}{1-|\varphi(z)|^2}\right)^{1-\frac{1}{p}} = E\end{aligned}$$

and

$$\begin{aligned}\|uC_\varphi - K\|_{B_p\to\mathcal{B}} &\gtrsim \limsup_{j\to\infty} \|uC_\varphi(l_j)\|_{\mathcal{B}} \gtrsim \limsup_{j\to\infty} \frac{\left(1-|z_j|^2\right)|\varphi'(z_j)||u(z_j)|}{1-|\varphi(z_j)|^2} \\ &= \limsup_{|\varphi(z)|\to 1} \frac{\left(1-|z|^2\right)|\varphi'(z)||u(z)|}{1-|\varphi(z)|^2} = F.\end{aligned}$$

Hence, we obtain (2.2).

Now, we show that

$$\|uC_\varphi\|_{e,B_p\to\mathcal{B}} \lesssim \max\left\{A, B, C\right\} \quad \text{and} \quad \|uC_\varphi\|_{e,B_p\to\mathcal{B}} \lesssim \max\left\{E, F\right\}. \tag{2.3}$$

For $r \in [0, 1)$, set $K_r : H(\mathbb{D}) \to H(\mathbb{D})$ by

$$(K_r f)(z) = f_r(z) = f(rz), \quad f \in H(\mathbb{D}).$$

It is clear that $f_r \to f$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1$. Moreover, the operator K_r is compact on B_p and $\|K_r\|_{B_p\to B_p} \le 1$. Let $\{r_j\} \subset (0, 1)$ be a sequence such that $r_j \to 1$ as $j \to \infty$. Then, for positive integer j, the operator $uC_\varphi K_{r_j} : B_p \to \mathcal{B}$ is compact. By the definition of the essential norm,

$$\|uC_\varphi\|_{e,B_p\to\mathcal{B}} \le \limsup_{j\to\infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{B_p\to\mathcal{B}}.$$

To give (2.3), we only need to show that

$$\limsup_{j\to\infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{B_p\to\mathcal{B}} \lesssim \max\left\{A, B, C\right\} \tag{2.4}$$

and

$$\limsup_{j\to\infty} \|uC_\varphi - uC_\varphi K_{r_j}\|_{B_p\to\mathcal{B}} \lesssim \max\left\{E, F\right\}. \tag{2.5}$$

For $f \in B_p$ with $\|f\|_{B_p} \le 1$, we consider

$$\begin{aligned}&\|(uC_\varphi - uC_\varphi K_{r_j})f\|_{\mathcal{B}}\\ &\quad= |u(0)f(\varphi(0)) - u(0)f_{r_j}(\varphi(0))| + \|u\cdot\left(f - f_{r_j}\right)\circ\varphi\|_\beta.\end{aligned}$$

It is obvious that $\lim_{j\to\infty} |u(0)f(\varphi(0)) - u(0)f_{r_j}(\varphi(0))| = 0$. Let $N \in \mathbb{N}$ be large enough such that $r_j \ge \frac{1}{2}$ for all $j \ge N$ and we have

$$\limsup_{j\to\infty} \|u\cdot\left(f - f_{r_j}\right)\circ\varphi\|_\beta \lesssim Q_1 + Q_2 + Q_3 + Q_4, \tag{2.6}$$

where

$$\begin{aligned}Q_1 &:= \limsup_{j\to\infty}\sup_{|\varphi(z)|\le r_N}\left(1-|z|^2\right)\left|\left(f - f_{r_j}\right)'(\varphi(z))||\varphi'(z)||u(z)\right|,\\ Q_2 &:= \limsup_{j\to\infty}\sup_{|\varphi(z)|> r_N}\left(1-|z|^2\right)\left|\left(f - f_{r_j}\right)'(\varphi(z))||\varphi'(z)||u(z)\right|,\\ Q_3 &:= \limsup_{j\to\infty}\sup_{|\varphi(z)|\le r_N}\left(1-|z|^2\right)\left|\left(f - f_{r_j}\right)(\varphi(z))||u'(z)\right|\end{aligned}$$

and

$$Q_4 := \limsup_{j\to\infty}\sup_{|\varphi(z)|> r_N}\left(1-|z|^2\right)\left|\left(f - f_{r_j}\right)(\varphi(z))||u'(z)\right|.$$

Since $uC_\varphi : B_p \to \mathcal{B}$ is bounded, from [2] we see that $u \in \mathcal{B}$ and

$$\widetilde{Q} := \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)|\varphi'(z)||u(z)| < \infty.$$

Since $f'_{r_j} \to f'$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$, we have

$$Q_1 \le \widetilde{Q}\limsup_{j\to\infty}\sup_{|w|\le r_N}\left|f'(w) - f'_{r_j}(w)\right| = 0. \tag{2.7}$$

Similarly, from the fact that $u \in \mathcal{B}$ and $f_{r_j} \to f$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$, we have

$$Q_3 \le \|u\|_{\mathcal{B}}\limsup_{j\to\infty}\sup_{|w|\le r_N}\left|f(w) - f(r_jw)\right| = 0. \tag{2.8}$$

It is clear that $Q_2 = \limsup_{j\to\infty} Q_{21}$, where

$$Q_{21} := \sup_{|\varphi(z)|>r_N} \left(1-|z|^2\right) \left|\left(f - f_{r_j}\right)'(\varphi(z))||\varphi'(z)||u(z)\right|.$$

Using Lemma 2.2 and the fact that $\|f\|_{B_p} \le 1$, we have

$$\begin{aligned}
Q_{21} &= \sup_{|\varphi(z)|>r_N} \left(1-|z|^2\right) \left|\left(f - f_{r_j}\right)'(\varphi(z))||\varphi'(z)||u(z)\right| \\
&\lesssim \|f - f_{r_j}\|_{B_p} \sup_{|\varphi(z)|>r_N} \frac{\left(1-|z|^2\right)|\varphi'(z)||u(z)|}{1-|\varphi(z)|^2} \\
&\lesssim \sup_{|\varphi(z)|>r_N} \left(1-|z|^2\right) \Big|u'(z)\frac{(\varphi(z)-\varphi(z))(1-|\varphi(z)|^2)}{(1-|\varphi(z)|^2)^2} \\
&\quad + u(z)\varphi'(z)\frac{(1-|\varphi(z)|^2)(2|\varphi(z)|^2-1-|\varphi(z)|^2)}{(1-|\varphi(z)|^2)^3}\Big| \\
&\lesssim \sup_{|\varphi(z)|>r_N} \sup_{|a|>r_N} \left(1-|z|^2\right) \Big|u'(z)\frac{(a-\varphi(z))\left(1-|a|^2\right)}{(1-\bar{a}\varphi(z))^2} \\
&\quad + u(z)\varphi'(z)\frac{\left(1-|a|^2\right)(2|a|^2-1-\bar{a}\varphi(z))}{(1-\bar{a}\varphi(z))^3}\Big| \\
&= \sup_{|\varphi(z)|>r_N} \sup_{|a|>r_N} \left(1-|z|^2\right) |(uC_\varphi h_a)'(z)| \\
&\lesssim \sup_{|a|>r_N} \left\|uC_\varphi(h_a)\right\|_{\mathcal{B}}.
\end{aligned}$$

Letting $N \to \infty$,

$$\limsup_{N\to\infty} Q_{21} \lesssim \limsup_{|\varphi(z)|\to 1} \frac{\left(1-|z|^2\right)|\varphi'(z)||u(z)|}{1-|\varphi(z)|^2} (= F) \lesssim \limsup_{|a|\to 1} \left\|uC_\varphi(h_a)\right\|_{\mathcal{B}}.$$

Hence,

$$Q_2 \lesssim F \lesssim C. \tag{2.9}$$

We know that $Q_4 = \limsup_{j\to\infty} Q_{41}$, where

$$Q_{41} := \sup_{|\varphi(z)|>r_N} \left(1-|z|^2\right) |\left(f - f_{r_j}\right)(\varphi(z))||u'(z)|.$$

Using a similar estimates to Q_{21}, Lemma 2.2 and the fact that $\|f\|_{B_p} \leq 1$, we get

$$\begin{aligned} Q_{41} &= \sup_{|\varphi(z)|>r_N} \left(1-|z|^2\right) |\left(f - f_{r_j}\right)(\varphi(z))||u'(z)| \\ &\lesssim \frac{1}{2} \sup_{|\varphi(z)|>r_N} \left(1-|z|^2\right) |u'(z)| \left(\log \frac{e}{1-|\varphi(z)|^2}\right)^{1-\frac{1}{p}} \\ &\lesssim \sup_{|a|>r_N} \left\| uC_\varphi \left(f_a - \frac{1}{2} g_a\right)\right\|_{\mathcal{B}} \lesssim \sup_{|a|>r_N} \left\|uC_\varphi(f_a)\right\|_{\mathcal{B}} + \sup_{|a|>r_N} \left\|uC_\varphi(g_a)\right\|_{\mathcal{B}}. \end{aligned}$$

Taking $N \to \infty$,

$$\begin{aligned} \limsup_{N\to\infty} Q_{41} &\lesssim \limsup_{|\varphi(z)|\to 1} \left(1-|z|^2\right) |u'(z)| \left(\log \frac{e}{1-|\varphi(z)|^2}\right)^{1-\frac{1}{p}} \ (= E) \\ &\lesssim \limsup_{|a|\to 1} \left\|uC_\varphi(f_a)\right\|_{\mathcal{B}} + \limsup_{|a|\to 1} \left\|uC_\varphi(g_a)\right\|_{\mathcal{B}} = A + B, \end{aligned}$$

and then

$$Q_4 \lesssim E \lesssim A + B. \tag{2.10}$$

Thus, by the above estimates (2.6)–(2.10) we get (2.4) and (2.5). Therefore,

$$\|uC_\varphi\|_{e, B_p\to\mathcal{B}} \lesssim E + F \lesssim \max\left\{E, F\right\}$$

and

$$\|uC_\varphi\|_{e, B_p\to\mathcal{B}} \lesssim A + B + C \lesssim \max\left\{A, B, C\right\}.$$

Hence, by (2.1)–(2.3) we get the desired result. This completes the proof of this theorem. □

Theorem 2.2 *Let* $1 < p < \infty$, $u \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$ *such that* $uC_\varphi : B_p \to \mathcal{B}$ *is bounded. Then*

$$\|uC_\varphi\|_{e, B_p\to\mathcal{B}} \approx \max\left\{\limsup_{|a|\to 1} \left\|uC_\varphi f_a\right\|_{\mathcal{B}}, \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{B}}\right\}.$$

Proof Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. Define

$$q_a(z) = \frac{1-|a|^2}{1-\bar{a}z}.$$

It is easy to check that the both $f_{\varphi(z_j)}$ and $q_{\varphi(z_j)}$ belong to B_p and converge to zero uniformly on compact subsets of $\mathbb{D}$. Here, f_a is defined in the proof of Theorem 2.1. Since

$$
\begin{aligned}
\|uC_\varphi f_{\varphi(z_j)}\|_{\mathcal{B}} &\geq \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)|u(z)f'_{\varphi(z_j)}(\varphi(z))\varphi'(z)+u'(z)f_{\varphi(z_j)}(\varphi(z))| \\
&\geq \left(1-|z_j|^2\right)|u'(z_j)|\left(\log\frac{e}{1-|\varphi(z_j)|^2}\right)^{1-\frac{1}{p}} \\
&\quad -\frac{\left(1-|z_j|^2\right)|u(z_j)\overline{\varphi(z_j)}\varphi'(z_j)|}{1-|\varphi(z_j)|^2}\left(\log\frac{e}{1-|\varphi(z_j)|^2}\right)^{-\frac{1}{p}}
\end{aligned}
$$

and

$$
\begin{aligned}
\|uC_\varphi q_{\varphi(z_j)}\|_{\mathcal{B}} &\geq \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)|u(z)q'_{\varphi(z_j)}(\varphi(z))\varphi'(z)+u'(z)q_{\varphi(z_j)}(\varphi(z))| \\
&\geq \frac{\left(1-|z_j|^2\right)|u(z_j)\,\overline{\varphi(z_j)}\varphi'(z_j)|}{1-|\varphi(z_j)|^2}-\left(1-|z_j|^2\right)|u'(z_j)|.
\end{aligned}
$$

Taking limit as $j\to\infty$ to the last two inequalities on both sides, we get

$$
\begin{aligned}
&\limsup_{|\varphi(z)|\to 1}\|uC_\varphi f_{\varphi(z)}\|_{\mathcal{B}}+\limsup_{|\varphi(z)|\to 1}\frac{\left(1-|z|^2\right)|u(z)||\varphi'(z)|}{1-|\varphi(z)|^2}\left(\log\frac{e}{1-|\varphi(z)|^2}\right)^{-\frac{1}{p}} \\
&\quad =\limsup_{j\to\infty}\|uC_\varphi f_{\varphi(z_j)}\|_{\mathcal{B}} \\
&\qquad +\limsup_{j\to\infty}\frac{\left(1-|z_j|^2\right)|u(z_j)\overline{\varphi(z_j)}\varphi'(z_j)|}{1-|\varphi(z_j)|^2}\left(\log\frac{e}{1-|\varphi(z_j)|^2}\right)^{-\frac{1}{p}} \\
&\quad \geq \limsup_{j\to\infty}\left(1-|z_j|^2\right)|u'(z_j)|\left(\log\frac{e}{1-|\varphi(z_j)|^2}\right)^{1-\frac{1}{p}} \\
&\quad \geq \limsup_{j\to\infty}\left(1-|z_j|^2\right)|u'(z_j)|=\limsup_{|\varphi(z)|\to 1}\left(1-|z|^2\right)|u'(z)|
\end{aligned}
$$

and

$$
\begin{aligned}
&\limsup_{|\varphi(z)|\to 1}\|uC_\varphi q_{\varphi(z)}\|_{\mathcal{B}}+\limsup_{|\varphi(z)|\to 1}\left(1-|z|^2\right)|u'(z)| \\
&\quad =\limsup_{j\to\infty}\|uC_\varphi q_{\varphi(z_j)}\|_{\mathcal{B}}+\limsup_{j\to\infty}\left(1-|z_j|^2\right)|u'(z_j)| \\
&\quad \geq \limsup_{j\to\infty}\frac{\left(1-|z_j|^2\right)|u(z_j)\overline{\varphi(z_j)}\varphi'(z_j)|}{1-|\varphi(z_j)|^2} \\
&\quad =\limsup_{|\varphi(z)|\to 1}\frac{\left(1-|z|^2\right)|u(z)||\varphi'(z)|}{1-|\varphi(z)|^2}.
\end{aligned}
$$

Since $uC_\varphi : B_p \to \mathcal{B}$ is bounded, we have (see [2])

$$\sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)|u(z)||\varphi'(z)|}{1-|\varphi(z)|^2} < \infty,$$

which implies that

$$\limsup_{|\varphi(z)|\to 1} \frac{\left(1-|z|^2\right)|u(z)||\varphi'(z)|}{1-|\varphi(z)|^2}\left(\log\frac{e}{1-|\varphi(z)|^2}\right)^{-\frac{1}{p}} = 0.$$

Thus, we have

$$\begin{aligned}\limsup_{|\varphi(z)|\to 1}\left(1-|z|^2\right)|u'(z)|\left(\log\frac{e}{1-|\varphi(z)|^2}\right)^{1-\frac{1}{p}} &\le \limsup_{|\varphi(z)|\to 1}\|uC_\varphi f_{\varphi(z)}\|_{\mathcal{B}}\\ &\le \limsup_{|a|\to 1}\|uC_\varphi f_a\|_{\mathcal{B}}, \qquad (2.11)\end{aligned}$$

$\limsup_{|\varphi(z)|\to 1}\left(1-|z|^2\right)|u'(z)| \le \limsup_{|a|\to 1}\|uC_\varphi f_a\|_{\mathcal{B}}$ and then

$$\begin{aligned}\limsup_{|\varphi(z)|\to 1}\frac{\left(1-|z|^2\right)|u(z)||\varphi'(z)|}{1-|\varphi(z)|^2} &\le \limsup_{|\varphi(z)|\to 1}\|uC_\varphi q_{\varphi(z)}\|_{\mathcal{B}} + \limsup_{|\varphi(z)|\to 1}\|uC_\varphi f_{\varphi(z)}\|_{\mathcal{B}}\\ &\le \limsup_{|a|\to 1}\|uC_\varphi q_a\|_{\mathcal{B}} + \limsup_{|a|\to 1}\|uC_\varphi f_a\|_{\mathcal{B}}.\end{aligned} \qquad (2.12)$$

By (2.11), (2.12) and Theorem 2.1, we obtain

$$\begin{aligned}\limsup_{|a|\to 1}\|uC_\varphi f_a\|_{\mathcal{B}} \le \|uC_\varphi\|_{e,B_p\to\mathcal{B}} &\approx \max\left\{E, F\right\}\\ &\le \max\left\{\limsup_{|a|\to 1}\|uC_\varphi f_a\|_{\mathcal{B}}, \limsup_{|a|\to 1}\|uC_\varphi q_a\|_{\mathcal{B}}\right\}. \qquad (2.13)\end{aligned}$$

To finish the proof, we only need to prove that

$$\limsup_{n\to\infty}\|u\varphi^n\|_{\mathcal{B}} \le \|uC_\varphi\|_{e,B_p\to\mathcal{B}}$$

and

$$\limsup_{|a|\to 1}\|uC_\varphi q_a\|_{\mathcal{B}} \lesssim \limsup_{n\to\infty}\|u\varphi^n\|_{\mathcal{B}}.$$

For each nonnegative integer n, let $p_n(z) = z^n$. Then, $p_n \in B_p$ and the sequence $\{p_n\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. Thus, for any compact operator $K : B_p \to \mathcal{B}$, by Lemma 2.1 we have $\lim_{n\to\infty}\|Kp_n\|_{\mathcal{B}} = 0$. Hence,

$$\|uC_\varphi - K\|_{B_p\to\mathcal{B}} \gtrsim \limsup_{n\to\infty} \|(uC_\varphi - K)p_n\|_{\mathcal{B}} \geq \limsup_{n\to\infty} \|uC_\varphi p_n\|_{\mathcal{B}}.$$

Thus,

$$\begin{aligned}\|uC_\varphi\|_{e,B_p\to\mathcal{B}} &= \inf_K \|uC_\varphi - K\|_{B_p\to\mathcal{B}} \geq \limsup_{n\to\infty} \|uC_\varphi p_n\|_{\mathcal{B}} \\ &= \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{B}}. \end{aligned} \tag{2.14}$$

On the other hand, let $a \in \mathbb{D}$, then

$$q_a(z) = \frac{1-|a|^2}{1-\bar{a}z} = \left(1-|a|^2\right)\sum_{k=0}^{\infty} \bar{a}^k z^k.$$

For fixed positive integer $n \geq 1$, it follows from the triangle inequality and the fact $\sup_{0\leq k<\infty} \|u\varphi^k\|_{\mathcal{B}} < \infty$ that

$$\begin{aligned}\|uC_\varphi q_a\|_{\mathcal{B}} &\leq \left(1-|a|^2\right)\sum_{k=0}^{\infty} |a|^k \|u\varphi^k\|_{\mathcal{B}} \\ &= \left(1-|a|^2\right)\sum_{k=0}^{n-1} |a|^k \|u\varphi^k\|_{\mathcal{B}} + \left(1-|a|^2\right)\sum_{k=n}^{\infty} |a|^k \|u\varphi^k\|_{\mathcal{B}} \\ &\leq n\left(1-|a|^2\right)\sup_{0\leq k\leq n-1} \|u\varphi^k\|_{\mathcal{B}} + \left(1-|a|^2\right)\sum_{k=n}^{\infty} |a|^k \sup_{j\geq n} \|u\varphi^j\|_{\mathcal{B}} \\ &\lesssim n\left(1-|a|^2\right) + 2\sup_{k\geq n} \|u\varphi^k\|_{\mathcal{B}}.\end{aligned}$$

Letting $|a| \to 1$ in the above inequality leads to

$$\limsup_{|a|\to 1} \|uC_\varphi q_a\|_{\mathcal{B}} \lesssim \sup_{k\geq n} \|u\varphi^k\|_{\mathcal{B}}$$

for any positive integer $n \geq 1$. Thus

$$\limsup_{|a|\to 1} \|uC_\varphi q_a\|_{\mathcal{B}} \lesssim \limsup_{n\to\infty} \|u\varphi^n\|_{\mathcal{B}}. \tag{2.15}$$

Therefore, by (2.13) , (2.14) and (2.15) we get the desired result. This completes the proof of this theorem. □

3 A New Characterization of $uC_\varphi : B_p \to \mathcal{B}$

In this section, we give a new characterization for the boundedness, compactness and essential norm of the operator $uC_\varphi : B_p \to \mathcal{B}$. For this purpose, we state some definitions and some lemmas which will be used.

Let $v : \mathbb{D} \to R_+$ be a continuous, strictly positive and bounded function. The weighted space H_v^∞ is the space which consisting of all $f \in H(\mathbb{D})$ such that $\|f\|_v = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty$. H_v^∞ is a Banach space under the norm $\|\cdot\|_v$. The weight v is called radial if $v(z) = v(|z|)$ for all $z \in \mathbb{D}$. The associated weight $\widetilde{v}$ of v is defined by (see [4])

$$\widetilde{v} = (\sup\{|f(z)| : f \in H_v^\infty, \|f\|_v \le 1\})^{-1}, z \in \mathbb{D}.$$

Let $0 < \alpha < \infty$. When $v = v_\alpha(z) = \left(1 - |z|^2\right)^\alpha$, it is easy to check that $\widetilde{v}_\alpha(z) = v_\alpha(z)$. In this case, we denote H_v^∞ by $H_{v_\alpha}^\infty$. Here

$$H_{v_\alpha}^\infty = \left\{ f \in H(\mathbb{D}) : \|f\|_{v_\alpha} = \sup_{z\in\mathbb{D}} |f(z)| \left(1 - |z|^2\right)^\alpha < \infty \right\}.$$

When

$$v = v_{\log,p}(z) = \left(\left(\log \frac{e}{1-|z|^2}\right)^{1-\frac{1}{p}}\right)^{-1},$$

it is not difficult to see that $\widetilde{v}_{\log,p} = v_{\log,p}$. Indeed, it is clear that $\widetilde{v}(z) = v(z)$, when (see [4])

$$v(z) = (\max\{|g(w)|; |w| = |z|\})^{-1}$$

is a weight for some $g \in H(\mathbb{D})$. Hence, the statement follows with $g(z) = (\log \frac{e}{1-|z|^2})^{1-\frac{1}{p}}$.

Lemma 3.1 ([5]) *For* $\alpha > 0$, *we have* $\lim_{k\to\infty} k^\alpha \|z^{k-1}\|_{v_\alpha} = (\frac{2\alpha}{e})^\alpha$.

After a calculation, we get the following result.

Lemma 3.2 *For* $1 < p < \infty$, *we have* $\lim_{k\to\infty} (\log k)^{1-\frac{1}{p}} \|z^k\|_{v_{\log,p}} \approx 1$.

Lemma 3.3 ([12]) *Let v and w be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Then, the following statements hold.*

(a) The weighted composition operator $uC_\varphi : H_v^\infty \to H_w^\infty$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{w(z)}{\widetilde{v}(\varphi(z))} |u(z)| < \infty.$$

(b) Suppose $uC_\varphi : H_v^\infty \to H_w^\infty$ is bounded. Then

$$\|uC_\varphi\|_{e, H_v^\infty \to H_w^\infty} = \lim_{s\to 1} \sup_{|\varphi(z)|>s} \frac{w(z)}{\widetilde{v}(\varphi(z))} |u(z)|.$$

Lemma 3.4 ([4]) *Let v and w be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Then, the following statements hold.*

(a) $uC_\varphi : H_v^\infty \to H_w^\infty$ *is bounded if and only if*

$$\sup_{k\geq 0} \frac{\|u\varphi^k\|_w}{\|z^k\|_v} < \infty,$$

with the norm comparable to the above supermum.

(b) Suppose $uC_\varphi : H_v^\infty \to H_w^\infty$ *is bounded. Then*

$$\|uC_\varphi\|_{e,H_v^\infty\to H_w^\infty} = \limsup_{k\to\infty} \frac{\|u\varphi^k\|_w}{\|z^k\|_v}.$$

Theorem 3.1 *Let $1 < p < \infty$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then, $uC_\varphi : B_p \to \mathcal{B}$ is bounded if and only if $u \in \mathcal{B}$,*

$$\sup_{j\geq 1} \|I_u(\varphi^j)\|_{\mathcal{B}} < \infty \qquad \textit{and} \qquad \sup_{j\geq 1}(\log j)^{1-\frac{1}{p}} \|J_u(\varphi^j)\|_{\mathcal{B}} < \infty.$$

Here

$$I_u f(z) = \int_0^z f'(\zeta)u(\zeta)d\zeta,\ J_u f(z) = \int_0^z f(\zeta)u'(\zeta)d\zeta.$$

Proof By Theorem 2.1 of [2], $uC_\varphi : B_p \to \mathcal{B}$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)|u'(z)|}{\left(\left(\log \frac{e}{1-|\varphi(z)|^2}\right)^{1-\frac{1}{p}}\right)^{-1}} < \infty \qquad \text{and} \qquad \sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)|u(z)||\varphi'(z)|}{1-|\varphi(z)|^2} < \infty. \tag{3.1}$$

By Lemma 3.3, the first inequality in (3.1) is equivalent to the operator $u'C_\varphi : H^\infty_{v_{\log,p}} \to H^\infty_{v_1}$ is bounded. By Lemma 3.4, this is equivalent to

$$\sup_{j\geq 1} \frac{\|u'\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_{\log,p}}} < \infty.$$

The second inequality in (3.1) is equivalent to the weighted composition operator $u\varphi'C_\varphi : H^\infty_{v_1} \to H^\infty_{v_1}$ is bounded. By Lemma 3.4, this is equivalent to

$$\sup_{j\geq 1} \frac{\|u\varphi'\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_1}} < \infty.$$

Since $I_u g(0) = 0$, $J_u g(0) = 0$,

$$\Big(I_u(\varphi^j)(z)\Big)' = ju(z)\varphi'(z)\varphi^{j-1}(z), \quad \Big(J_u(\varphi^{j-1})(z)\Big)' = u'(z)\varphi^{j-1}(z),$$

by Lemmas 3.1 and 3.2, we see that $uC_\varphi : B_p \to \mathcal{B}$ is bounded if and only if

$$\sup_{j\geq 1} \|I_u(\varphi^j)\|_{\mathcal{B}} = \sup_{j\geq 1} j\|u\varphi'\varphi^{j-1}\|_{v_1} \approx \sup_{j\geq 1} \frac{j\|u\varphi'\varphi^{j-1}\|_{v_1}}{j\|z^{j-1}\|_{v_1}} < \infty$$

and

$$\begin{aligned}
\infty > \sup_{j\geq 1} \frac{\|u'\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_{\log,p}}} &= \sup_{j\geq 1} \frac{\|J_u(\varphi^{j-1})\|_{\mathcal{B}}}{\|z^{j-1}\|_{v_{\log,p}}} \\
&\approx \max\Big\{\|u\|_{\mathcal{B}}, \quad \sup_{j\geq 2}(\log(j-1))^{1-\frac{1}{p}}\|J_u(\varphi^{j-1})\|_{\mathcal{B}}\Big\} \\
&= \max\Big\{\|u\|_{\mathcal{B}}, \quad \sup_{j\geq 1}(\log j)^{1-\frac{1}{p}}\|J_u(\varphi^{j})\|_{\mathcal{B}}\Big\}.
\end{aligned}$$

The proof is completed. □

Theorem 3.2 *Let* $1 < p < \infty$, $u \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$ *such that* $uC_\varphi : B_p \to \mathcal{B}$ *is bounded. Then*

$$\|uC_\varphi\|_{e,B_p\to\mathcal{B}} \approx \max\Big\{\limsup_{j\to\infty}\|I_u(\varphi^j)\|_{\mathcal{B}}, \quad \limsup_{j\to\infty}(\log j)^{1-\frac{1}{p}}\|J_u(\varphi^j)\|_{\mathcal{B}}\Big\}.$$

Proof From the proof of Theorem 3.1 we see that the boundedness of $uC_\varphi : B_p \to \mathcal{B}$ is equivalent to the boundedness of the operators $u\varphi'C_\varphi : H^\infty_{v_1} \to H^\infty_{v_1}$ and $u'C_\varphi : H^\infty_{v_{\log,p}} \to H^\infty_{v_1}$. By Lemmas 3.1, 3.2 and 3.4, we get

$$\begin{aligned}
\|u\varphi'C_\varphi\|_{e,H^\infty_{v_1}\to H^\infty_{v_1}} &= \limsup_{j\to\infty}\frac{\|u\varphi'\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_1}} = \limsup_{j\to\infty}\frac{j\|u\varphi'\varphi^{j-1}\|_{v_1}}{j\|z^{j-1}\|_{v_1}} \\
&\approx \limsup_{j\to\infty} j\|u\varphi'\varphi^{j-1}\|_{v_1} = \limsup_{j\to\infty}\|I_u(\varphi^j)\|_{\mathcal{B}} \qquad (3.2)
\end{aligned}$$

and

$$\begin{aligned}
\|u'C_\varphi\|_{e,H^\infty_{v_{\log,p}}\to H^\infty_{v_1}} &= \limsup_{j\to\infty}\frac{\|u'\varphi^{j-1}\|_{v_1}}{\|z^{j-1}\|_{v_{\log,p}}} = \limsup_{j\to\infty}\frac{(\log(j-1))^{1-\frac{1}{p}}\|u'\varphi^{j-1}\|_{v_1}}{(\log(j-1))^{1-\frac{1}{p}}\|z^{j-1}\|_{v_{\log,p}}} \\
&\approx \limsup_{j\to\infty}(\log(j-1))^{1-\frac{1}{p}}\|u'\varphi^{j-1}\|_{v_1} \\
&= \limsup_{j\to\infty}(\log j)^{1-\frac{1}{p}}\|J_u(\varphi^j)\|_{\mathcal{B}}. \qquad (3.3)
\end{aligned}$$

The upper estimate From the fact that

$$(uC_\varphi f)'(z) = u'(z)f(\varphi(z)) + u(z)\varphi'(z)f'(\varphi(z)),$$

it is easy to see that

$$\|uC_\varphi\|_{e,B_p\to\mathcal{B}} \le \|u'C_\varphi\|_{e,H^\infty_{v_{\log,p}}\to H^\infty_{v_1}} + \|u\varphi'C_\varphi\|_{e,H^\infty_{v_1}\to H^\infty_{v_1}}. \tag{3.4}$$

Then, by (3.2), (3.3) and (3.4) we get

$$\begin{aligned}\|uC_\varphi\|_{e,B_p\to\mathcal{B}} &\lesssim \limsup_{j\to\infty}\|I_u(\varphi^j)\|_{\mathcal{B}} + \limsup_{j\to\infty}(\log j)^{1-\frac{1}{p}}\|J_u(\varphi^j)\|_{\mathcal{B}}\\ &\lesssim \max\left\{\limsup_{j\to\infty}\|I_u(\varphi^j)\|_{\mathcal{B}},\quad \limsup_{j\to\infty}(\log j)^{1-\frac{1}{p}}\|J_u(\varphi^j)\|_{\mathcal{B}}\right\}.\end{aligned}$$

The lower estimate From Theorem 2.1, (3.2), (3.3) and Lemma 3.3, we have

$$\|uC_\varphi\|_{e,B_p\to\mathcal{B}} \gtrsim F = \|u\varphi'C_\varphi\|_{e,H^\infty_{v_1}\to H^\infty_{v_1}} \approx \limsup_{j\to\infty}\|I_u(\varphi^j)\|_{\mathcal{B}}$$

and

$$\|uC_\varphi\|_{e,B_p\to\mathcal{B}} \gtrsim E = \|u'C_\varphi\|_{e,H^\infty_{v_{\log,p}}\to H^\infty_{v_1}} \approx \limsup_{j\to\infty}(\log j)^{1-\frac{1}{p}}\|J_u(\varphi^j)\|_{\mathcal{B}}.$$

Therefore,

$$\|uC_\varphi\|_{e,B_p\to\mathcal{B}} \gtrsim \max\left\{\limsup_{j\to\infty}\|I_u(\varphi^j)\|_{\mathcal{B}},\quad \limsup_{j\to\infty}(\log j)^{1-\frac{1}{p}}\|J_u(\varphi^j)\|_{\mathcal{B}}\right\}.$$

This completes the proof of this Theorem. □

From Theorem 3.2, we immediately get the following result.

Theorem 3.3 *Let* $1 < p < \infty$, $u \in H(\mathbb{D})$ *and* $\varphi \in S(\mathbb{D})$ *such that* $uC_\varphi : B_p \to \mathcal{B}$ *is bounded. Then* $uC_\varphi : B_p \to \mathcal{B}$ *is compact if and only if*

$$\limsup_{j\to\infty}\|I_u(\varphi^j)\|_{\mathcal{B}} = 0 \quad \textit{and} \quad \limsup_{j\to\infty}(\log j)^{1-\frac{1}{p}}\|J_u(\varphi^j)\|_{\mathcal{B}} = 0.$$

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