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Normal Criteria for Family Meromorphic Functions Sharing Holomorphic Function

Nguyen Van Thin[1]

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Abstract In this paper, we study the value distribution of differential polynomial with the form $f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}$, where f is a transcendental meromorphic function. Namely, we prove that $f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)} - P(z)$ has infinitely zeros, where $P(z)$ is a nonconstant polynomial and $n \in \mathbb{N}$, $k, n_1, \dots, n_k, t_1, \dots, t_k$ are positive integer numbers satisfying $n + \sum_v^k n_v \geq \sum_{v=1}^k t_v + 3$. Using it, we establish some normality criterias for family of meromorphic functions under a condition where differential polynomials generated by the members of the family share a holomorphic function with zero points. Our results generalize some previous results on normal family of meromorphic functions.

Keywords Meromorphic function · Normal family · Nevanlinna theory

Mathematics Subject Classification Primary 30D35 · 30D45

1 Introduction

Let D be a domain in the complex plane $\mathbb{C}$ and $\mathcal{F}$ be a family of meromorphic functions in D. The family $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if for any sequence $\{f_v\} \subset \mathcal{F}$, there exists a subsequence $\{f_{v_i}\}$ such that $\{f_{v_i}\}$ converges spherically locally uniformly in D, to a meromorphic function or ∞.

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✉ Nguyen Van Thin
nguyenvanthintn@gmail.com

1 Department of Mathematics, Thai Nguyen University of Education, Luong Ngoc Quyen Street, Thai Nguyen City, Vietnam

In 1989, Schwick [8] proved that

Theorem A *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D and k, n are positive integer numbers satisfying $n \geq k+3$. If $(f^n)^{(k)} \neq 1$ for every $f \in \mathcal{F}$, then $\mathcal{F}$ is normal.*

In 2014, Dethloff et al. [4] came up with new normality criteria, which extended the result given by Schwick.

Theorem B *Let a be a nonzero complex value, let n be a non-negative integer and $n_1, n_2, \ldots, n_k$, $t_1, t_2, \ldots, t_k$ be positive integers. Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain D such that for every $f \in \mathcal{F}$, $f^n(f^{n_1})^{(t_1)} \cdots (f^{n_k})^{(t_k)} - a$ is nowhere vanishing on D. Assume that*

(a) $n_v \geq t_v$ for all $1 \leqslant v \leqslant k$,
(b) $n + \sum_{v=1}^{k} n_v \geq 3 + \sum_{v=1}^{k} t_v$.

Then $\mathcal{F}$ is normal on D.

In 2009, Li and Gu [7] improved Theorem A in the following manner

Theorem C *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, $k, n (n \geq k+2)$ be positive integers and $a \in \mathbb{C}\backslash\{0\}$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share the value $a - IM$ in D for each pair of functions $f, g \in \mathcal{F}$, then $\mathcal{F}$ is normal.*

In 2014, Datt and Kumar [3], by idea sharing value, they proved the result corresponding Theorem B.

Theorem D *Let $\alpha(z)$ be a holomorphic function defined in $D \subset C$ such that $\alpha(z) \neq 0$. Let n be a non-negative integer and $n_1, n_2, \ldots, n_k$, $t_1, t_2, \ldots, t_k$ be positive integers such that*

(a) $n_v \geq t_v$ for all $1 \leqslant v \leqslant k$;
(b) $n + \sum_{v=1}^{k} n_v \geq 3 + \sum_{v=1}^{k} t_v$.

Let $\mathcal{F}$ be a family of meromorphic functions in a domain D such that for every pair $f, g \in \mathcal{F}$, $f^n(z)(f^{n_1})^{(t_1)}(z) \cdots (f^{n_k})^{(t_k)}(z)$ and $g^n(z)(g^{n_1})^{(t_1)}(z) \cdots (g^{n_k})^{(t_k)}(z)$ share $\alpha(z) - IM$ on D. Then $\mathcal{F}$ is normal in D.

In 2012, Zeng and Lahiri [12] proved the result concerning Theorem C.

Theorem E *Let $\mathcal{F}$ be a family of meromorphic functions defined in a domain D, $a \in \mathbb{C}\backslash\{0\}$ and k, n be positive integers such that $n \geq 1$ if $k = 1$ and $n \geq 2$ if $k \geq 2$. If $f^n(f^{k+1})^{(k)}$ and $g^n(g^{k+1})^{(k)}$ share the value $a - IM$ in D for each pair of functions $f, g \in \mathcal{F}$, then $\mathcal{F}$ is normal.*

We see that the value $a \neq 0$ in Theorems C and E is a holomorphic function nowhere vanishing.

Question 1 Can we extend Theorems C, D and E by idea sharing a holomorphic function with zero point?

In 2012, Yunbo and Zongsheng [10] proved that

Theorem F *Let $n, k \geq 2$, $m \geq 0$ be three integers, and m be divisible by $n+1$. Suppose that $a(z) \not\equiv 0$ is a holomorphic function with zeros of multiplicity m in a domain D. Let $\mathcal{F}$ be a family of holomorphic functions in D, for each $f \in \mathcal{F}$, f has only zeros of multiplicity $k+m$ at least. For each pair $(f, g) \in F$, $f(f^{(k)})^n$ and $g(g^{(k)})^n$ share $a(z) - IM$, then $\mathcal{F}$ is normal in D.*

For each meromorphic function f on D, we call that $N(f, f', \ldots, f^{(t_1)}, \ldots, f^{(t_k)})$ is a monomial differential polynomial of f and defined by

$$N(f, f', \ldots, f^{(t_1)}, \ldots, f^{(t_k)}) = f^n (f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)},$$

where $n \in \mathbb{N}, n_1, \ldots, n_k, t_1, \ldots, t_k$ are positive integer numbers and $k \in \mathbb{N}^*$. We denote

$$\Gamma_N = n + n_1 + \cdots + n_k, \Upsilon_N = t_1 + \cdots + t_k.$$

In this paper, we consider the differential polynomial with the form

$$\begin{aligned} & f^n (f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)} + \sum_I a_I f^{n_I} (f^{n_{1I}})^{(t_{1I})} \ldots (f^{n_{kI}})^{(t_{kI})} \\ & = N(f, f', \ldots, f^{(t_1)}, \ldots, f^{(t_k)}) + \sum_I a_I N_I(f, f', \ldots, f^{(t_{1I})}, \ldots, f^{(t_{kI})}), \quad (1.1) \end{aligned}$$

where a_I are holomorphic functions on D, and $n_I, n_{jI}, t_{jI}, j = 1, \ldots, k$ are non-negative integer numbers, and $I \subset \mathbb{N}$ is the set index finitely.

Now, connection with result of Theorem F, we prove the results as following:

Theorem 1 *Let $n, m \in \mathbb{N}$ and n_v, t_v, k $(v = 1, 2, \ldots, k)$ be positive integer numbers such that m is divisible by $n + \sum_{v=1}^{k} n_v$ and*

$$n_v \geqslant t_v, v = 1, \ldots, k, n + \sum_{v=1}^{k} n_v \geqslant \sum_{v=1}^{k} t_v + 3, n + \sum_{v=1}^{k} n_v = \Gamma_{N_I}.$$

Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain D with all poles and zeros of multiplicity at least $\left[\frac{2m + 2 + \sum_{v=1}^{k} t_v}{n + \sum_{v=1}^{k} n_v}\right] + 1$. Let $a(z) \not\equiv 0$ be a holomorphic functions with zeros of multiplicity m in a domain D. If

$$f^n (f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)} + \sum_I a_I f^{n_I} (f^{n_{1I}})^{(t_{1I})} \ldots (f^{n_{kI}})^{(t_{kI})},$$

$$g^n (g^{n_1})^{(t_1)} \ldots (g^{n_k})^{(t_k)} + \sum_I a_I g^{n_I} (g^{n_{1I}})^{(t_{1I})} \ldots (g^{n_{kI}})^{(t_{kI})}$$

share $a(z)-IM$ in D for each pair (f,g) in $\mathcal{F}$, where t_{jI} satisfy

$$\sum_{v=1}^{k} t_v > \sum_{v=1}^{k} t_{vI},$$

then $\mathcal{F}$ is a normal family. Here, we denote $[x]$ by integer part of the number x.

Remark 2 Theorem 1 is an extension of Theorems D and E for case sharing holomorphic function with zero point.

From Theorem 1, we get a corollary as following:

Corollary 3 *Let $n,m\in\mathbb{N}$ and n_v,t_v,k $(v=1,2,\ldots,k)$ be positive integer numbers such that m is divisible by $n+\sum_{v=1}^{k}n_v$ and*

$$n_v\geqslant t_v,\ v=1,\ldots,k,\ n+\sum_{v=1}^{k}n_v\geqslant\sum_{v=1}^{k}t_v+3,\ n+\sum_{v=1}^{k}n_v=\Gamma_{N_I}.$$

Let $\mathcal{F}$ be a family of meromorphic functions in a complex domain D with all poles and zeros of multiplicity at least $\left[\frac{2m+2+\sum_{v=1}^{k}t_v}{n+\sum_{v=1}^{k}n_v}\right]+1$. Let $a(z)\not\equiv 0$ be a holomorphic functions with zeros of multiplicity m in a domain D. If

$$f^n(f^{n_1})^{(t_1)}\ldots(f^{n_k})^{(t_k)}+\sum_I a_I f^{n_I}(f^{n_{1I}})^{(t_{1I})}\ldots(f^{n_{kI}})^{(t_{kI})}\neq a(z)$$

for every f in $\mathcal{F}$, where t_{jI} satisfy

$$\sum_{v=1}^{k} t_v > \sum_{v=1}^{k} t_{vI},$$

then $\mathcal{F}$ is a normal family.

We see that Corollary 3 is an extension of Theorems A and B.

Theorem 4 *Let $n,m\in\mathbb{N}$ and n_v,t_v,k $(v=1,2,\ldots,k)$ be positive integer numbers such that m is divisible by $n+\sum_{v=1}^{k}n_v$ and*

$$n_v\geqslant t_v,\ v=1,\ldots,k,\ n+\sum_{v=1}^{k}n_v\geqslant\sum_{v=1}^{k}t_v+2,\ n+\sum_{v=1}^{k}n_v=\Gamma_{N_I}.$$

Let $\mathcal{F}$ be a family of entire functions in a complex domain D with all zeros of multiplicity at least $\left[\frac{2m+2+\sum_{v=1}^{k}t_v}{n+\sum_{v=1}^{k}n_v}\right]+1$. Let $a(z)\not\equiv 0$ be a holomorphic functions with zeros of multiplicity m in a domain D. If

$$f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}+\sum_I a_I f^{n_I}(f^{n_{1I}})^{(t_{1I})}\dots(f^{n_{kI}})^{(t_{kI})},$$
$$g^n(g^{n_1})^{(t_1)}\dots(g^{n_k})^{(t_k)}+\sum_I a_I g^{n_I}(g^{n_{1I}})^{(t_{1I})}\dots(g^{n_{kI}})^{(t_{kI})}$$

share $a(z)-$ IM in D for each pair (f, g) in $\mathcal{F}$, where t_{jI} satisfy

$$\sum_{v=1}^{k} t_v > \sum_{v=1}^{k} t_{vI},$$

then $\mathcal{F}$ is a normal family.

2 Some Lemmas

To prove our results, we need the following lemmas.

Lemma 1 (Zalcman's Lemma, [11]) *Let $\mathcal{F}$ be a family of meromorphic functions defined in the unit disc $\triangle$. Then if $\mathcal{F}$ is not normal at a point $z_0 \in \triangle$, there exist, for each real number α satisfying $-1 < \alpha < 1$,*

1. *a real number r, $0 < r < 1$,*
2. *points z_n, $|z_n| < r$, $z_n \to z_0$,*
3. *positive numbers $\rho_n \to 0^+$,*
4. *functions $f_n \in \mathcal{F}$*

such that

$$g_n(\xi) = \frac{f_n(z_n + \rho_n \xi)}{\rho_n^{\alpha}} \to g(\xi)$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $g(\xi)$ is a nonconstant meromorphic function and $g^{\#}(\xi) \leqslant g^{\#}(0) = 1$. Moreover, the order of g is not greater than 2. Here, as usual, $g^{\#}(z) = \frac{|g'(z)|}{1+|g(z)|^2}$ is the spherical derivative.

Lemma 2 [2] *Let g be a entire function, and M is a positive constant. If $g^{\#}(\xi) \leqslant M$ for all $\xi \in \mathbb{C}$, then g has the order at most one.*

Remark 5 In Lemma 1, if $\mathcal{F}$ is a family of holomorphic functions, then by Hurwitz's Theorem, g is a holomorphic function. Therefore, by Lemma 2, the order of g is not greater than 1.

We consider a nonconstant meromorphic function g in the complex plane $\mathbb{C}$, and its first p derivatives. A differential polynomial P of g is defined by

$$P(z) := \sum_{i=1}^{n} \alpha_i(z) \prod_{j=0}^{p} (g^{(j)}(z))^{S_{ij}},$$

where S_{ij} $(0 \leqslant i, j \leqslant n)$ are non-negative integers, and α_i $(1 \leqslant i \leqslant n)$ are small (with respect to g) meromorphic functions. Set

$$d(P) := \min_{1 \leqslant i \leqslant n} \sum_{j=0}^{p} S_{ij} \text{ and } \theta(P) := \max_{1 \leqslant i \leqslant n} \sum_{j=0}^{p} j S_{ij}.$$

In 2002, Hinchliffe [6] generalized theorems of Hayman [5] and Chuang [1] and obtained the following result.

Proposition 1 *Let g be a transcendental meromorphic function, let $P(z)$ be a nonconstant differential polynomial in g with $d(P) \geq 2$. Then*

$$T(r, g) \leqslant \frac{\theta(P)+1}{d(P)-1} \overline{N}\left(r, \frac{1}{g}\right) + \frac{1}{d(P)-1} \overline{N}\left(r, \frac{1}{P-1}\right) + o(T(r, g)),$$

for all $r \in [1, +\infty)$ excluding a set of finite Lebesgues measure.

By argument as Proposition 1, we are easy to get the result as following for small function. However, for convenience of the reader, we prove it here.

Lemma 3 *Let g be a nonconstant meromorphic function and $P(z)$ be a nonconstant differential polynomial in g with $d(P) \geq 1$. Let $a(z) \not\equiv 0$ be a small function of $P(g)$. Then*

$$T(r, g) \leqslant \frac{\theta(P)+1}{d(P)} \overline{N}\left(r, \frac{1}{g}\right) + \frac{1}{d(P)} \overline{N}(r, g) + \frac{1}{d(P)} \overline{N}\left(r, \frac{1}{P-a}\right) + o(T(r, g)),$$

for all $r \in [1, +\infty)$ excluding a set of finite Lebesgues measure.
Moreover, in the case where g is a nonconstant entire function, we have

$$T(r, g) \leqslant \frac{\theta(P)+1}{d(P)} \overline{N}\left(r, \frac{1}{g}\right) + \frac{1}{d(P)} \overline{N}\left(r, \frac{1}{P-a}\right) + o(T(r, g)),$$

for all $r \in [1, +\infty)$ excluding a set of finite Lebesgues measure.

Remark 6 Let g be a nonconstant meromorphic function and $P(z)$ be a nonconstant differential polynomial in g with $d(P) \geq 2$. Let $a(z) \not\equiv 0$ be a small function of $P(g)$. Then

$$T(r, g) \leqslant \frac{\theta(P)+1}{d(P)-1} \overline{N}\left(r, \frac{1}{g}\right) + \frac{1}{d(P)-1} \overline{N}\left(r, \frac{1}{P-a}\right) + o(T(r, g)),$$

for all $r \in [1, +\infty)$ excluding a set of finite Lebesgues measure.

Proof of Lemma 3 and Remark 6 For any z such that $|g(z)| \leqslant 1$, since $\sum_{j=0}^{p} S_{ij} \geq d(P)$ $(1 \leqslant i \leqslant n)$, we have

$$\frac{1}{|g(z)|^{d(P)}} = \frac{1}{|P(z)|} \cdot \frac{|P(z)|}{|g(z)|^{d(P)}}$$
$$\leqslant \frac{1}{|P(z)|} \cdot \sum_{i=1}^{n} \big(|\alpha_i(z)| \prod_{j=0}^{p} \big|\frac{g^{(j)}(z)}{g(z)}\big|^{S_{ij}}\big).$$

This implies that for all $z \in \mathbb{C}$,

$$\log^+ \frac{1}{|g(z)|^{d(P)}} \leqslant \log^+ \big(\frac{1}{|P(z)|} \cdot \sum_{i=1}^{n} \big(|\alpha_i(z)| \prod_{j=0}^{p} \big|\frac{g^{(j)}(z)}{g(z)}\big|^{S_{ij}}\big)\big).$$

Therefore, by the Lemma on logarithmic derivative and by the first main theorem, we have

$$d(P)m\left(r, \frac{1}{g}\right) \leqslant m\left(r, \frac{1}{P}\right) + o(T(r, g))$$
$$= T\left(r, \frac{1}{P}\right) - N(r, \frac{1}{P}) + o(T(r, g))$$
$$= T(r, P) - N\left(r, \frac{1}{P}\right) + o(T(r, g)).$$

On the other hand, by the second main theorem for small function [5,9], we have

$$T(r, P) \leqslant \overline{N}(r, P) + \overline{N}\left(r, \frac{1}{P}\right) + \overline{N}\left(r, \frac{1}{P-a}\right) + o(T(r, g)).$$

Hence,

$$d(P)m(r, \frac{1}{g}) \leqslant \big(\overline{N}(r, P) + \overline{N}\big(r, \frac{1}{P}\big) + \overline{N}\big(r, \frac{1}{P-a}\big)\big)$$
$$- N\big(r, \frac{1}{P}\big) + o(T(r, g)).$$

By First Main Theorem, we have

$$d(P)T(r, g) = d(P)T\big(r, \frac{1}{g}\big) + O(1)$$
$$= d(P)m\big(r, \frac{1}{g}\big) + d(P)N(r, \frac{1}{g}) + O(1)$$
$$\leqslant \big(\overline{N}(r, P) + \overline{N}\big(r, \frac{1}{P}\big) + \overline{N}\big(r, \frac{1}{P-a}\big)\big)$$
$$+ d(P)N\big(r, \frac{1}{g}\big) - N\big(r, \frac{1}{P}\big) + o(T(r, g)). \tag{2.1}$$

We see

$$\frac{1}{g^{d(P)}} = \frac{1}{P(z)} \sum_{i=1}^{n} \left(\alpha_i g^{S_{ij}-d(P)} \prod_{j=0}^{p} \left(\frac{g^{(j)}}{g}\right)^{S_{ij}}\right).$$

Note that $\sum_{j=0}^{p} S_{ij} - d(P) \geq 0$, and therefore we get

$$\begin{aligned} d(P)\nu_{\frac{1}{g}} &\leqslant \nu_{\frac{1}{P}} + \max_{1\leqslant i\leqslant n} \left\{ \nu_{\alpha_i} + \sum_{j=0}^{p} j S_{ij} \overline{\nu}_{\frac{1}{g}} \right\} \\ &\leqslant \nu_{\frac{1}{P}} + \sum_{i=1}^{n} \nu_{\alpha_i} + \theta(P)\overline{\nu}_{\frac{1}{g}}, \end{aligned}$$

where ν_ϕ is the pole divisor of the meromorphic ϕ and $\overline{\nu}_\phi := \min\{\nu_\phi, 1\}$.
This implies

$$d(P)\nu_{\frac{1}{g}} - \nu_{\frac{1}{P}} + \overline{\nu}_{\frac{1}{P}} \leqslant (\theta(P)+1)\overline{\nu}_{\frac{1}{g}} + \sum_{i=1}^{n} \nu_{\alpha_i}$$

(note that for any z_0, if $\nu_{\frac{1}{g}}(z_0) = 0$ then $d(P)\nu_{\frac{1}{g}}(z_0) - \nu_{\frac{1}{P}}(z_0) + \overline{\nu}_{\frac{1}{P}}(z_0) \leqslant 0$). Then, we obtain

$$\begin{aligned} d(P)N\left(r, \frac{1}{g}\right) - N\left(r, \frac{1}{P}\right) + \overline{N}\left(r, \frac{1}{P}\right) &\leqslant (\theta(P)+1)\overline{N}(r, \frac{1}{g}) + \sum_{i=1}^{n} N(r, \alpha_i) \\ &= (\theta(P)+1)\overline{N}\left(r, \frac{1}{g}\right) + o(T(r, g)). \end{aligned}$$

Combining with (2.1), we have

$$d(P)T(r, g) \leqslant \left(\overline{N}(r, P) + \overline{N}\left(r, \frac{1}{P-a}\right)\right) + (\theta(P)+1)\overline{N}\left(r, \frac{1}{g}\right) + o(T(r, g)).$$

On the other hand, by the definition of the differential polynomial P, $\text{Pole}(P) \subset \cup_{i=1}^{n}$ $\text{Pole}(\alpha_i) \cup$ $\text{Pole}(g)$. Hence,

$$\begin{aligned} d(P)T(r, g) \leqslant &\left(\overline{N}(r, g) + \overline{N}\left(r, \frac{1}{P-a}\right)\right) \\ &+ (\theta(P)+1)\overline{N}\left(r, \frac{1}{g}\right) + o(T(r, g)). \end{aligned} \quad (2.2)$$

This implies that

$$T(r,g) \leqslant \frac{(\theta(P)+1)}{d(P)}\overline{N}\big(r,\frac{1}{g}\big) + \frac{1}{d(P)}\overline{N}(r,g) + \frac{1}{d(P)}\overline{N}\big(r,\frac{1}{P-a}\big)\big) + o(T(r,g)). \tag{2.3}$$

From (2.3), we conclude the statement of Lemma 3 for nonconstant meromorphic function.

From (2.2), we have

$$d(P)T(r,g) \leqslant \big(T(r,g) + \overline{N}\big(r,\frac{1}{P-a}\big)\big) + (\theta(P)+1)\overline{N}\big(r,\frac{1}{g}\big) + o(T(r,g)).$$

Therefore, if $d(P) \geq 2$, we get

$$T(r,g) \leqslant \frac{\theta(P)+1}{d(P)-1}\overline{N}\left(r,\frac{1}{g}\right) + \frac{1}{d(P)-1}\overline{N}\left(r,\frac{1}{P-a}\right) + o(T(r,g)). \tag{2.4}$$

From (2.4), we obtain Remark 6.

In the case where g is nonconstant holomorphic function, the inequality in (2.2) becomes

$$d(P)T(r,g) \leqslant \overline{N}\left(r,\frac{1}{P-a}\right) + (\theta(P)+1)\overline{N}\left(r,\frac{1}{g}\right) + o(T(r,g)).$$

This implies that

$$T(r,g) \leqslant \frac{\theta(P)+1}{d(P)})\overline{N}\left(r,\frac{1}{g}\right) + \frac{1}{d(P)}\overline{N}\left(r,\frac{1}{P-a}\right) + o(T(r,g)).$$

We have completed the proof of Lemma 3. □

Lemma 4 *Let f be a nonconstant rational function, $P(z) = a_d z^d + a_{d-1}z^{d-1} + \cdots + a_0$, $d \in \mathbb{N}$, $a_d \neq 0$, $a_{d-1}, \ldots, a_0$ be complex numbers and $n \in \mathbb{N}$, $k, n_v, t_v \in \mathbb{N}^*$, $v = 1, \ldots, k$.*
If $d = 0$, $P(z) = a_0 \neq 0$,

$$n_v \geqslant t_v, n + \sum_{v=1}^{k} n_v \geqslant \sum_{v=1}^{k} t_v + 2, v = 1, \ldots, k;$$

and if $d \geq 1$,

$$n_v \geqslant t_v, n + \sum_{v=1}^{k} n_v \geqslant \sum_{v=1}^{k} t_v + 2, v = 1, \ldots, k,$$

suppose that all zeros and poles of f having multiplicity at least

$$\left[\frac{2d+2+\sum_{v=1}^{k}t_v}{n+\sum_{v=1}^{k}n_v}\right]+1,$$

then the equation

$$f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}=P(z)$$

have at least two distinct zeros.

Proof We consider two cases as following:
Case 1. f is a polynomial. Then $f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}-P(z)$ is a polynomial with degree at least $2d+2$ and when $P(z)=a_0\neq 0$, $f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}-P(z)$ is polynomial with degree at least 2. Indeed, when $\deg P\geq 1$, then all zeros of f have multiple at least

$$\left[\frac{2d+2+\sum_{v=1}^{k}t_v}{n+\sum_{v=1}^{k}n_v}\right]+1>\frac{2d+2+\sum_{v=1}^{k}t_v}{n+\sum_{v=1}^{k}n_v}.$$

Hence $\deg f>\dfrac{2d+2+\sum_{v=1}^{k}t_v}{n+\sum_{v=1}^{k}n_v}$. This implies that $f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}-P(z)$ is polynomial with degree at least

$$\left(n+\sum_{v=1}^{k}n_v\right)\deg f-\sum_{j=1}^{k}t_v>2d+2+\sum_{v=1}^{k}t_v-\sum_{v=1}^{k}t_v=2d+2.$$

We suppose that $f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}-P(z)$ has unique zero z_0, then

$$f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}-P(z)=A(z-z_0)^l,\ l\geq 2d+2 \tag{2.5}$$

and $A\neq 0$ is a constant. Take derivative both sides (2.5) to d and $d+1$ times, we have

$$\left(f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}\right)^{(d)}-Al(l-1)\dots(l-d+1)(z-z_0)^{l-d}=P^{(d)}(z). \tag{2.6}$$

and

$$\left(f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}\right)^{(d+1)}=Al(l-1)\dots(l-d+1)(l-d)(z-z_0)^{l-d-1}. \tag{2.7}$$

Since $l\geq 2d+2>d+1$, from (2.7), we get that z_0 is uniqueness zero of $\left(f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}\right)^{(d+1)}$. We see that all zeros of f belong to zeros of

$\left(f^n(f^{n_1})^{(t_1)}\ldots(f^{n_k})^{(t_k)}\right)^{(d+1)}$. Thus, f has uniqueness zero z_0. From (2.7), we get

$$0=\left(f^n(f^{n_1})^{(t_1)}\ldots(f^{n_k})^{(t_k)}\right)^{(d)}(z_0)=P^{(d)}(z_0)\neq 0.$$

This is a contradiction. Hence,

$$f^n(f^{n_1})^{(t_1)}\ldots(f^{n_k})^{(t_k)}-P(z)$$

have at least distinct two zeros in two cases $P(z)=a_0\neq 0$ and $\deg P\geq 1$.
Case 2. f is not a polynomial. By hypothesis, we can express f as following

$$f=A\frac{(z-a_1)^{p_1}(z-a_2)^{p_2}\ldots(z-a_s)^{p_s}}{(z-b_1)^{q_1}(z-b_2)^{q_2}\ldots(z-b_t)^{q_t}}, \tag{2.8}$$

where $p_i\geq 1, i=1,\ldots,s,\ q_j\geq 1,\ j=1,\ldots,t$ if $P(z)=a_0\neq 0$ and $p_i\geq \left[\frac{2d+2+\sum_{j=1}^k t_j}{n+\sum_{j=1}^k n_j}\right]+1, i=1,\ldots,s, q_j\geq\left[\frac{2d+2+\sum_{j=1}^k t_j}{n+\sum_{j=1}^k n_j}\right]+1, j=1,\ldots,t$ if $\deg P\geq 1$.

Take

$$p=\sum_{i=1}^{s}p_i\geq s\frac{2d+2+\sum_{v=1}^k t_v}{n+\sum_{v=1}^k n_v}, q=\sum_{j=1}^{t}q_j\geq t\frac{2d+2+\sum_{v=1}^k t_v}{n+\sum_{v=1}^k n_v} \tag{2.9}$$

if $\deg P\geq 1$, and $p\geq s, q\geq t$ if $P(z)=a_0\neq 0$.

From (2.8), we have

$$f^{n_v}=A^{n_v}\frac{(z-a_1)^{p_1n_v}(z-a_2)^{p_2n_v}\ldots(z-a_s)^{p_sn_v}}{(z-b_1)^{q_1n_v}(z-b_2)^{q_2n_v}\ldots(z-b_t)^{q_tn_v}},\quad v=1,\ldots,k.$$

Then

$$(f^{n_v})^{(t_v)}=A^{n_v}\frac{(z-a_1)^{p_1n_v-t_v}(z-a_2)^{p_2n_v-t_v}\ldots(z-a_s)^{p_sn_v-t_v}}{(z-b_1)^{q_1n_v+t_v}(z-b_2)^{q_2n_v+t_v}\ldots(z-b_t)^{q_tn_v+t_v}}g_v(z),\quad v=1,\ldots,k, \tag{2.10}$$

where

$$g_v(z)=(n_vp-n_vq)(n_vp-n_vq-1)\ldots(n_vp-n_vq-t_v+1)z^{t_v(s+t-1)}+b_{t_v(s+t-1)-1}z^{t_v(s+t-1)-1}+\cdots+b_0$$

and b_e, $e=0,\ldots,t_v(s+t-1)-1$ are complex numbers.

From (2.10), we see

$$
\begin{aligned}
& f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)} \\
&= A^{n+\sum_{v=1}^k n_v}\frac{\prod_{i=1}^s(z-a_i)^{(n+\sum_{v=1}^k n_v)p_i-\sum_{v=1}^k t_v}}{\prod_{j=1}^t(z-b_j)^{(n+\sum_{v=1}^k n_v)q_j+\sum_{v=1}^k t_v}}g(z) \\
&= \frac{P_1(z)}{Q_1(z)},
\end{aligned}
\tag{2.11}
$$

where $g(z)=\prod_{v=1}^k g_v(z)$, $\deg g\le(\sum_{v=1}^k t_v)(s+t-1)$.
Case 2.1. $f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}-P(z)$ has uniqueness a zero, we denote by z_0. Thus, we can write

$$
f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}-P(z)=\frac{B(z-z_0)^l}{\prod_{j=1}^t(z-b_j)^{(n+\sum_{v=1}^k n_v)q_j+\sum_{v=1}^k t_v}},
\tag{2.12}
$$

where $l\in\mathbb{N}^*$ and $B\neq 0$ is a complex number. From (2.11), taking derivative both sides d times, we get

$$
\begin{aligned}
& (f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)})^{(d)} \\
&= A^{n+\sum_{v=1}^k n_v}\frac{\prod_{i=1}^s(z-a_i)^{(n+\sum_{v=1}^k n_v)p_i-\sum_{v=1}^k t_v-d}}{\prod_{j=1}^t(z-b_j)^{(n+\sum_{v=1}^k n_v)q_j+\sum_{v=1}^k t_v+d}}G_1(z),
\end{aligned}
\tag{2.13}
$$

where $\deg G_1\le(\sum_{v=1}^k t_v+d)(s+t-1)$. Similar to (2.13), taking derivative both sides (2.11) $d+1$ times, we get

$$
\begin{aligned}
& (f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)})^{(d+1)} \\
&= A^{n+\sum_{v=1}^k n_v}\frac{\prod_{i=1}^s(z-a_i)^{(n+\sum_{v=1}^k n_v)p_i-\sum_{v=1}^k t_v-d-1}}{\prod_{j=1}^t(z-b_j)^{(n+\sum_{v=1}^k n_v)q_j+\sum_{v=1}^k t_v+d+1}}G_2(z),
\end{aligned}
\tag{2.14}
$$

where $\deg G_2\le(\sum_{v=1}^k t_v+d+1)(s+t-1)$.

Note that in the case $P(z)=a_0\neq 0$, we take the derivative both sides (2.11) with 0, 1 times, respectively, we obtain the (2.13) and (2.14), respectively.
Case 2.1.1. $d\ge l$. Then from (2.12), we have

$$
(f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)})^{(d+1)}=\frac{BR_{d+1}(z)}{\prod_{j=1}^t(z-b_j)^{(n+\sum_{v=1}^k n_v)q_j+\sum_{v=1}^k t_v+d+1}},
\tag{2.15}
$$

where

$$R_{d+1}(z) = \prod_{h=0}^{d}\left(l - \left(n + \sum_{v=1}^{k} n_v\right) q - \left(\sum_{v=1}^{k} t_v\right) t - h\right) z^{(d+1)t-(d-l+1)} + c_{(d+1)t-(d-l+1)-1} z^{(d+1)t-(d-l+1)-1} + \cdots + c_0$$

and c_h, $h = 0, \ldots, (d+1)t - (d-l+1) - 1$ are complex numbers.

From (2.11) and (2.12), compare degree of the numerator after computing, we get

$$\left(n + \sum_{v=1}^{k} n_v\right) p - \left(\sum_{v=1}^{k} t_v\right) s + \deg g = \max\{l, \left(n + \sum_{v=1}^{k} n_v\right) q + \left(\sum_{v=1}^{k} t_v\right) t + d\}. \quad (2.16)$$

From (2.16) and $\deg g \le (\sum_{v=1}^{k} t_v)(s+t-1)$, we obtain

$$\left(n + \sum_{v=1}^{k} n_v\right) p - \left(\sum_{v=1}^{k} t_v\right) s + \left(\sum_{v=1}^{k} t_v\right)(s+t-1) \ge \left(n + \sum_{v=1}^{k} n_v\right) q + \left(\sum_{v=1}^{k} t_v\right) t + d.$$

This implies

$$\left(n + \sum_{v=1}^{k} n_v\right)(p-q) \ge \sum_{v=1}^{k} t_v + d.$$

Hence, $p \ge q + \dfrac{\sum_{v=1}^{k} t_v + d}{n + \sum_{v=1}^{k} n_v} > q$. From (2.14) and (2.15), we see

$$\deg \prod_{i=1}^{s} (z - a_i)^{(n+\sum_{v=1}^{k} n_v)p_i - \sum_{v=1}^{k} t_v - d - 1} \le \deg R_{d+1}(z).$$

Thus,

$$\left(n + \sum_{v=1}^{k} n_v\right) p - \left(\sum_{v=1}^{k} t_v + d + 1\right) s \le (d+1)t - (d-l+1).$$

This implies

$$l - d \ge \left(n + \sum_{v=1}^{k} n_v\right) p + 1 - \left(\sum_{v=1}^{k} t_v + d + 1\right) s - (d+1)t. \quad (2.17)$$

From (2.9) and $p > q$, we have

$$\left(\sum_{v=1}^{k} t_v + d + 1\right) s + (d+1)t \leq \left(\sum_{v=1}^{k} t_v + d + 1\right) p \frac{n + \sum_{v=1}^{k} n_v}{2d + 2 + \sum_{v=1}^{k} t_v} + (d+1)q \frac{n + \sum_{v=1}^{k} n_v}{2d + 2 + \sum_{v=1}^{k} t_v} < \left(n + \sum_{v=1}^{k} n_v\right) p. \quad (2.18)$$

Combining (2.17) and (2.18), we have $l - d \geq 1$. This contradicts with $d \geq l$.
Case 2.1.2. $d < l$. If $d \geq 1$, we have

$$(f^n (f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)})^{(d)} - (P(z))^{(d)} = \frac{(z - z_0)^{l-d} U_d(z)}{\prod_{j=1}^{t} (z - b_j)^{(n + \sum_{v=1}^{k} n_v) q_j + \sum_{v=1}^{k} t_v + d}}, \quad (2.19)$$

where $U_d(z) = \prod_{h=0}^{d-1} (l - (n + \sum_{v=1}^{k} n_v) q - (\sum_{v=1}^{k} t_v) t - h) z^{dt} + y_{dt-1} z^{dt-1} + \cdots + y_0$, y_j, $j = 0, \ldots, dt - 1$ are complex numbers. We also have

$$(f^n (f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)})^{(d+1)} - (P(z))^{(d+1)} = \frac{(z - z_0)^{l-d-1} U_{d+1}(z)}{\prod_{j=1}^{t} (z - b_j)^{(n + \sum_{v=1}^{k} n_v) q_j + \sum_{v=1}^{k} t_v + d}}, \quad (2.20)$$

where $U_{d+1}(z) = \prod_{h=0}^{d} (l - (n + \sum_{v=1}^{k} n_v) q - (\sum_{v=1}^{k} t_v) t - h) z^{(d+1)t} + x_{(d+1)t-1} z^{(d+1)t-1} + \cdots + x_0$, x_j, $j = 0, \ldots, (d+1)t - 1$ are complex numbers.

We distinguish two subcase:
Case 2.1.2.1. $l \neq (n + \sum_{v=1}^{k} n_v) q + (\sum_{v=1}^{k} t_v) t + d$. From (2.11) and (2.12), we see $\deg P_1 \geq \deg Q_1$. This implies

$$\left(n + \sum_{v=1}^{k} n_v\right) p - \left(\sum_{v=1}^{k} t_v\right) s + \deg g \geq \left(n + \sum_{v=1}^{k} n_v\right) q + \left(\sum_{v=1}^{k} t_v\right) t.$$

From $\deg g \leq (\sum_{v=1}^{k} t_v)(s + t - 1)$. Thus,

$$p \geq q + \frac{\sum_{v=1}^{k} t_v}{n + \sum_{v=1}^{k} n_v} > q. \quad (2.21)$$

From (2.13) and (2.19), we see $z_0 \neq a_i, i = 1, \ldots, s$. Thus, from (2.14) and (2.20), we get

$$\deg \prod_{i=1}^{s} (z - a_i)^{(n+\sum_{v=1}^{k} n_v)p_i - \sum_{v=1}^{k} t_v - d - 1} \leq \deg U_{d+1}(z).$$

Thus, we have

$$\left(n + \sum_{v=1}^{k} n_v\right) p - \left(\sum_{v=1}^{k} t_v\right) s - (d+1)s \leq (d+1)t.$$

From (2.9) and (2.21), we obtain

$$\begin{aligned}\left(n + \sum_{v=1}^{k} n_v\right) p &\leq \left(\sum_{v=1}^{k} t_v + d + 1\right) s + (d+1)t \\ &\leq \left(\sum_{v=1}^{k} t_v + d + 1\right) p \frac{n + \sum_{v=1}^{k} n_v}{2d + 2 + \sum_{v=1}^{k} t_v} \\ &\quad + (d+1)q \frac{n + \sum_{v=1}^{k} n_v}{2d + 2 + \sum_{v=1}^{k} t_v} \\ &< \left(n + \sum_{v=1}^{k} n_v\right) p.\end{aligned}$$

This is a contradiction.
Case 2.1.2.2. $l = (n + \sum_{v=1}^{k} n_v)q + (\sum_{v=1}^{k} t_v)t + d.$
If $p > q$, by argument as Case 2.1.2.1, we obtain the contradiction.
If $p \leq q$, from (2.14) and (2.20), we have

$$l - d - 1 \leq \deg G_2 \leq \left(\sum_{v=1}^{k} t_v + d + 1\right) (s + t - 1).$$

Therefore,

$$\begin{aligned}\left(n + \sum_{v=1}^{k} n_v\right) q &= l - \left(\sum_{v=1}^{k} t_v\right) t - d \\ &\leq \deg G_2 - \left(\sum_{v=1}^{k} t_v\right) t + 1 \\ &\leq \left(\sum_{v=1}^{k} t_v + d + 1\right) (s + t - 1) - \left(\sum_{v=1}^{k} t_v\right) t + 1\end{aligned}$$

$$< \left(\sum_{v=1}^{k} t_v + d + 1\right) s + (d+1)t$$
$$\leq \left(\sum_{v=1}^{k} t_v + d + 1\right) p \frac{n + \sum_{v=1}^{k} n_v}{2d + 2 + \sum_{v=1}^{k} t_v}$$
$$+ (d+1)q \frac{n + \sum_{v=1}^{k} n_v}{2d + 2 + \sum_{v=1}^{k} t_v}$$
$$\leq \left(n + \sum_{v=1}^{k} n_v\right) q.$$

This is an impossible.

If $d = 0$, $P(z) = a_0 \neq 0$, from (2.12), we have

$$(f^n (f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)})' = \frac{(z - z_0)^{l-1} H_1(z)}{\prod_{j=1}^{t} (z - b_j)^{(n + \sum_{v=1}^{k} n_j) q_j + \sum_{v=1}^{k} t_v + 1}} \tag{2.22}$$

where $H_1(z) = B(l - (n + \sum\limits_{j=1}^{k} n_j)q - (\sum\limits_{j=1}^{k} t_j)t)z^t + w_1 z^{t-1} + \cdots + w_t$, $w_1, \ldots, w_t$ are complex numbers and B is a nonzero constant. From (2.11), we see

$$(f^n (f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)})' = \frac{\prod_{i=1}^{s} (z - a_i)^{(n + \sum_{v=1}^{k} n_v) p_i + \sum_{v=1}^{k} t_v - 1} H_2(z)}{\prod_{j=1}^{t} (z - b_j)^{(n + \sum_{v=1}^{k} n_v) q_j + \sum_{v=1}^{k} t_v + 1}}, \tag{2.23}$$

where $s + t - 1 \leq \deg H_2(z) \leq (\sum_{v=1}^{k} t_v + 1)(s + t - 1)$. By argument as $d \geq 1$, and remark that $p \geq s$, $q \geq t$, we get a contradiction.

Case 2.2. $f^n (f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)} - P(z)$ has no zeros. Thus, we can write

$$f^n (f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)} - P(z) = \frac{C}{\prod_{j=1}^{t} (z - b_j)^{(n + \sum_{v=1}^{k} n_v) q_j + \sum_{v=1}^{k} t_v}}, \tag{2.24}$$

where $C \neq 0$ is a complex number. Thus, (2.20) can be replaced by

$$(f^n (f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)})^{(d+1)} - (P(z))^{(d+1)}$$
$$= \frac{U_{d+1}^*(z)}{\prod_{j=1}^{t} (z - b_j)^{(n + \sum_{v=1}^{k} n_v) q_j + \sum_{v=1}^{k} t_v + d}}, \tag{2.25}$$

where $U_{d+1}^*(z) = \prod_{h=0}^{d} (-(n + \sum_{v=1}^{k} n_v)q - (\sum_{v=1}^{k} t_v)t - h) z^{(d+1)(t-1)} + x_{dt}^* z^{dt} + \cdots + x_0^*$, x_j^*, $j = 0, \ldots, (d+1)(t-1) - 1$ are complex numbers.

From (2.24) and (2.11), we have $\deg P_1 \geq \deg Q_1$. From (2.21), we see that $p \geq q + \dfrac{\sum_{v=1}^{k} t_v}{n + \sum_{v=1}^{k} n_v} > q$. Thus, combine (2.14) and (2.25), we get

$$\deg \prod_{i=1}^{s} (z - a_i)^{(n+\sum_{v=1}^{k} n_v) p_i - \sum_{v=1}^{k} t_v - d - 1} \leq \deg U_{d+1}^*(z).$$

Hence,

$$\left(n + \sum_{v=1}^{k} n_v\right) p - \left(\sum_{v=1}^{k} t_v\right) s - (d+1)s \leq (d+1)(t-1).$$

This implies

$$\begin{aligned} \left(n + \sum_{v=1}^{k} n_v\right) p &\leq \left(\sum_{v=1}^{k} t_v + d + 1\right) s + (d+1)(t-1) \\ &< \left(\sum_{v=1}^{k} t_v + d + 1\right) s + (d+1)t \end{aligned} \tag{2.26}$$

From (2.26), and compute similarly to Case 2.1.2.1, we get a contradiction. □

Lemma 5 *Let f be a nonconstant rational function, $f \neq 0$, $P(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0$, $d \in \mathbb{N}^*$, $a_d \neq 0$, $a_{d-1}, \ldots, a_0$ be complex numbers and $n \in \mathbb{N}$, $k, n_j, t_j \in \mathbb{N}^*$, $j = 1, \ldots, k$.*
If $d = 0$, $P(z) = a_0 \neq 0$,

$$n_j \geqslant t_j, n + \sum_{j=1}^{k} n_j \geqslant \sum_{j=1}^{k} t_j + 2, \quad j = 1, \ldots, k;$$

and if $d \geq 1$,

$$n_j \geqslant t_j, n + \sum_{j=1}^{k} n_j \geqslant \sum_{j=1}^{k} t_j + 2, \quad j = 1, \ldots, k,$$

suppose that all poles of f having multiplicity at least

$$\left[\frac{2d + 2 + \sum_{j=1}^{k} t_j}{n + \sum_{j=1}^{k} n_j}\right] + 1.$$

Then the equation

$$f^n (f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)} = P(z)$$

has at least distinct two zeros.

Proof Since f has not zeros, then we can write

$$f=\frac{A}{\prod_{j=1}^{t}(z-b_j)^{q_j}}, \tag{2.27}$$

where $q_j\geq\dfrac{2d+2+\sum_{v=1}^{k}t_v}{n+\sum_{v=1}^{k}n_v}$, $j=1,\ldots,t$ if $\deg P\geq 1$ and $q_j\geq 1$, $j=1,\ldots,t$ if $P(z)=a_0\neq 0$. Similar to (2.10), from (2.27), we have

$$(f^{n_v})^{(t_v)}=\frac{A^{n_v}}{\prod_{j=1}^{t}(z-b_j)^{q_jn_v+t_v}}g_v(z),\quad v=1,\ldots,k, \tag{2.28}$$

where

$$g_v(z)=(-n_vq)(-n_vq-1)\ldots(-n_vq-t_v+1)z^{t_v(t-1)}\\ +b^*_{t_v(t-1)-1}z^{t_v(s+t-1)-1}+\cdots+b^*_0$$

and b^*_e, $e=0,\ldots,t_v(t-1)-1$ are complex numbers.

We consider two cases:

Case 1.1. $f^n(f^{n_1})^{(t_1)}\ldots(f^{n_k})^{(t_k)}-P(z)$ has uniqueness a zero, we denote by z_0. Thus, we can write

$$f^n(f^{n_1})^{(t_1)}\ldots(f^{n_k})^{(t_k)}-P(z)=\frac{B(z-z_0)^l}{\prod_{j=1}^{t}(z-b_j)^{(n+\sum_{v=1}^{k}n_v)q_j+\sum_{v=1}^{k}t_v}}, \tag{2.29}$$

where $l\in\mathbb{N}^*$ and $B\neq 0$ is a complex number.

From (2.28), we have

$$f^n(f^{n_1})^{(t_1)}\ldots(f^{n_v})^{(t_v)}=\frac{A^{n+\sum_{v=1}^{k}n_v}g(z)}{\prod_{j=1}^{t}(z-b_j)^{(n+\sum_{v=1}^{k}n_v)q_j+\sum_{v=1}^{k}t_v}}\\ =\frac{P_2(z)}{Q_2(z)}, \tag{2.30}$$

where $g(z)=\prod_{v=1}^{k}g_v(z)$, $\deg g\leq(\sum_{v=1}^{k}t_v)(t-1)$. Similar to (2.14), we have

$$(f^n(f^{n_1})^{(t_1)}\ldots(f^{n_k})^{(t_k)})^{(d+1)}=\frac{A^{n+\sum_{v=1}^{k}n_v}G_2^*(z)}{\prod_{j=1}^{t}(z-b_j)^{(n+\sum_{v=1}^{k}n_v)q_j+\sum_{v=1}^{k}t_v+d+1}}, \tag{2.31}$$

where $\deg G_2^*\leq(\sum_{v=1}^{k}t_v+d+1)(t-1)$.

Case 1.1.1. $d\geq l$.

From (2.29) and (2.30), we see

$$\deg g = \max\left\{l, \left(n + \sum_{v=1}^{k} n_v\right) q + \left(\sum_{v=1}^{k} t_v\right) t + d\right\}. \tag{2.32}$$

From $\deg g \leq (\sum_{v=1}^{k} t_v)(t-1)$ and (2.32), we obtained

$$\left(\sum_{v=1}^{k} t_v\right)(t-1) \geq \left(n + \sum_{v=1}^{k} n_v\right) q + \left(\sum_{v=1}^{k} t_v\right) t + d.$$

This is a contradiction.

Case 1.1.2. $d < l$.

If $l \neq (n + \sum_{v=1}^{k} n_v)q + (\sum_{v=1}^{k} t_v)t + d$. From (2.29) and (2.30), we have $\deg P_2 = \deg g \geq \deg Q_2$. By argument Case 1.1.1, we get a contradiction. We have the expression as following

$$\begin{aligned}&(f^n(f^{n_1})^{(t_1)} \ldots (f^{n_k})^{(t_k)})^{(d+1)} - (P(z))^{(d+1)}\\ &= \frac{(z-z_0)^{l-d-1} U_{d+1}(z)}{\prod_{j=1}^{t} (z-b_j)^{(n+\sum_{v=1}^{k} n_v)q_j + \sum_{v=1}^{k} t_v + d}},\end{aligned} \tag{2.33}$$

where

$$\begin{aligned}U_{d+1}(z) = &\prod_{h=0}^{d} \left(l - \left(\left(n + \sum_{v=1}^{k} n_v\right) q\right) - \left(\sum_{v=1}^{k} t_v\right) t - h\right) z^{(d+1)t}\\ &+ x_{(d+1)t-1} z^{(d+1)t-1} + \cdots + x_0,\end{aligned}$$

x_j, $j = 0, \ldots, (d+1)t - 1$ are complex numbers.

If $l = (n + \sum_{v=1}^{k} n_v)q + (\sum_{v=1}^{k} t_v)t + d$. From (2.31) and (2.33), we obtain

$$\deg G_2^* = l - d - 1 + \deg U_{d+1} \geq l - d - 1. \tag{2.34}$$

From (2.34) and $\deg G_2^* \leq (\sum_{v=1}^{k} t_v + d + 1)(t-1)$, we obtain

$$\begin{aligned}\left(n + \sum_{v=1}^{k} n_v\right) q &= l - \left(\sum_{v=1}^{k} t_v\right) t - d\\ &\leq \deg G_2^* - \left(\sum_{v=1}^{k} t_v\right) t + 1\\ &\leq \left(\sum_{v=1}^{k} t_v + d + 1\right)(t-1) - \left(\sum_{v=1}^{k} t_v\right) t + 1\end{aligned}$$

$$< (n + \sum_{v=1}^{k} n_v)q.$$

This is a impossible.

Case 1.2. $f^n(f^{n_1})^{t_1}\dots(f^{n_k})^{t_k} - P(z)$ has no zeros. Thus, we can write

$$f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)} - P(z) = \frac{C}{\prod_{j=1}^{t}(z-b_j)^{(n+\sum_{v=1}^{k} n_v)q_j + \sum_{v=1}^{k} t_v}}, \quad (2.35)$$

where $C \neq 0$ is a constant complex number. From (2.30) and (2.35), we have

$$\deg g = \left(n + \sum_{v=1}^{k} n_v\right) q + \left(\sum_{v=1}^{k} t_v\right) t + d. \quad (2.36)$$

From (2.36) and $\deg g \leq (\sum_{v=1}^{k} t_v)(t-1)$, we get a contradiction. □

Lemma 6 *Let f be a transcendental meromorphic function and $a(z) = a_d z^d + a_{d-1}z^{d-1} + \cdots + a_0$, $d \in \mathbb{N}^*$, $a_d \neq 0, a_{d-1}, \dots, a_0$ be constant numbers complex. Let $n \in \mathbb{N}$, $k, n_v, t_v \in \mathbb{N}^*$, $v = 1, \dots, k$ satistfy*

$$n + \sum_{j=1}^{k} n_v \geqslant \sum_{v=1}^{k} t_v + 3.$$

Then the equation

$$f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)} = a(z)$$

has infinitely zeros. Furthermore, if f is a transcendental entire function, then the statement holds with

$$n + \sum_{j=1}^{k} n_v \geqslant \sum_{v=1}^{k} t_v + 2.$$

Proof We see $P(f) = f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)}$ is a transcendental meromorphic function. By Remark 6, we have

$$T(r,f) \leqslant \frac{\theta(P)+1}{d(P)-1}\overline{N}\left(r,\frac{1}{f}\right) + \frac{1}{d(P)-1}\overline{N}\left(r,\frac{1}{P-a}\right) + o(T(r,f)). \quad (2.37)$$

By easy computing, we have $d(P)=\sum_{v=1}^{k} n_v, \theta(P)=\sum_{v=1}^{k} t_v$. From (2.37) we get

$$\left(n+\sum_{v=1}^{k} n_v-\sum_{v=1}^{k} t_v-2\right) T(r, f) \leqslant \overline{N}\left(r, \frac{1}{P-a}\right)+o(T(r, f)). \tag{2.38}$$

By $n+\sum_{v=1}^{k} n_v \geqslant \sum_{v=1}^{k} t_v+3$ and (2.38), we obtain that the equation

$$f^n\left(f^{n_1}\right)^{(t_1)} \ldots\left(f^{n_k}\right)^{(t_k)}=a(z)$$

has infinitely zeros. In the case f is a transcendental entire function, by Lemma 3, we have

$$T(r, f) \leqslant \frac{\theta(P)+1}{d(P)} \overline{N}\left(r, \frac{1}{f}\right)+\frac{1}{d(P)} \overline{N}\left(r, \frac{1}{P-a}\right)+o(T(r, f)).$$

Thus,

$$\left(n+\sum_{v=1}^{k} n_v-\sum_{v=1}^{k} t_v-1\right) T(r, f) \leqslant \overline{N}\left(r, \frac{1}{P-a}\right)+o(T(r, f)). \tag{2.39}$$

By $n+\sum_{v=1}^{k} n_v \geqslant \sum_{v=1}^{k} t_v+2$ and (2.39), we obtain that the equation

$$f^n\left(f^{n_1}\right)^{(t_1)} \ldots\left(f^{n_k}\right)^{(t_k)}=a(z)$$

has infinitely zeros. We have completed the proof of Lemma 6. □

3 Proof of Our Results

Proof of Theorem 1 and Theorem 4 First, we prove Theorem 1. Without loss of generality, we may assume that D is the unit disc and $\mathcal{F}$ is not normal at $z_0=0 \in D$. Then $a(0)=0$ or $a(0) \neq 0$.

Case 1. $a(0)=0$, then we may assume that $a(z)=a_m z^m+a_{m+1} z^{m+1}+\cdots=z^m h(z)$, where $h(z)$ is a holomorphic function on neighbourhood of 0, $h(0)=a_m \neq 0$ and

$$m=\left(n+\sum_{v=1}^{k} n_v\right) s=\left(n_I+\sum_{v=1}^{k} n_{v I}\right) s.$$

We consider the family $\mathcal{G}$ which defined as following

$$\mathcal{G}=\left\{H_j: H_j(z)=\frac{f_j(z)}{z^s}\right\}.$$

If $\mathcal{G}$ is not normal at $z=0$, apply to Lemma 1, for $\alpha=\frac{\sum_{v=1}^{k} t_v}{n+\sum_{v=1}^{k} n_v}$ there exist

(1) a real number r, $0 < r < 1$,
(2) points z_j, $|z_v| < r$, $z_j \to 0$,
(3) positive numbers $\rho_j \to 0^+$,
(4) functions $H_j \in \mathcal{G}$

such that

$$g_j(\xi) = \frac{H_j(z_j + \rho_j \xi)}{\rho_j^{\alpha}} \to g(\xi) \tag{3.1}$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $g(\xi)$ is a nonconstant meromorphic function and $g^{\#}(\xi) \leqslant g^{\#}(0) = 1$.

We consider two subcases:

Case 1.1. There exists the subsequence of $\frac{z_j}{\rho_j}$, we also still denote by $\frac{z_j}{\rho_j}$ such that $\frac{z_j}{\rho_j} \to c \in \mathbb{C}$. Then

$$F_j(\xi) = \frac{f_j(\rho_j \xi)}{\rho_j^{s + \frac{\sum_{v=1}^k t_v}{n + \sum_{v=1}^k n_v}}} = \frac{\xi^s H_j\left(z_j + \rho_j\left(\xi - \frac{z_j}{\rho_j}\right)\right)}{\rho_j^{\alpha}} \tag{3.2}$$

$$\to \xi^s g(\xi - c) := H(\xi).$$

On the other hand, we see

$$\left(F_j^{n_v}(\xi)\right)^{(t_v)} = \left(\left(\frac{f_j(\rho_j\xi)}{\rho_j^{\beta}}\right)^{n_v}\right)^{(t_v)} = \frac{1}{\rho_j^{n_v\beta - t_v}}\left(f_j^{n_v}\right)^{(t_v)}(\rho_j\xi),$$

where $\beta = s + \frac{\sum_{v=1}^k t_v}{n + \sum_{v=1}^k n_v}$. From $\sum_{v=1}^k t_v > \sum_{v=1}^k t_{vI}$, we have

$$\begin{aligned} &F_j^n (F_j^{n_1})^{(t_1)} \ldots (F_j^{n_k})^{(t_k)} \\ &\quad + \sum_I \rho_j^{\frac{(n_I + \sum_{v=1}^k n_{vI})(\sum_{v=1}^k t_v) - (n + \sum_{v=1}^k n_v)(\sum_{v=1}^k t_{vI})}{n + \sum_{v=1}^k n_v}} a_I(\rho_j \xi) \\ &\quad \times F_j^{n_I} (F_j^{n_{1I}})^{(t_{1I})} \ldots (F_j^{n_{kI}})^{(t_{kI})} - \frac{a(\rho_j\xi)}{\rho_j^m} \\ &\quad = \frac{f_j^n(\rho_j\xi)(f_j^{n_1})^{(t_1)}(\rho_j\xi) \ldots (f_j^{n_k})^{(t_k)}(\rho_j\xi)}{\rho_j^m} \end{aligned}$$

$$+\sum_I \rho_j^{\frac{(n_I+\sum_{v=1}^k n_{vI})(\sum_{v=1}^k t_v)-(n+\sum_{v=1}^k n_v)(\sum_{v=1}^k t_{vI})}{n+\sum_{v=1}^k n_v}} a_I(\rho_j\xi)$$
$$\times \frac{\rho_j^{\sum_{v=1}^k t_{vI}}}{\rho_j^{(n_I+\sum_{v=1}^k n_{vI})\left(s+\frac{\sum_{v=1}^k t_v}{n+\sum_{v=1}^k n_v}\right)}} f_j^{n_I}(\rho_j\xi)(f_j^{n_{1I}})^{(t_{1I})}(\rho_j\xi)\ldots(f_j^{n_{kI}})^{(t_{kI})}(\rho_j\xi)$$
$$-\frac{a(\rho_j\xi)}{\rho_j^m}.$$

Thus,

$$F_j^n(F_j^{n_1})^{(t_1)}\ldots(F_j^{n_k})^{(t_k)}$$
$$+\sum_I \rho_j^{\frac{(n_I+\sum_{v=1}^k n_{vI})(\sum_{v=1}^k t_v)-(n+\sum_{v=1}^k n_v)(\sum_{v=1}^k t_{vI})}{n+\sum_{v=1}^k n_v}} a_I(\rho_j\xi)$$
$$\times F_j^{n_I}(F_j^{n_{1I}})^{(t_{1I})}\ldots(F_j^{n_{kI}})^{(t_{kI})}-\frac{a(\rho_j\xi)}{\rho_j^m}$$

$$=\frac{f_j^n(\rho_j\xi)(f_j^{n_1})^{(t_1)}(\rho_j\xi)\ldots(f_j^{n_k})^{(t_k)}(\rho_j\xi)}{\rho_j^m}$$
$$+\sum_I \rho_j^{\frac{(n_I+\sum_{v=1}^k n_{vI})(\sum_{v=1}^k t_v)-(n+\sum_{v=1}^k n_v)(\sum_{v=1}^k t_{vI})}{n+\sum_{v=1}^k n_v}} a_I(\rho_j\xi)$$
$$\times\ \rho_j^{\frac{(\sum_{v=1}^k t_{vI})(n+\sum_{v=1}^k n_v)-(n_I+\sum_{v=1}^k n_{vI})(\sum_{v=1}^k t_v)}{n+\sum_{v=1}^k n_v}}$$
$$\times\frac{f_j^{n_I}(\rho_j\xi)(f_j^{n_{1I}})^{(t_{1I})}(\rho_j\xi)\ldots(f_j^{n_{kI}})^{(t_{kI})}(\rho_j\xi)}{\rho_j^m}-\frac{a(\rho_j\xi)}{\rho_j^m}$$
$$=\frac{f_j^n(\rho_j\xi)(f_j^{n_1})^{(t_1)}(\rho_j\xi)\ldots(f_j^{n_k})^{(t_k)}(\rho_j\xi)}{\rho_j^m}$$
$$+\frac{\sum_I a_I(\rho_j\xi)f_j^{n_I}(\rho_j\xi)(f_j^{n_{1I}})^{(t_{1I})}(\rho_j\xi)\ldots(f_j^{n_{kI}})^{(t_{kI})}(\rho_j\xi)}{\rho_j^m}-\frac{a(\rho_j(\xi))}{\rho_j^m}$$
$$\to H^n(\xi)(H^{n_1})^{(t_1)}(\xi)\ldots(H^{n_k})^{(t_k)}(\xi)-a_m\xi^m \tag{3.3}$$

spherically uniformly on compact subsets of $\mathbb{C}\backslash\{\text{poles of } H\}$, all zeros and poles of H are multiple at least $\left[\dfrac{2m+2+\sum_{v=1}^{k}t_v}{n+\sum_{v=1}^{k}n_v}\right]+1$. We see that

$$H^n(\xi)(H^{n_1})^{(t_1)}(\xi)\ldots(H^{n_k})^{(t_k)}(\xi)\not\equiv a_m\xi^m.$$

Indeed, if

$$H^n(\xi)(H^{n_1})^{(t_1)}(\xi)\ldots(H^{n_k})^{(t_k)}(\xi)\equiv a_m\xi^m, \tag{3.4}$$

then H has not poles on $\mathbb{C}$ and H has a unique zero $z=0$. Thus, from (3.4), we have

$$\begin{aligned}
T(r,H^{n+\sum_{v=1}^{k}n_v}(\xi)) &= T\left(r,\frac{1}{a_m\xi^m}\frac{(H^{n_1})^{(t_1)}(\xi)}{H^{n_1}}\ldots\frac{(H^{n_k})^{(t_k)}(\xi)}{H^{n_k}}\right)+O(1)\\
&= m\left(r,\frac{(H^{n_1})^{(t_1)}(\xi)}{H^{n_1}}\ldots\frac{(H^{n_k})^{(t_k)}(\xi)}{H^{n_k}}\right)\\
&\quad + N\left(r,\frac{(H^{n_1})^{(t_1)}(\xi)}{H^{n_1}}\ldots\frac{(H^{n_k})^{(t_k)}(\xi)}{H^{n_k}}\right)+O(\log r)\\
&\le O(\log r).
\end{aligned}$$

Thus, H is polynomial with the form $H(z)=az^p$, $a\neq 0$, where

$$\begin{aligned}
p &\ge \left[\frac{2m+2+\sum_{v=1}^{k}t_v}{n+\sum_{v=1}^{k}n_v}\right]+1>\frac{2m+2+\sum_{v=1}^{k}t_v}{n+\sum_{v=1}^{k}n_v}\\
&= \frac{s(2m+2+\sum_{v=1}^{k}t_v)}{m}>2s.
\end{aligned}$$

From (3.4), we see

$$\left(n+\sum_{v=1}^{k}n_v\right)p-\sum_{v=1}^{k}t_v=m=\left(n+\sum_{v=1}^{k}n_v\right)s.$$

Thus $\sum_{v=1}^{k}t_v$ divisible by $n+\sum_{v=1}^{k}n_v$, this contradicts with

$$n+\sum_{v=1}^{k}n_v>\sum_{v=1}^{k}t_v.$$

Then, by Lemma 4 to Lemma 6, we see

$$H^n(\xi)(H^{n_1})^{(t_1)}(\xi)\ldots(H^{n_k})^{(t_k)}(\xi)-a_m\xi^m$$

having at least two distinct zeros, we denote by ξ_1, ξ_2. Thus, there exists $\delta > 0$ such that $D(\xi_1, \delta) \cap D(\xi_2, \delta) = \varnothing$. From (3.3), by Hurwitz's Theorem there exist two sequences $\xi_j \to \xi_1$ and $\xi_j^* \to \xi_2$ satisfying

$$\begin{aligned} &f_j^n(\rho_j\xi_j)(f_j^{n_1})^{(t_1)}(\rho_j\xi_j)\dots(f_j^{n_k})^{(t_k)}(\rho_j\xi_j) \\ &\qquad + \sum_I a_I(\rho_j\xi_j) f_j^{n_I}(\rho_j\xi_j)(f_j^{n_{1I}})^{(t_{1I})}(\rho_j\xi_j)\dots \\ &\qquad (f_j^{n_{kI}})^{(t_{kI})}(\rho_j\xi_j) = a(\rho_j\xi_j), \\ &f_j^n(\rho_j\xi_j^*)(f_j^{n_1})^{(t_1)}(\rho_j\xi_j^*)\dots(f_j^{n_k})^{(t_k)}(\rho_j\xi_j^*) \\ &\qquad + \sum_I a_I(\rho_j\xi_j^*) f_j^{n_I}(\rho_j\xi_j^*)(f_j^{n_{1I}})^{(t_{1I})}(\rho_j\xi_j^*)\dots \\ &\qquad (f_j^{n_{kI}})^{(t_{kI})}(\rho_j\xi_j^*) = a(\rho_j\xi_j^*). \end{aligned}$$

By hypothesis

$$\begin{aligned} &f^n(f^{n_1})^{(t_1)}\dots(f^{n_k})^{(t_k)} + \sum_I a_I f^{n_I}(f^{n_{1I}})^{(t_{1I})}\dots(f^{n_{kI}})^{(t_{kI})}, \\ &g^n(g^{n_1})^{(t_1)}\dots(g^{n_k})^{(t_k)} + \sum_I a_I g^{n_I}(g^{n_{1I}})^{(t_{1I})}\dots(g^{n_{kI}})^{(t_{kI})} \end{aligned}$$

share $a(z)-\ IM$ in D for each pair (f, g) in $\mathcal{F}$, then for any $r \in \mathbb{Z}^+$, we have

$$\begin{aligned} &f_r^n(\rho_j\xi_j)(f_r^{n_1})^{(t_1)}(\rho_j\xi_j)\dots(f_r^{n_k})^{(t_k)}(\rho_j\xi_j) \\ &\qquad + \sum_I a_I(\rho_j\xi_j) f_r^{n_I}(\rho_j\xi_j)(f_r^{n_{1I}})^{(t_{1I})}(\rho_j\xi_j)\dots \\ &\qquad (f_r^{n_{kI}})^{(t_{kI})}(\rho_j\xi_j) = a(\rho_j\xi_j), \\ &f_r^n(\rho_j\xi_j^*)(f_r^{n_1})^{(t_1)}(\rho_j\xi_j^*)\dots(f_r^{n_k})^{(t_k)}(\rho_j\xi_j^*) \\ &\qquad + \sum_I a_I(\rho_j\xi_j^*) f_r^{n_I}(\rho_j\xi_j^*)(f_r^{n_{1I}})^{(t_{1I})}(\rho_j\xi_j^*)\dots \\ &\qquad (f_r^{n_{kI}})^{(t_{kI})}(\rho_j\xi_j^*) = a(\rho_j\xi_j^*). \end{aligned}$$

Fix r, taking $j \to \infty$, we get

$$\begin{aligned} &f_r^n(0)(f_r^{n_1})^{(t_1)}(0)\dots(f_r^{n_k})^{(t_k)}(0) \\ &\quad + \sum_I a_I(0) f_r^{n_I}(0)(f_r^{n_{1I}})^{(t_{1I})}(0)\dots(f_r^{n_{kI}})^{(t_{kI})}(0) = a(0). \end{aligned}$$

Since the zeros of

$$f_r^n(z)(f_r^{n_1})^{(t_1)}(z)\dots(f_r^{n_k})^{(t_k)}(z) + \sum_I a_I(z) f_r^{n_I}(z)(f_r^{n_{1I}})^{(t_{1I})}(z)\dots(f_r^{n_{kI}})^{(t_{kI})}(z) = a(z).$$

have no accumulation points, in fact we have

$$\rho_j\xi_j = \rho_j\xi_j^* = 0$$

or equivalently

$$\xi_j = \xi_j^* = 0.$$

This contradicts with $D(\xi_1, \delta) \cap D(\xi_2, \delta) = \varnothing$. Hence, $\mathcal{G}$ is a normal family at 0.

Case 1.2. There exists the subsequence of $\frac{z_j}{\rho_j}$, we also still denote by $\frac{z_j}{\rho_j}$ such that $\frac{z_j}{\rho_j} \to \infty$. By definition of H, we have

$$f_j(z_j + \rho_j\xi) = (z_j + \rho_j\xi)^s H_j(z_j + \rho_j\xi).$$

This implies

$$\left(f_j^{n_v}(z_j + \rho_j\xi)\right)^{(t_v)} = \sum_{l_v=0}^{t_v} C_{t_v}^{l_v}\left[(z_j + \rho_j\xi)^{n_v s}\right]^{(l_v)}\left[H_j^{n_v}(z_j + \rho_j\xi)\right]^{(t_v-l_v)},$$

for all $v = 1, \dots, k$. Hence,

$$(f_j^{n_v})^{(t_v)}(z_j + \rho_j\xi) = (z_j + \rho_j\xi)^{n_v s}(H_j^{n_v})^{(t_v)}(z_j + \rho_j\xi) + \sum_{l_v=1}^{t_v} b_{l_v}(z_j + \rho_j\xi)^{n_v s - l_v}(H_j^{n_v})^{(t_v-l_v)}(z_j + \rho_j\xi), \tag{3.5}$$

for all $v = 1, \dots, k$, where $b_{l_v}, l_v = 1, \dots, t_v$ are nonzero complex numbers. We also have

$$\rho_j^\alpha g_j(\xi) = H_j(z_j + \rho_j\xi).$$

Thus,

$$(H_j^{n_v})^{(l_v)}(z_j + \rho_j\xi) = \rho_j^{n_v\alpha - l_v}(g_j^{n_v})^{(l_v)}(\xi), \tag{3.6}$$

for all $l_v = 1, \ldots, t_v$, $v = 1, \ldots, k$. From (3.5) and (3.6), we get

$$(f_j^{n_v})^{(t_v)}(z_j + \rho_j \xi) = \rho_j^{n_v \alpha - t_v}(z_j + \rho_j \xi)^{n_v s}(g_j^{n_v})^{(t_v)}(\xi) + \sum_{l_v=1}^{t_v} b_{l_v} \rho_j^{n_v \alpha}(z_j + \rho_j \xi)^{n_v s - l_v} \rho_j^{-(t_v - l_v)}(g_j^{n_v})^{(t_v - l_v)}(z_j + \rho_j \xi), \tag{3.7}$$

for all $v = 1, \ldots, k$. Thus,

$$\begin{aligned} &f_j^n(z_j + \rho_j \xi)(f_j^{n_1})^{(t_1)}(z_j + \rho_j \xi) \ldots (f_j^{n_k})^{(t_k)}(z_j + \rho_j \xi) \\ &= (z_j + \rho_j \xi)^m g_j^n(\xi)(g_j^{n_1})^{(t_1)}(\xi) \ldots (g_j^{n_k})^{(t_k)}(\xi) \\ &+ \sum_{0 \le l_1 \le t_1, \ldots, 0 \le l_k \le t_k, \sum_{v=1}^k l_v \ge 1} C_{l_1, \ldots, l_k}(z_j + \rho_j \xi)^m \\ &\times \prod_{v=1}^k \frac{1}{\left(\frac{z_j}{\rho_j} + \xi\right)^{l_v}} g_j^n(\xi)(g_j^{n_1})^{(t_1)}(\xi) \ldots (g_j^{n_k})^{(t_k)}(\xi). \end{aligned} \tag{3.8}$$

Similar to (3.8), we also have

$$\begin{aligned} &f_j^{n_I}(z_j + \rho_j \xi)(f_j^{n_{1I}})^{(t_{1I})}(z_j + \rho_j \xi) \ldots (f_j^{n_{kI}})^{(t_{kI})}(z_j + \rho_j \xi) \\ &= \rho_j^{(n_I + \sum_{v=1}^k n_{vI})\alpha - \sum_{v=1}^k t_{vI}}(z_j + \rho_j \xi)^m g_j^{n_I}(\xi)(g_j^{n_{1I}})^{(t_{1I})}(\xi) \ldots (g_j^{n_{kI}})^{(t_{kI})}(\xi) \\ &+ \rho_j^{(n_I + \sum_{v=1}^k n_{vI})\alpha - \sum_{v=1}^k t_{vI}} \sum_{0 \le l_{1I} \le t_{1I}, \ldots, 0 \le l_{kI} \le t_{kI}, \sum_{v=1}^k l_{vI} \ge 1} C_{l_{1I}, \ldots, l_{kI}}(z_j + \rho_j \xi)^m \\ &\times \prod_{v=1}^k \frac{1}{\left(\frac{z_j}{\rho_j} + \xi\right)^{l_{vI}}} g_j^{n_I}(\xi)(g_j^{n_{1I}})^{(t_{1I})}(\xi) \ldots (g_j^{n_{kI}})^{(t_{kI})}(\xi). \end{aligned} \tag{3.9}$$

From (3.8) and (3.9) and condition

$$\sum_{v=1}^k t_v > \sum_{v=1}^k t_{vI},$$

we conclude that

$$\frac{a_m f_j^n(z_j + \rho_j \xi)(f_j^{n_1})^{(t_1)}(z_j + \rho_j \xi) \ldots (f_j^{n_k})^{(t_k)}(z_j + \rho_j \xi)}{a(z_j + \rho_j \xi)} + \frac{a_m \sum_I f_j^{n_I}(z_j + \rho_j \xi)(f_j^{n_{1I}})^{(t_{1I})}(z_j + \rho_j \xi) \ldots (f_j^{n_{kI}})^{(t_{kI})}(z_j + \rho_j \xi)}{a(z_j + \rho_j \xi)} - a_m$$

$$
\begin{aligned}
&= \frac{a_m}{h(z_j+\rho_j\xi)} g_j^n(\xi)(g_j^{n_1})^{(t_1)}(\xi)\dots(g_j^{n_k})^{(t_k)}(\xi)\\
&\quad + \frac{a_m}{h(z_j+\rho_j\xi)} \sum_{0\le l_1\le t_1,\dots,0\le l_k\le t_k,\sum_{v=1}^k l_v\ge 1}\\
&\quad C_{l_1,\dots,l_k}\prod_{v=1}^k \frac{1}{\left(\frac{z_j}{\rho_j}+\xi\right)^{l_v}} g_j^n(\xi)(g_j^{n_1})^{(t_1)}(\xi)\dots(g_j^{n_k})^{(t_k)}(\xi)\\
&\quad + \rho_j^{(n_I+\sum_{v=1}^k n_{vI})\alpha-\sum_{v=1}^k t_{vI}} \frac{a_m}{h(z_j+\rho_j\xi)} g_j^{n_I}(\xi)(g_j^{n_{1I}})^{(t_{1I})}(\xi)\dots(g_j^{n_{kI}})^{(t_{kI})}(\xi)\\
&\quad + \rho_j^{(n_I+\sum_{v=1}^k n_{vI})\alpha-\sum_{v=1}^k t_{vI}} \sum_{0\le l_{1I}\le t_{1I},\dots,0\le l_{kI}\le t_{kI},\sum_{v=1}^k l_{vI}\ge 1} \frac{a_m}{h(z_j+\rho_j\xi)} C_{l_{1I},\dots,l_{kI}}\\
&\quad \times\prod_{v=1}^k \frac{1}{\left(\frac{z_j}{\rho_j}+\xi\right)^{l_{vI}}} g_j^{n_I}(\xi)(g_j^{n_{1I}})^{(t_{1I})}(\xi)\dots(g_j^{n_{kI}})^{(t_{kI})}(\xi)\\
&\quad \to g^n(\xi)(g^{n_1})^{(t_1)}(\xi)\dots(g^{n_k})^{(t_k)}(\xi)-a_m
\end{aligned}
$$

spherically uniformly on compact subsets of $\mathbb{C}\backslash\{\text{poles of } g\}$.

Claim $g^n(\xi)(g^{n_1}(\xi))^{(t_1)}\dots(g^{n_k}(\xi))^{(t_k)}$ *is nonconstant.*

Since g is nonconstant and $n_j \ge t_j$ $(j=1,\dots,k)$, it is easy to see that $(g^{n_j}(\xi))^{(t_j)}\not\equiv 0$, for all $j\in\{1,\dots,k\}$. Hence, $g^n(\xi)(g^{n_1}(\xi))^{(t_1)}\dots(g^{n_k}(\xi))^{(t_k)}\not\equiv 0$.

Suppose that $g^n(\xi)(g^{n_1}(\xi))^{(t_1)}\dots(g^{n_k}(\xi))^{(t_k)}\equiv a$, $a\in\mathbb{C}\backslash\{0\}$. From conditions of Theorem 1, we have that in the case $n=0$, there exists $i\in\{1,\dots,k\}$ such that $n_i>t_i$. Therefore, since $a\ne 0$, it is easy to see that g is entire having no zero. So, by Lemma 2, $g(\xi)=e^{c\xi+d}$, $c\ne 0$. Then

$$
\begin{aligned}
g^n(\xi)(g^{n_1}(\xi))^{(t_1)}\cdots(g^{n_k}(\xi))^{(t_k)} &= e^{nc\xi+nd}(e^{n_1c\xi+n_1d})^{(t_1)}\cdots(e^{n_kc\xi+n_kd})^{(t_k)}\\
&= (n_1c)^{t_1}\cdots(n_kc)^{t_k}e^{(n+\sum_{j=1}^k n_j)c\xi+(n+\sum_{j=1}^k n_j)d}.
\end{aligned}
$$

Then $(n_1c)^{t_1}\cdots(n_kc)^{t_k}e^{(n+\sum_{j=1}^k n_j)c\xi+(n+\sum_{j=1}^k n_j)d}\equiv a$, which is impossible. So,

$$
g^n(\xi)(g^{n_1}(\xi))^{(t_1)}\dots(g^{n_k}(\xi))^{(t_k)}
$$

is nonconstant.

By Lemma 4 to Lemma 6, $g^n(\xi)(g^{n_1})^{(t_1)}(\xi)\dots(g^{n_k})^{(t_k)}(\xi)-a_m$ has at least two distinct zeros. Similar to Case 1.1, we have $\mathcal{G}$ is normal at $z=0$.

Hence, there exists $\Delta_\rho=\{z:|z|<\rho\}$ and a subsequence $\{H_{j_k}\}$ of $\{H_j\}$ such that H_{j_k} converges spherically locally uniformly to a meromorphic function $U(z)$ or ∞ $(k\to\infty)$ in Δ_ρ. Now, we consider two case as following:

Case (i). When k is sufficiently large, $f_{j_k}(0)\ne 0$. So $U(0)=\infty$. Then, for arbitrary constant $R>0$, there exists $\sigma\in(0,\rho)$, when $z\in\Delta_\sigma$, we have $|U(z)|>R$. Hence, for sufficiently large k, $|H_{j_k}(z)| > \frac{R}{2}$. So $\frac{1}{f_{j_k}}$ is holomorphic in Δ_σ and when $\sigma = \frac{R}{2}$, we have

$$\Big|\frac{1}{f_{j_k}(z)}\Big| = \Big|\frac{1}{z^s H_{j_k}(z)}\Big| \leq \frac{2^{s+1}}{R\sigma^s}.$$

By maximum principle and Montel's theorem, $\mathcal{F}$ is normal at $z = 0$.
Case (ii). There exists a subsequence of f_{j_k}, still denoted as f_{j_k} such that $f_{j_k}(0) = 0$. Since the multiplicity of every zero of f_{j_k} is at least

$$\Big[\frac{2m+2+\sum_{v=1}^{k} t_v}{n+\sum_{v=1}^{k} n_v}\Big] + 1 > \frac{2m+2+\sum_{v=1}^{k} t_v}{n+\sum_{v=1}^{k} n_v} = \frac{s(2m+2+\sum_{v=1}^{k} t_v)}{m} > 2s,$$

and $H_{j_k} = \frac{f_{j_k}}{z^s}$, then $H_{j_k}(0) = 0$. Thus, there exists $0 < \omega < \rho$ such that H_{j_k} is holomorphic in $\Delta_\omega = \{z : |z| < \omega\}$. Then H_{j_k} converges spherically locally uniformly to a holomorphic function $U(z)$ in $\Delta_\omega = \{z : |z| < \omega\}$. Since $H_{j_k}(0) = 0$, then $U(0) = 0$. Hence, there exists $0 < r < \rho$ such that $U(z)$ is holomorphic in $\Delta_r = \{z : |z| < r\}$ and has a unique zero $z = 0$ in Δ_r. Thus, H_{j_k} converges spherically locally uniformly to a holomorphic function $U(z)$ in Δ_r, then f_{j_k} converges spherically locally uniformly to a holomorphic function $z^s U(z)$ in Δ_r. Hence, $\mathcal{F}$ is normal at $z = 0$. From Case (i) and Case (ii), we see $\mathcal{F}$ is normal at 0.
Case 2. $a(0) \neq 0$.

Apply to Lemma 1 with $\alpha = \frac{\sum_{v=1}^{k} t_v}{n+\sum_{v=1}^{k} n_v}$, we have

$$h_j(\xi) = \frac{f_j(z_j + \rho_j \xi)}{\rho_j^\alpha} \to h(\xi)$$

spherically uniformly on compact subsets of $\mathbb{C}$, where $h(\xi)$ is a nonconstant meromorphic function. We have

$$\begin{aligned}
& h_j^n(\xi)(h_j^{n_1})^{(t_1)}(\xi)\ldots(h_j^{n_k})^{(t_k)}(\xi) \\
&\quad + \sum_I \rho_j^{(n_I+\sum_{v=1}^{k} n_{vI})\alpha - \sum_{v=1}^{k} t_{vI}} a_I(z_j+\rho_j\xi) h_j^{n_I}(\xi)(h_j^{n_{1I}})^{(t_{1I})}(\xi)\ldots \\
&\quad (h_j^{n_{kI}})^{(t_{kI})}(\xi) - a(z_j+\rho_j\xi) \\
&\quad = f_j^n(z_j+\rho_j\xi)(f_j^{n_1})^{(t_1)}(z_j+\rho_j\xi)\ldots(f_j^{n_k})^{(t_k)}(z_j+\rho_j\xi) \\
&\quad + \sum_I a_I(z_j+\rho_j\xi) f_j^{n_I}(z_j+\rho_j\xi)(f_j^{n_{1I}})^{(t_{1I})}(z_j+\rho_j\xi)\ldots \\
&\quad (f_j^{n_{kI}})^{(t_{kI})}(z_j+\rho_j\xi) - a(z_j+\rho_j\xi).
\end{aligned}$$

By the condition

$$\sum_{v=1}^{k} t_{vI} < \sum_{v=1}^{k} t_v,$$

we get

$$h^n(\xi)(h^{n_1}(\xi))^{(t_1)} \dots (h^{n_k}(\xi))^{(t_k)} - a(0)$$

is the uniform limit (with metric spherical) of

$$f_j^n(z_j+\rho_j\xi)(f_j^{n_1})^{(t_1)}(z_j+\rho_j\xi)\dots(f_j^{n_k})^{(t_k)}(z_j+\rho_j\xi) \\ + \sum_I a_I(z_j+\rho_j\xi) f_j^{n_I}(z_j+\rho_j\xi)(f_j^{n_{1I}})^{(t_{1I})}(z_j+\rho_j\xi)\dots(f_j^{n_{kI}})^{(t_{kI})}(z_j+\rho_j\xi) \\ - a(z_j+\rho_j\xi)$$

on each compact subset of $\mathbb{C}\backslash\{\text{pole of} \quad h\}$. By Lemma 4 to Lemma 6 and Lemma 2, the equation

$$h^n(\xi)(h^{n_1}(\xi))^{(t_1)} \dots (h^{n_k}(\xi))^{(t_k)} = a(0).$$

has at least two distinct zeros $\xi_1 \neq \xi_2$. By argument as case 1.1, we get a contradiction.

By an argument of Theorem 1, we are easy to prove Theorem 4. □

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