Bull. Malays. Math. Sci. Soc. (2019) 42:337–353
https://doi.org/10.1007/s40840-017-0489-5

BULLETIN of the MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

Concomitants of Ordered Variables from Huang–Kotz FGM Type Bivariate Generalized Exponential Distribution

H. M. Barakat[1] · **E. M. Nigm**[1] · **A. H. Syam**[2]

Received: 28 November 2016 / Revised: 13 March 2017 / Published online: 5 April 2017

Abstract We introduce the Huang–Kotz Morgenstern type bivariate generalized exponential distribution. Some distributional properties of concomitants of order statistics as well as record values for this family are studied. Recurrence relations between single and product moments of concomitants are obtained. Moreover, the rank and the asymptotic behavior of concomitants of order statistics are investigated.

Keywords Concomitants · Order statistics · Record values · Generalized exponential distribution · Huang–Kotz FGM family

Mathematical Subject Classification 62B10 · 62G30

1 Introduction

Huang and Kotz [24] considered a polynomial-type single parameter extension of the classical Farlie–Gumble–Morgenstern (FGM) family of distributions. The distribution function (*df*) which they suggested is

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)\left[1 + \lambda(1 - F_X^p(x))(1 - F_Y^p(y))\right], p \geq 1, \qquad (1.1)$$

denoted by HK–FGM(λ, p). The corresponding probability density function (pdf) is given by

✉ H. M. Barakat
hbarakat2@hotmail.com

1 Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt

2 Higher Technological Institute, 10th of Ramadan city, Egypt

$$f_{X,Y}(x,y)=f_X(x)f_Y(y)\left[1+\lambda((1+p)F_X^p(x)-1)((1+p)F_Y^p(y)-1)\right], \quad (1.2)$$

where $F_X(x)$ and $F_Y(y)$ are *df*'s, while $f_X(x)$ and $f_Y(y)$ are pdf's of the random variables (rv's) X and Y, respectively. The admissible range of the associated parameter λ is $-\max(1,p)^{-2}\leq\lambda\leq p^{-1}$, and since $p\geq 1$, this admissible becomes $-p^{-2}\leq\lambda\leq p^{-1}$. When the marginals are uniform then, while for the classical FGM the correlation between components does not exceed $\frac{1}{3}$, the modified version HK–FGM allows correlation up to 0.39. Actually, Huang and Kotz [24] modification of the FGM distribution paved the way for many research papers on modifications of FGM distributions allowing high correlation. Meanwhile, the simple analytical form of HK–FGM family aroused interest of many researchers, e.g., Amblard and Girard [6], Bairamov and Kotz [8], Fischer and Klein [19] and Mokhlis and Khames [25,26] among others.

The generalized exponential (GE) distribution is defined as a particular case of the Gompertz–Verhulst *df* $G(x)=(1-\rho\exp(-\theta x))^\alpha$, for $x>\frac{1}{\theta}\log\rho,\ \rho,\theta,\alpha>0$ (see, Gompertz [21] and Verhulst [32–34]), when $\rho=1$. For more detail on the Gompertz–Verhulst *df*, see Ahsanullah et al. [3], Ahuja [4] and Ahuja and Nash [5]. Therefore, X is a two-parameter generalized exponential rv if it has the df

$$F_X(x)=(1-\exp(-\theta x))^\alpha;\ x>0;\theta>0;\alpha>0,$$

denoted by GE$(\theta;\alpha)$. This distribution is a generalization of the exponential distribution and is more flexible, for being that, the hazard function of the exponential distribution is constant, but the hazard function of GE distribution can be constant, increasing or decreasing. Gupta and Kundu [22] showed that the kth moment of GE$(\theta;\alpha)$ is

$$\mu_k=\frac{\alpha k!}{\theta^k}\sum_{i=0}^{\aleph(\alpha-1)}\frac{(-1)^i}{(i+1)^{k+1}}\binom{\alpha-1}{i},$$

where $\aleph(x)=\infty$, if x is noninteger and $\aleph(x)=x$, if x is integer. Moreover, the mean, variance and moment generating function of GE$(\theta;\alpha)$ are given, respectively, by

$$\mu_1=\mathrm{E}(X)=\frac{B(\alpha)}{\theta},\quad \mathrm{Var}(X)=\frac{C(\alpha)}{\theta^2},\quad M_X(t)=\alpha\beta\left(\alpha,1-\frac{t}{\theta}\right), \quad (1.3)$$

where $B(\alpha)=\Psi(\alpha+1)-\Psi(1)$, $C(\alpha)=\Psi'(1)-\Psi'(\alpha+1)$, $\beta(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ and $\Psi(.)$ is the digamma function, while $\Psi'(.)$ is its derivation (the trigamma function). Recently, Tahmasebi and Jafari [29] studied some properties of the Morgenstern type bivariate generalized exponential distribution (denoted by MTBGED). Also, they studied some distributional properties of concomitants of order statistics as well as record values of this df. Moreover, they obtained some recurrence relations between moments of concomitants of order statistics.

In this paper, all the results of Tahmasebi and Jafari [29] are extended to HK–FGM family with two marginals F_X and F_Y, where $X \sim \mathrm{GE}(\theta_1; \alpha_1)$ and $Y \sim \mathrm{GE}(\theta_2; \alpha_2)$ (denoted by HK–FGM-GE$(\theta_1, \alpha_1; \theta_2, \alpha_2)$). Moreover, some new results, which were not obtained by Tahmasebi and Jafari [29] for FGM family, are given such as recurrence relations for the single, as well as the product, moments of bivariate concomitants of order statistics, the concomitant rank order statistics and the asymptotic behavior of the concomitants of order statistics. Finally, some essential corrections of Tahmasebi and Jafari [29] are made. Namely, both the variance and the correlation of concomitants of order statistics, as well as all the results of Sect. 4, concerning the concomitants of record values, are corrected.

It is worth mentioning that some of the results presented in this paper are related to paper of Beg and Ahsanullah [12]. Namely, Beg and Ahsanullah [12] considered concomitants of generalized order statistics (the generalized order statistics constitute a unified model for ordered random variables that includes order statistics and record values among others) for the FGM family and derived the joint distribution of concomitants of two generalized order statistics and obtain their product moments. Tahmasebi and Behboodian [27], Tahmasebi et al. [30], Tahmasebi and Jafari [28] and Tahmasebi et al. [31] are further recent relevant works on this subject.

2 The HK–FGM-GE and Some of its Properties

The joint df and pdf of (X, Y) are defined by (1.1) and (1.2), respectively, where $X \sim \mathrm{GE}(\theta_1; \alpha_1)$ and $Y \sim \mathrm{GE}(\theta_2; \alpha_2)$. Therefore, it is easy to show that the (n, m)th joint moments of HK–FGM-GE$(\theta_1, \alpha_1; \theta_2, \alpha_2)$ are given by

$$\mathrm{E}(X^n Y^m) = \mathrm{E}(X^n)E(Y^m) + \lambda(\mathrm{E}(U^n) - \mathrm{E}(X^n))(\mathrm{E}(V^m) - \mathrm{E}(Y^m)),\ n,\quad m = 1, 2, \ldots, \tag{2.1}$$

where $U \sim \mathrm{GE}(\theta_1; \alpha_1(p+1))$ and $V \sim \mathrm{GE}(\theta_2; \alpha_2(p+1))$. Thus, by combining (2.1) and (1.3), we get

$$E(XY) = \frac{B(\alpha_1)B(\alpha_2) + \lambda D(\alpha_1, p)D(\alpha_2, p)}{\theta_1\theta_2},$$

where $D(\alpha_i, p) = B(\alpha_i(1+p)) - B(\alpha_i), i = 1, 2$. Therefore, the coefficient of correlation between X and Y is

$$\rho_{X,Y} = \frac{\lambda D(\alpha_1, p)D(\alpha_2, p)}{\sqrt{C(\alpha_1)C(\alpha_2)}} = \lambda g(\alpha_1, \alpha_2, p).$$

Clearly, for any $p \geq 1$, the function $g(\alpha_1, \alpha_2, p)$ is increasing and positive function with respect to each of $\alpha_i, i = 1, 2$. Therefore, if $\lambda > 0$, then $\rho_{X,Y}$ is increasing and positive function, and if $\lambda < 0$, then $\rho_{X,Y}$ decreasing and negative function with respect to each of α_1 and α_2. Moreover, we can show that

$$\lim_{\substack{\alpha_1\to\infty\\ \alpha_2\to\infty}} g(\alpha_1,\alpha_2,p)=\frac{6(\log(p+1))^2}{\pi^2},\quad \lim_{\substack{\alpha_1\to 0^+\\ \alpha_2\to 0^+}} g(\alpha_1,\alpha_2,p)=0.$$

Therefore, $\underline{\rho}(p)=-\frac{6(\log(p+1))^2}{\pi^2p^2}\leq\rho_{X,Y}\leq\frac{6(\log(p+1))^2}{\pi^2p}=\overline{\rho}(p)$ (note that $-p^{-2}\leq\lambda\leq p^{-1}$), which yields $\rho_{X,Y}\to 0$, as $p\to\infty$ and $|\rho_{X,Y}|\leq 0.2921$, when $p=1$ (see Tahmasebi and Jafari [29]). However, we can show that the upper bound $\overline{\rho}(p)$ as a function of p is increasing function in the interval $(1, 3.9241)$ and is decreasing function in the interval $(3.9242,\infty)$. Therefore, we get $\max_{p\geq 1}\overline{\rho}(p)\approx\overline{\rho}(3.9241)=0.3937$. On the other hand, since $\frac{\log(1+p)}{p}$ is strictly decreasing function of p, then $\min_{p\geq 1}\underline{\rho}(p)=\underline{\rho}(1)$. Consequently, we get $\underline{\rho}(1)\leq\rho_{X,Y}\leq 0.3937$, which is a significant improvement comparing with the upper bound "0.2921" obtained by Tahmasebi and Jafari [29]. This fact gives a satisfactory motivation to deal with HK–FGM-GE rather than MTBGED. It is worth mentioning that the interval for p when $\overline{\rho}(p)$ is better than Tahmasebi and Jafari [29] is $(1, 18.1]$ ($\overline{\rho}(18.1)=0.2922302$), while the interval for p when $\overline{\rho}(p)$ is worse than Tahmasebi and Jafari [29] is $[18.2,\infty)$ ($\overline{\rho}(18.2)=0.291645$) and $\underline{\rho}(p)$ is not better than lower bound given by Tahmasebi and Jafari [29].

The conditional df of Y given $X=x$ is given by

$$F_{Y|X}(y|x)=F_Y(y)\left[1-\lambda(1-F_Y^p(y))((1+p)F_X^p(x)-1)\right]. \tag{2.2}$$

Therefore, the regression curve of Y given $X=x$ for HK–FGM-GE is

$$\begin{aligned}\mathrm{E}(Y|X=x)&=\mathrm{E}(Y)+\lambda((p+1)F_X^p(x)-1)(\mathrm{E}(V)-\mathrm{E}(Y))\\ &=\frac{1}{\theta_2}\left[B(\alpha_2)+\lambda D(\alpha_2,p)((p+1)(1-\mathrm{e}^{-\theta_1 x})^{\alpha_1 p}-1)\right],\end{aligned}$$

where $V\sim\mathrm{GE}(\theta_2;\alpha_2(p+1))$ and the conditional expectation is nonlinear with respect to x.

3 Concomitants of Order Statistics Based on HK–FGM-GE

The concept of concomitants of order statistics was first introduced by David [15] and almost simultaneously under the name of induced order statistics by Bhattacharya [13]. Suppose (X_i, Y_i), $i=1,2,\ldots,n$ is a random sample from a bivariate df $F_{X,Y}(x,y)$. If we order the sample by the $X-$variate and obtain the order statistics, $X_{1:n}\leq X_{1:n}\leq\cdots\leq X_{n:n}$, for the X sample, then the $Y-$variate associated with the rth order statistic $X_{r:n}$ is called the concomitant of the rth order statistic and is denoted by $Y_{[r:n]}$. Concomitants of order statistics can arise in several applications. In selection procedures, items or subjects may be chosen on the basis of their X characteristic, and an associated characteristic Y that is hard to measure or can be observed only later may be of interest. Another application of concomitants of order statistics is in ranked set sampling. It is a sampling scheme for situations where measurement of the variable of primary interest for sampled items is expensive or time-consuming while ranking of a set of items related to the variable of interest can be easily done. A comprehensive review of ranked set sampling can be found in Chen et al. [14]. Concomitants of order statistics have also been used in estimation and hypotheses testing problems. Another natural application of concomitants of order statistics is in dealing with the estimation of parameters for multivariate data sets that are subject to some form of type II censoring. For a recent comprehensive review of these applications, see David and Nagaraja [16] and Sects. 9.8 and 11.7 of David and Nagaraja [17].

3.1 Marginal Distribution of Concomitants of Order Statistics Based on HK–FGM-GE

Let $X \sim \mathrm{GE}(\theta_1; \alpha_1)$ and $Y \sim \mathrm{GE}(\theta_2; \alpha_2)$. Since the conditional pdf of $Y_{[r:n]}$ given $X_{[r:n]} = x$ is $f_{Y_{[r:n]}|X_{r:n}}(y|x) = f_{Y|X}(y|x)$ (cf. Galambos [20], see also Tahmasebi and Jafari [29]), then the pdf of $Y_{[r:n]}$ is given by

$$\begin{aligned} f_{[r:n]}(y) &= f_Y(y)\left[1 + (1 - (1+p)F_Y^p(y))\Delta_{r,n:p}\right] \\ &= (1 + \Delta_{r,n:p}) f_Y(y) - \Delta_{r,n:p} f_V(y), \;\; y > 0, \end{aligned} \tag{3.1}$$

where $V \sim \mathrm{GE}(\theta_2; \alpha_2(p+1))$ and

$$\Delta_{r,n:p} = \lambda\left(1 - \frac{(1+p)\beta(r+p, n-r+1)}{\beta(r, n-r+1)}\right).$$

Therefore, the moment generating function of $Y_{[r:n]}$ is given by

$$\begin{aligned} M_{[r:n]}(t) = \alpha_2 \Bigg[&(1 + \Delta_{r,n:p})\beta\left(\alpha_2, 1 - \frac{t}{\theta_2}\right) \\ &-(p+1)\Delta_{r,n:p}\beta\left(\alpha_2(p+1), 1 - \frac{t}{\theta_2}\right)\Bigg]. \end{aligned} \tag{3.2}$$

Thus, by using (3.1) (or by using (3.2)), the kth moment of $Y_{[r:n]}$ is given by

$$\begin{aligned} \mu_{[r:n]}^{(k)} &= \mathrm{E}[Y_{[r:n]}^k] = (1 + \Delta_{r,n:p})\mathrm{E}[Y^k] - \Delta_{r,n:p}\mathrm{E}[V^k] \\ &= (1 + \Delta_{r,n:p}) \sum_{i=0}^{\aleph(\alpha_2 - 1)} \frac{\alpha_2 k!(-1)^i}{\theta_2^k (i+1)^{k+1}} \binom{\alpha_2 - 1}{i} \\ &\quad -\Delta_{r,n:p} \sum_{i=0}^{\aleph(\alpha_2(p+1)-1)} \frac{\alpha_2(p+1)k!(-1)^i}{\theta_2^k (i+1)^{k+1}} \binom{\alpha_2(p+1)-1}{i}. \end{aligned}$$

Clearly, all the moments exist for integer values of α_2 and $\alpha_2 p$. Moreover, by putting $k = 1$ we get the mean of $Y_{[r:n]}$

$$\mu_{[r:n]} = \frac{1}{\theta_2}\left[B(\alpha_2) - \Delta_{r,n:p} D(\alpha_2, p)\right]. \tag{3.3}$$

Thus, the difference between the means of Y and $Y_{[r:n]}$ is $h(r, \lambda, \alpha_2, p) = -\frac{\Delta_{r,n:p} D(\alpha_2, p)}{\theta_2}$, which implies that $h(r, \lambda, \alpha_2, p) = 0$, if $\lambda = 0$ or $\frac{(p+1)\beta(r+p, n-r+1)}{\beta(r, n-r+1)} = 1$. Since $B(\alpha)$ is increasing function of α, then $D(\alpha, p) \geq 0$. Therefore, $h(r, \lambda, \alpha_2, p)$ has the same sign of $-\Delta_{r,n:p}$, which means that $h(r, \lambda, \alpha_2, p) > 0$, if and only if $\lambda > 0$, $(p+1)\beta(r+p, n-r+1) > \beta(r, n-r+1)$, or $\lambda < 0$, $(p+1)\beta(r+p, n-r+1) < \beta(r, n-r+1)$. Finally, by using (3.3) we get the following general recurrence relations:

Theorem 3.1 *For any* $1 \leq r \leq n-3$, *we get*

$$(r+1)\mu_{[r+2:n]} = (2r+p+1)\mu_{[r+1:n]} - (p+r)\mu_{[r:n]}. \tag{3.4}$$

Moreover, for all $n > 2$, *we get*

$$(n+p)\mu_{[r:n]} = (2n+p-1)\mu_{[r:n-1]} - (n-1)\mu_{[r:n-2]}. \tag{3.5}$$

Proof It is easy to check that

$$\Delta_{r+1,n:p} = \Delta_{r,n:p} - \frac{\lambda p(p+1)}{r} \frac{\beta(r+p, n-r+1)}{\beta(r, n-r+1)}$$

and

$$\Delta_{r+2,n:p} = \Delta_{r,n:p} - \frac{\lambda p(p+1)(2r+p+1)}{r(r+1)} \frac{\beta(r+p, n-r+1)}{\beta(r, n-r+1)},$$

which yield, after some algebra, the first recurrence relation (3.4). Also, we can check that

$$\Delta_{r,n-1:p} = \Delta_{r,n:p} - \frac{\lambda p(p+1)}{n} \frac{\beta(r+p, n-r+1)}{\beta(r, n-r+1)}$$

and

$$\Delta_{r,n-2:p} = \Delta_{r,n:p} - \frac{\lambda p(p+1)(2n+p-1)}{n(n-1)} \frac{\beta(r+p, n-r+1)}{\beta(r, n-r+1)}.$$

The second recurrence relation (3.5) is followed by combining the last two relations, after some algebra. □

Remark 3.1 When $p = 1$, we get the recurrence relation $\mu_{[r+2:n]} = 2\mu_{[r+1:n]} - \mu_{[r:n]}$ of $Y_{[r:n]}$ based on MTBGED, which is obtained by Tahmasebi and Jafari [29]. Moreover, it is worth mentioning that the second recurrence relation (3.5) is new even for MTBGED.

Remark 3.2 The recurrence relation (3.4) can provide us with an estimate of p. Namely, based on the relation $p = \frac{(r+1)\mu_{[r+2:n]}-(2r+1)\mu_{[r+1:n]}+r\mu_{[r:n]}}{\mu_{[r+1:n]}-\mu_{[r:n]}}$, we can suggest the estimator

$$\hat{p} = \frac{1}{n-2}\sum_{i=1}^{n-3} \frac{(i+1)Y_{[i+2:n]} - (2i+1)Y_{[i+1:n]} + iY_{[i:n]}}{Y_{[i+1:n]} - Y_{[i:n]}}.$$

Actually, the suggested estimator $\hat{p}$ does not consistent or even unbiased, but bearing in mind that there is no any known estimator of the power parameter p in the literature, we can use it; nevertheless, it needs further theoretical and practical investigation.

By multiplying the both sides of (3.1) by $(y - \mu_{[r:n]})^2$ and integrating, we obtain the variance of $Y_{[r:n]}$ as

$$\sigma^2_{[r:n]} = \frac{1}{\theta_2^2}\Big[C(\alpha_2) + \Delta_{r,n:p}(C(\alpha_2) - C(\alpha_2(p+1))) \\ -\Delta_{r,n:p}(1+\Delta_{r,n:p})D^2(\alpha_2, p)\Big]. \tag{3.6}$$

Clearly, when $p = 1$, we get

$$\sigma^2_{[r:n]} = \frac{1}{\theta_2^2}\left[C(\alpha_2) + \delta_r(C(\alpha_2) - C(2\alpha_2)) - \delta_r(1+\delta_r)D^2(\alpha_2)\right], \tag{3.7}$$

where $\delta_r = \frac{\lambda(n-2r+1)}{n+1}$ and $D(\alpha_2) = B(2\alpha_2) - B(\alpha_2)$. The formula (3.7) is the correction of the formula

$$\sigma^2_{[r:n]} = \frac{1}{\theta_2^2}\left[C(\alpha_2) + \delta_r(C(2\alpha_2) - C(\alpha_2))\right],$$

which is obtained by Tahmasebi and Jafari [29].

It is well known that in many cases, the concomitants of the extremes among the X's are not extremes among the Y's (with high probability) (cf. Galambos [20]). This fact aroused interest of some researchers to investigate the rank (of $Y_{[r:n]}$) $\mathcal{R}_{[r:n]} = \sum_{j=1}^{n} \mathrm{I}(Y_{[r:n]} - Y_j)$, where $\mathrm{I}(x) = 1$, if $x \geq 0$, $\mathrm{I}(x) = 0$, if $x < 0$. The distribution of $R_{r:n}$ is obtained by David et al. [18]. Barakat and El-Shandidy [9] gave a new representation of the *df* and the expected value of $\mathcal{R}_{[r:n]}$. Namely, for all $r, s = 2, 3, \ldots, n-1$, we have

$$A_{r:n}(s) = P(\mathcal{R}_{[r:n]} = s) = n[\mathrm{E}(\mathcal{C}(W_{r:n-1}, Z_{s:n-1})) - \mathrm{E}(\mathcal{C}(W_{r-1:n-1}, Z_{s:n-1})) \\ -\mathrm{E}(\mathcal{C}(W_{r:n-1}, Z_{s-1:n-1})) + \mathrm{E}(\mathcal{C}(W_{r-1:n-1}, Z_{s-1:n-1}))], \tag{3.8}$$

where $\mathcal{C}(.,.)$ is the copula of the bivariate df $F_{X,Y}(x, y)$, i.e., $\mathcal{C}(w, z) = wz(1+\lambda(1-w^p)(1-z^p))$. Moreover, $W_{j:n} = F_X(X_{j:n})$ and $Z_{j:n} = F_Y(Y_{j:n})$ are the jth uniform order statistics with expectation $\mathrm{E}(W_{j:n}) = \mathrm{E}(Z_{j:n}) = \frac{j}{n+1}$. The representation (3.8) enables us to use the $\delta-$method (with one-step Taylor approximation) to compute an approximate formula for the df $A_{r:n}(s)$, by

$$A_{r:n}(s) \sim n\left[\mathcal{C}\left(\frac{r}{n},\frac{s}{n}\right)-\mathcal{C}\left(\frac{r-1}{n},\frac{s}{n}\right)-\mathcal{C}\left(\frac{r}{n},\frac{s-1}{n}\right)+\mathcal{C}\left(\frac{r-1}{n},\frac{s-1}{n}\right)\right]$$
$$=\frac{1+\lambda}{n}$$
$$-\frac{\lambda}{n^{p+1}}[rs(r^p+s^p)-(r-1)s((r-1)^p+s^p)-r(s-1)(r^p+(s-1)^p)$$
$$+(r-1)(s-1)((r-1)^p+(s-1)^p)]$$
$$+\frac{\lambda}{n^{2p+1}}[r^{p+1}s^{p+1}-(r-1)^{p+1}s^{p+1}$$
$$-r^{p+1}(s-1)^{p+1}+(r-1)^{p+1}(s-1)^{p+1}].$$

The limiting distribution of $Y_{[n:n]}$, as $n\to\infty$, depends on the conditional distribution of Y given X and the marginal distribution of X, and it is given by the following theorem.

Theorem 3.2 *Let* $A_n=\frac{1}{\theta_2}$. *Then*

$$F_{[n:n]}(A_n y)\xrightarrow[n]{w}F_Y(y)\left(1-\lambda p\left(1-F_Y^p(y)\right)\right),$$

where "$\xrightarrow[n]{w}$" *denotes the weak convergence, as* $n\to\infty$ *(for the definition of the weak convergence, see Galambos* [20]*) and* $Y\sim \mathrm{GE}(\theta_2;\alpha_2)$.

Proof First, by applying Theorem 2.1, Part II, in Barakat et al. [10], by putting $b=1, a=n$, we get

$$P(X_{n:n}\le a_n x+b_n)=F_X^n(a_n x+b_n)\xrightarrow[n]{w}\mathrm{e}^{-\mathrm{e}^{-x}},\forall x, \tag{3.9}$$

where $X\sim \mathrm{GE}(\theta_1;\alpha_1)$, $a_n=\frac{1}{\theta_1}$, $b_n=-\log[\alpha_1 n]$ and $[x]$ means the integer part of x. On the other hand, in view of (2.2), we get

$$F_{Y|X}(A_n y|X=a_n x+b_n)\xrightarrow[n]{w}=T(x,y)=F_Y(y)(1-\lambda p(1-F_Y^p(y))). \tag{3.10}$$

Finally, we can easily check that the df $F_X(x)$ satisfies the von Mises condition: Namely,

$$\lim_{x\to\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1-F_X(x)}{f_X(x)}\right]=-1-\lim_{x\to\infty}\frac{1-(1-\mathrm{e}^{-\theta_1 x})^{\alpha_1}}{\alpha_1\mathrm{e}^{-\theta_1 x}}=0. \tag{3.11}$$

Therefore, in view of Theorem 5.5.1, in Galambos [20], (3.9), (3.10) and (3.11) are sufficient conditions for the relation

$$F_{[n:n]}(A_n y) \xrightarrow[n]{w} \int_{-\infty}^{\infty} T(y,x)\mathrm{e}^{-\mathrm{e}^{-x}}\mathrm{d}x = F_Y(y)(1-\lambda p(1-F_Y^p(y))).$$

This completes the proof. □

3.2 Joint Distribution of Concomitants of Order Statistics Based on HK–FGM-GE

The joint pdf of concomitants $Y_{[r:n]}$ and $Y_{[s:n]}$, $r < s$, is (cf. Tahmasebi and Jafari [29])

$$f_{[r,s:n]}(y_1,y_2) = \int_0^{\infty}\int_0^{x_2} f_{Y|X}(y_1|x_1) f_{Y|X}(y_2|x_2) f_{r,s:n}(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2,$$

where $\beta(a,b,c) = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}$ and

$$\begin{aligned} f_{r,s:n}(x_1,x_2) = &\frac{1}{\beta(r,s-r,n-s+1)} F_X^{r-1}(x_1) \\ &\times (F_X(x_2)-F_X(x_1))^{s-r-1}(1-F_X(x_2))^{n-s} f_X(x_1) f_X(x_2),\ x_1 < x_2. \end{aligned}$$

Therefore,

$$\begin{aligned} f_{[r,s:n]}(y_1,y_2) = &\int_0^{\infty}\int_0^{x_2}\left[f_Y(y_1)(1+\lambda((1+p)F_X^p(x_1)-1)((1+p)F_Y^p(y_1)-1))\right] \\ &\times\left[f_Y(y_2)(1+\lambda((1+p)F_X^p(x_2)-1)((1+p)F_Y^p(y_2)-1))\right] \\ &\times\left[\frac{F_X^{r-1}(x_1)(F_X(x_2)-F_X(x_1))^{s-r-1}(1-F_X(x_2))^{n-s}}{\beta(r,s-r,n-s+1)} f_X(x_1) f_X(x_2)\right]\mathrm{d}x_1\mathrm{d}x_2. \end{aligned} \tag{3.12}$$

On the other hand, we have

$$\begin{aligned} I_1 &= \lambda\int_0^{\infty}\int_0^{x_2}((1+p)F_X^p(x_1)-1) \\ &\quad\times\left[\frac{F_X^{r-1}(x_1)(F_X(x_2)-F_X(x_1))^{s-r-1}(1-F_X(x_2))^{n-s}}{\beta(r,s-r,n-s+1)}\frac{f_X(x_1)f_X(x_2)}{1-F_X(x_1)}\right]\mathrm{d}x_1\mathrm{d}x_2 \\ &= \frac{\lambda(1+p)}{\beta(r,s-r,n-s+1)}\int_0^1\int_0^v\left[u^{p+r-1}(v-u)^{s-r-1}(1-v)^{n-s}\right]\mathrm{d}u\mathrm{d}v - \lambda \\ &= -\lambda\frac{\beta(r,s-r,n-s+1)-(p+1)\beta(r+p,s-r,n-s+1)}{\beta(r,s-r,n-s+1)} = -\Delta_{r,s,n:p}^{(1)} \end{aligned} \tag{3.13}$$

(upon substituting $u = F_X(x_1)$ and $v = F_X(x_2)$). Moreover, we have

$$I_2 = \lambda \int_0^\infty \int_0^{x_2} ((1+p)F_X^p(x_2) - 1) \times \left[\frac{F_X^{r-1}(x_1)(F_X(x_2) - F_X(x_1))^{s-r-1}(1 - F_X(x_2))^{n-s}}{\beta(r, s-r, n-s+1)} f_X(x_1) f_X(x_2) \right] \mathrm{d}x_1 \mathrm{d}x_2.$$

Upon substituting $u = F_X(x_1)$ and $v = F_X(x_2)$, we get

$$\begin{aligned} I_2 &= \lambda \int_0^1 \int_0^v \big((1+p)v^p - 1\big) \left[\frac{u^{r-1}(v-u)^{s-r-1}(1-v)^{n-s}}{\beta(r, s-r, n-s+1)} \right] \mathrm{d}u\mathrm{d}v \\ &= \frac{\lambda(1+p)}{\beta(r, s-r, n-s+1)} \int_0^1 \int_0^v v^p u^{r-1}(v-u)^{s-r-1}(1-v)^{n-s} \mathrm{d}u\mathrm{d}v - \lambda. \end{aligned}$$

Moreover, upon substituting $\frac{u}{v} = w$, we get

$$\begin{aligned} I_2 &= \frac{\lambda(1+p)}{\beta(r, s-r, n-s+1)} \int_0^1 \int_0^1 v^{s+p-1}(1-v)^{n-s} w^{r-1}(1-w)^{s-r-1} \mathrm{d}w\mathrm{d}v - \lambda \\ &= -\lambda \frac{\beta(r, s-r, n-s+1) - (p+1)\beta(s+p, n-s+1)\beta(r, s-r)}{\beta(r, s-r, n-s+1)} \\ &= -\Delta^{(2)}_{r,s,n:p}. \end{aligned} \tag{3.14}$$

Finally, consider

$$\begin{aligned} I_3 &= \lambda^2 \int_0^\infty \int_0^{x_2} ((1+p)F_X^p(x_1) - 1)((1+p)F_X^p(x_2) - 1) \\ &\quad \times \left[\frac{F_X^{r-1}(x_1)(F_X(x_2) - F_X(x_1))^{s-r-1}(1 - F_X(x_2))^{n-s}}{\beta(r, s-r, n-s+1)} f_X(x_1) f_X(x_2) \right] \mathrm{d}x_1 \mathrm{d}x_2 \\ &= \lambda(I_3' - I_1 - I_2) = \Delta_{r,s,n:p}, \end{aligned} \tag{3.15}$$

where

$$I_3' = \lambda(1+p)^2 \int_0^\infty \int_0^{x_2} F_X^p(x_1) F_X^p(x_2) \left[\frac{F_X^{r-1}(x_1)(F_X(x_2) - F_X(x_1))^{s-r-1}(1 - F_X(x_2))^{n-s}}{\beta(r, s-r, n-s+1)} f_X(x_1) f_X(x_2) \right] \mathrm{d}x_1 \mathrm{d}x_2. \tag{3.16}$$

Put $u = F_X(x_1)$ and $v = F_X(x_2)$ in the double integration (3.16) and then put $\frac{u}{v} = t$, we get (by using (3.13) and (3.14))

$$
\begin{aligned}
I_3' &= \lambda(1+p)^2 \int_0^1 \int_0^1 v^{s+2p-1}(1-v)^{n-s} t^{r+p-1}(1-t)^{s-r-1} \mathrm{d}t\mathrm{d}v - \lambda. \\
&= \lambda \frac{(p+1)^2 \beta(s+2p, n-s+1)\beta(r+p, s-r) - \beta(r, s-r, n-s+1)}{\beta(r, s-r, n-s+1)} \\
&= \Delta^{(3)}_{r,s,n:p}.
\end{aligned}
$$

Thus,

$$
\Delta_{r,s,n:p} = \lambda \left(\Delta^{(3)}_{r,s,n:p} - \Delta^{(1)}_{r,s,n:p} - \Delta^{(2)}_{r,s,n:p} \right). \tag{3.17}
$$

Now, combining (3.12)–(3.15), with (3.17), we get

$$
\begin{aligned}
f_{[r,s:n]}(y_1, y_2) &= \left(1 + \Delta^{(1)}_{r,s,n:p} + \Delta^{(2)}_{r,s,n:p} + \Delta_{r,s,n:p}\right) f_Y(y_1) f_Y(y_2) \\
&- \left(\Delta^{(1)}_{r,s,n:p} + \Delta_{r,s,n:p}\right) f_V(y_1) f_Y(y_2) - \left(\Delta^{(2)}_{r,s,n:p} + \Delta_{r,s,n:p}\right) f_V(y_2) f_Y(y_1) \\
&+ \Delta_{r,s,n:p} f_V(y_1) f_V(y_2),
\end{aligned} \tag{3.18}
$$

where

$$
\begin{aligned}
\Delta_{r,s,n:p} &= \lambda \left(\Delta^{(3)}_{r,s,n:p} - \Delta^{(1)}_{r,s,n:p} - \Delta^{(2)}_{r,s,n:p} \right), \\
\Delta^{(1)}_{r,s,n:p} &= \lambda \frac{\beta(r, s-r, n-s+1) - (p+1)\beta(r+p, s-r, n-s+1)}{\beta(r, s-r, n-s+1)}, \\
\Delta^{(2)}_{r,s,n:p} &= \lambda \frac{\beta(r, s-r, n-s+1) - (p+1)\beta(s+p, n-s+1)\beta(r, s-r)}{\beta(r, s-r, n-s+1)}, \\
\Delta^{(3)}_{r,s,n:p} &= \lambda \frac{(p+1)^2 \beta(s+2p, n-s+1)\beta(r+p, s-r) - \beta(r, s-r, n-s+1)}{\beta(r, s-r, n-s+1)}.
\end{aligned} \tag{3.19}
$$

The product moment $\mathrm{E}[Y_{[r:n]} Y_{[s:n]}] = \mu_{[r,s:n]}$ is obtained directly from (3.18) as

$$
\begin{aligned}
\mu_{[r,s:n]} = \frac{1}{\theta_2^2} \Big[&\left(1 + \Delta^{(1)}_{r,s,n:p} + \Delta^{(2)}_{r,s,n:p} + \Delta_{r,s,n:p}\right) B^2(\alpha_2) \\
&- \left(\Delta^{(1)}_{r,s,n:p} + \Delta^{(2)}_{r,s,n:p} + 2\Delta_{r,s,n:p}\right) B(\alpha_2) B(\alpha_2(p+1)) \\
&+ \Delta_{r,s,n:p} B^2(\alpha_2(p+1)) \Big].
\end{aligned} \tag{3.20}
$$

Therefore, by using (3.3) and (3.20) we can after some algebra calculate the covariance between $Y_{[r:n]}$ and $Y_{[s:n]}$ as

$$\sigma_{[r,s:n,p]} = \Delta_{r,s,n:p} D^2(\alpha_2, p) - (\Delta^{(1)}_{r,s,n:p} + \Delta^{(2)}_{r,s,n:p}) B(\alpha_2) D(\alpha_2, p) + (\Delta_{r,n:p} + \Delta_{s,n:p}) B(\alpha_2) D(\alpha_2, p) - \Delta_{r,n:p}\Delta_{s,n:p} D^2(\alpha_2, p). \tag{3.21}$$

It is easily to verify that

$$\sigma_{[r,s:n,1]} = \frac{1}{\theta_2^2} D^2(\alpha_2)(\delta_{r,s} - \delta_r \delta_s), \tag{3.22}$$

where $\delta_{r,s} = \lambda^2 \left[\frac{n-2s+1}{n+1} - \frac{2r(n-2s)}{(n+1)(n+2)}\right]$ (we can easily verify that $\Delta^{(1)}_{r,s,n:1} = \delta_r$, $\Delta^{(2)}_{r,s,n:1} = \delta_s$ and $\Delta_{r,s,n:1} = \delta_{r,s}$). The relation (3.22) is obtained by Tahmasebi and Jafari [29] for MTBGED.

We can now use (3.21) and (3.6) to obtain the coefficient of correlation between $Y_{[r:n]}$ and $Y_{[s:n]}$ as

$$\rho_{[r,s:n,p]} = \frac{\Delta_{r,s,n:p} D^2 - (\Delta^{(1)}_{r,s,n:p} + \Delta^{(2)}_{r,s,n:p}) BD + (\Delta_{r,n:p} + \Delta_{s,n:p}) BD - \Delta_{r,n:p}\Delta_{s,n:p} D^2}{\sqrt{\prod_{i=1}^{2} [C(\alpha_2) + \Delta_{i,n:p}(C(\alpha_2) - C(\alpha_2(p+1))) - \Delta_{i,n:p}(1+\Delta_{i,n:p}) D^2]}}, \tag{3.23}$$

where in formula (3.23) we abbreviated $B(\alpha_2)$ and $D(\alpha_2, p)$ by B and D, respectively. Moreover, we use the abbreviations $\Delta_{1,n:p} = \Delta_{r,n:p}$ and $\Delta_{2,n:p} = \Delta_{s,n:p}$. It is easily to verify that

$$\rho_{[r,s:n,1]} = \frac{D^2(\alpha_2)(\delta_{r,s} - \delta_r \delta_s)}{\sqrt{\prod_{i=1}^{2} [C(\alpha_2) + \delta_i (C(\alpha_2) - C(2\alpha_2)) - \delta_i(1+\delta_i) D^2(\alpha_2)]}},$$

which is the correction formula of the formula (3.20) obtained by Tahmasebi and Jafari [29] for MTBGED, where $\delta_1 = \delta_r$ and $\delta_2 = \delta_s$.

Now, by using (3.19) and the representation (3.18), we get the following general recurrence relations for the product moment $\mu_{[r,s:n]}$.

Theorem 3.3 *For any* $1 \le r \le n-3$, *we get*

$$(r+1)\mu_{[r+2,s:n]} = (2r+p+1)\mu_{[r+1,s:n]} - (p+r)\mu_{[r,s:n]}. \tag{3.24}$$

Moreover, $1 \le s \le n-3$, *we get*

$$(s+1)\mu_{[r,s+2:n]} = (2s+p+1)\mu_{[r,s+1:n]} - (p+s)\mu_{[r,s:n]} + \xi_n(r, s, \alpha_2, \lambda : p). \tag{3.25}$$

$\xi_n(s, n, \alpha_2, \lambda : p) = \frac{\lambda p(1-p)}{\theta_2^2(s+p+1)} D^2(\alpha_2, p)\left(\Delta^{(3)}_{r,s+1,n:p} - \Delta^{(3)}_{r,s,n:p}\right)$. *Finally, for all* $n > 2$, *we get*

$$(n + p)\mu_{[r,s:n]} = (2n + p - 1)\mu_{[r,s:n-1]} - (n - 1)\mu_{[r,s:n-2]} + \zeta_n(r, s, \alpha_2, \lambda : p), \tag{3.26}$$

where $\zeta_n(s, n, \alpha_2, \lambda : p) = \frac{\lambda p}{\theta_2^2} D^2(\alpha_2, p)\left(\Delta^{(3)}_{r,s,n-1:p} - \Delta^{(3)}_{r,s,n:p}\right)$.

Proof It is easy to check that

$$\Delta^{(i)}_{r+2,s,n:p} - \Delta^{(i)}_{r,s,n:p} = \left(\Delta^{(i)}_{r+1,s,n:p} - \Delta^{(i)}_{r,s,n:p}\right) \frac{2r + p + 1}{r + 1}, i = 1, 3, \tag{3.27}$$

and

$$\Delta^{(2)}_{r,s,n:p} = \Delta^{(2)}_{r+1,s,n:p} = \Delta^{(2)}_{r+2,s,n:p}. \tag{3.28}$$

Therefore,

$$\Delta_{r+2,s,n:p} - \Delta_{r,s,n:p} = (\Delta_{r+1,s,n:p} - \Delta_{r,s,n:p})\frac{2r + p + 1}{r + 1}. \tag{3.29}$$

The recurrence relation (3.24) is now followed by combining (3.27), (3.28) and (3.29) with (3.20). Now, we turn to prove (3.25). First, we notice that

$$\Delta^{(1)}_{r,s,n:p} = \Delta^{(1)}_{r,s+1,n:p} = \Delta^{(1)}_{r,s+2,n:p}. \tag{3.30}$$

Moreover, it is easy to check that

$$\Delta^{(i)}_{r,s+2,n:p} - \Delta^{(i)}_{r,s,n:p} = \left(\Delta^{(i)}_{r,s+1,n:p} - \Delta^{(i)}_{r,s,n:p}\right) \frac{2s + p + 1}{s + 1} + \phi_i, i = 2, 3, \tag{3.31}$$

where $\phi_2 = 0$ and $\phi_3 = -\frac{p(1-p)}{(s+1)(s+p+1)}$. Therefore,

$$\begin{aligned}\Delta_{r,s+2,n:p} - \Delta_{r,s,n:p} = (\Delta_{r,s+1,n:p} - \Delta_{r,s,n:p})\frac{2s + p + 1}{s + 1} \\ + \lambda\phi_3(\Delta^{(3)}_{r,s+1,n:p} - \Delta^{(3)}_{r,s,n:p}).\end{aligned} \tag{3.32}$$

The recurrence relation (3.25) is now followed by combining (3.30), (3.31) and (3.32) with (3.20). In order to prove the recurrence relation (3.26), we first notice that

$$\Delta^{(i)}_{r,s,n-2:p} - \Delta^{(i)}_{r,s,n:p} = \left(\Delta^{(i)}_{r,s,n-1:p} - \Delta^{(i)}_{r,s,n:p}\right) \frac{2n + p - 1}{n - 1}, i = 1, 2, \tag{3.33}$$

and

$$\Delta^{(3)}_{r,s,n-2:p} - \Delta^{(3)}_{r,s,n:p} = \left(\Delta^{(3)}_{r,s,n-1:p} - \Delta^{(3)}_{r,s,n:p}\right)\frac{2n+2p-1}{n-1}. \tag{3.34}$$

Therefore,

$$\Delta_{r,s,n-2:p} - \Delta_{r,s,n:p} = (\Delta_{r,s,n-1:p} - \Delta_{r,s,n:p})\frac{2n+p-1}{n-1} + \frac{\lambda p}{n-1}\left(\Delta^{(3)}_{r,s,n-1:p} - \Delta^{(3)}_{r,s,n:p}\right). \tag{3.35}$$

The recurrence relation (3.26) is now followed by combining (3.33), (3.34) and (3.35) with (3.20). The theorem is established. □

4 Concomitants of Record Values Based on HK–FGM-GE

Let $\{(X_i, Y_i)\}, i = 1, 2, \ldots$ be a random sample from HK–FGM-GE$(\theta_1, \alpha_1; \theta_2, \alpha_2)$. When the experimenter interests in studying just the sequence of records of the first component X_i's, the second component associated with the record value of the first one is termed as the concomitant of that record value. The concomitants of record values arise in a wide variety of practical experiments, e.g., see Bdair and Raqab [11] and Arnold et al. [7]. Some properties from concomitants of record values were discussed in Ahsanullah [1] and Ahsanullah and Shakil [2]. Let $\{R_n, n \geq 1\}$ be the sequence of record values in the sequence of X's, while $R_{[n]}$ be the corresponding concomitant. Houchens [23] has obtained the pdf of concomitant of nth record value for $n \geq 1$, as $h_{[n]}(y) = \int_0^\infty f_Y(y|x) g_n(x) \mathrm{d}x$, where $g_n(x) = \frac{1}{\Gamma(n)}(-\log(1 - F_X(x)))^{n-1} f_X(x)$ is the pdf of R_n. Therefore, after some algebra, we get

$$h_{[n]}(y) = (1 + \Upsilon_{n:p}) f_Y(y) - \Upsilon_{n:p} f_V(y), \tag{4.1}$$

where $V \sim \mathrm{GE}(\theta_2; \alpha_2(p+1))$ and

$$\Upsilon_{n:p} = \lambda\left[1 - (1+p)\sum_{i=0}^{\aleph(p)} \frac{(-1)^i \binom{p}{i}}{(i+1)^n}\right].$$

Clearly, $\Upsilon_{n:1} = \lambda(2^{-(n-1)} - 1) = \lambda_{n-1}$ and the representation (4.1) becomes $h_{[n]}(y) = (1 + \lambda_{n-1}) f_Y(y) - \lambda_{n-1} f_V(y)$, which is the essential correction of the representation (4.1) due to Tahmasebi and Jafari [29] for MTBGED, which is given by $h_{[n]}(y) = (1 + \lambda_n) f_Y(y) - 2\lambda_n f_V(y)$.

The representation (4.1) enables us to derive the mean and the variance of $R_{[n]}$ as

$$\mu_{[R_n]:p} = \frac{1}{\theta_2}\left[B(\alpha_2) - \Upsilon_{n:p} D(\alpha_2, p)\right]$$

and

$$\sigma^2_{[R_n]:p} = \frac{1}{\theta_2^2}\left[C(\alpha_2) + \Upsilon_{n:p}(C(\alpha_2) - C(\alpha_2(p+1))) - \Upsilon_{n:p}(1 + \Upsilon_{n:p})D^2(\alpha_2, p)\right]. \tag{4.2}$$

Clearly,

$$\mu_{[R_n]:1} = \frac{1}{\theta_2}\left[B(\alpha_2) - \lambda_{n-1}D(\alpha_2)\right]$$

and

$$\sigma^2_{[R_n]:1} = \frac{1}{\theta_2^2}\left[C(\alpha_2) + \lambda_{n-1}(C(\alpha_2) - C(2\alpha_2)) - \lambda_{n-1}(1 + \lambda_{n-1})D^2(\alpha_2)\right],$$

which are the correction formulas of the mean and the variance, respectively, of $R_{[n]}$ given by Tahmasebi and Jafari [29] for MTBGED.

The joint pdf of the concomitants $R_{[n]}$ and $R_{[m]}$, $n < m$, is given by

$$h_{[n,m]}(y_1, y_2) = \int_0^\infty \int_{x_1}^\infty f_{Y|X}(y_1|x_1) f_{Y|X}(y_2|x_2) g_{m,n}(x_1, x_2) \mathrm{d}x_2 \mathrm{d}x_1,$$

where

$$g_{m,n}(x) = \frac{1}{\Gamma(n)\Gamma(m-n)}(-\log(1 - F_X(x_1)))^{n-1}\left(-\log\frac{1 - F_X(x_2)}{1 - F_X(x_1)}\right)^{m-n-1} \frac{f_X(x_1)f_X(x_1)}{1 - F_X(x_1)}$$

is the joint pdf of R_n and R_m. Therefore, after some algebra, we get

$$\begin{aligned} h_{[n,m]}(y_1, y_2) = {} & (1 + \Upsilon_{m:p} + \Upsilon_{m:p} + \Upsilon_{n,m:p}) f_Y(y_1) f_Y(y_2) \\ & -(\Upsilon_{n:p} + \Upsilon_{n,m:p}) f_V(y_1) f_Y(y_2) - (\Upsilon_{m:p} + \Upsilon_{n,m:p}) f_V(y_2) f_Y(y_1) \\ & +\Upsilon_{n,m:p} f_V(y_1) f_V(y_2), \end{aligned} \tag{4.3}$$

where $\Upsilon_{n,m:p} = \lambda(\Upsilon_{n:p} + \Upsilon_{m:p} + \Upsilon^\star_{n:p})$ and

$$\Upsilon^\star_{n,m:p} = \lambda\left[(1+p)^2 \sum_{i=0}^{\aleph(p)} \sum_{j=0}^{\aleph(p)} \frac{(-1)^{i+j}\binom{p}{i}\binom{p}{j}}{(i+j+1)^n (j+1)^{m-n}} - 1\right].$$

Clearly, $\Upsilon_{n:1} = \lambda^2\left[3^{-n}(2^{n-m+2} - 3^n) - \frac{\lambda_{n-1}+\lambda_{m-1}}{\lambda}\right]$, which is the correction of the term $\lambda_{n,m}$ by Tahmasebi and Jafari [29] to compute the joint pdf $h_{[n,m]}(y_1, y_2)$.

The representation (4.3) enables us to derive the product moment and the covariance of $R_{[n]}$ and $R_{[m]}$ as

$$\mu_{[R_n,R_m]:p} = \frac{1}{\theta_2^2}[(1+\Upsilon_{n:p}+\Upsilon_{m:p}+\Upsilon_{n,m:p})B^2(\alpha_2) \\ -(\Upsilon_{n:p}+\Upsilon_{m:p}+2\Upsilon_{n,m:p})B(\alpha_2)B(\alpha_2(p+1)) \\ +\Upsilon_{n,m:p}B^2(\alpha_2(p+1))]$$

and

$$\sigma_{[R_n,R_m]:p} = \frac{D^2(\alpha_2,p)}{\theta_2^2}\left[\Upsilon_{n,m:p}-\Upsilon_{n:p}\Upsilon_{m:p}\right]. \tag{4.4}$$

Clearly, $\sigma_{[R_n,R_m]:1} = \frac{D^2(\alpha_2)}{\theta_2^2}\left[\Upsilon_{n,m:1}-\lambda_{n-1}\lambda_{m-1}\right]$, which again is the correction of wrong relation (4.6) given by Tahmasebi and Jafari [29] to compute the covariance of the concomitants $R_{[n]}$ and $R_{[m]}$, $n < m$. Finally, combining (4.2) with (4.4), we get the correlation coefficient of the concomitants $R_{[n]}$ and $R_{[m]}$, as

$$\rho_{[R_n,R_m]:p} = \frac{D^2(\alpha_2,p)\left[\Upsilon_{n,m:p}-\Upsilon_{n:p}\Upsilon_{m:p}\right]}{\sqrt{\prod_{i=1}^{2}\left[C(\alpha_2)+\Upsilon_{i:p}(C(\alpha_2)-C(\alpha_2(p+1)))-\Upsilon_{i:p}(1+\Upsilon_{i:p})D^2(\alpha_2,p)\right]}},$$

where in the above formula we use the abbreviation $\Upsilon_{1:p}=\Upsilon_{n:p}$ and $\Upsilon_{2:p}=\Upsilon_{m:p}$. Again, $\rho_{[r,s:n,1]}$ gives the correction formula of the correlation given by Tahmasebi and Jafari [29] for MTBGED.

Acknowledgements The authors are grateful to the Editor in Chief, Professor Rosihan M. Ali, and the anonymous referees for suggestions and comments that improved the presentation substantially.

References

1. Ahsanullah, M.: Records and concomitants. Bull. Malays. Math. Sci. Soc. **32**(2), 101–117 (2009)
2. Ahsanullah, M., Shakil, M.: Characterizations of Rayleigh distribution based on order statistics and record values. Bull. Malays. Math. Sci. Soc. **36**(3), 625–635 (2013)
3. Ahsanullah, M., Shakil, M., Kibria, G.M.: A note on a characterization of Gompeertz–Verhulst distribution. J. Stat. Theory Appl. **13**(1), 17–26 (2013)
4. Ahuja, J.C.: On certain properties of the generalized Gompertz distribution. Indian J. Stat. Ser. B **31**, 541–544 (1969)
5. Ahuja, J.C., Nash, S.W.: The generalized Gompertz–Verhulst family of distributions. Sankhya Ser. A **29**, 141–156 (1967)
6. Amblard, C., Girard, S.: Symmetry and dependence properties within a semiparametric family of bivariate copulas. J. Nonparametric Stat. **14**(6), 715–727 (2002)
7. Arnold, B.C., Balakrishnan, N., Nagaraja, H.N.: Records, Wiley Series in Probability and Statistics: Probability and Statistics. Wiley, New York (1998)
8. Bairamov, I., Kotz, S.: Dependence structure and symmetry of Huang–Kotz FGM distributions and their extensions. Metrika **56**(1), 55–72 (2002)

9. Barakat, H.M., El-Shandidy, M.A.: Computing the distribution and expected value of the concomitant rank order statistics. Commun. Stat. Theory Meth. **33**(11), 2575–2594 (2004)
10. Barakat, H.M., Nigm, E.M., Harpy, M.H.: Limit theorems of order statistics and record values from the gamma and Kumaraswamy-generated-distributions. Bull. Malays. Math. Sci. (2016). doi:10.1007/s40840-016-0356-9
11. Bdair, O.M., Raqab, M.Z.: Mean residual life of kth records under double monitoring. Bull. Malays. Math. Sci. Soc. **37**(2), 457–464 (2014)
12. Beg, M.I., Ahsanullah, M.: Concomitants of generalized order statistics from Farlie–Gumbel–Morgenstern distributions. Stat. Methodol. **5**, 1–20 (2008)
13. Bhattacharya, P.K.: Convergence of sample paths of normalized sums of induced order statistics. Ann. Stat. **2**, 1034–1039 (1974)
14. Chen, Z., Bai, Z., Sinha, B.K.: Ranked Set Sampling: Theory and Applications. Springer, New York (2004)
15. David, H.A.: Concomitants of order statistics. Bull. Int. Stat. Inst. **45**, 295–300 (1973)
16. David, H.A., Nagaraja, H.N.: Concomitants of order statistics. In: Balakrishnan, N., Rao, C.R. (eds.) Order Statistics: Theory & Methods, pp. 487–513. Elsevier, Amsterdam (1998)
17. David, H.A., Nagaraja, H.N.: Order Statistics, 3rd edn. Wiley, New York (2003)
18. David, H.A., O'Connell, M.J., Yang, S.S.: Distribution and expected value of the rank of a concomitant and an order statistic. Ann. Stat. **5**, 216–223 (1977)
19. Fischer, M., Klein, I.: Constructing generalized FGM copulas by means of certain univariate distributions. Metrika **65**(243), 260 (2007)
20. Galambos, J.: The Asymptotic Theory of Extreme Order Statistics, 2nd edn. Krieger, Malabar (1987)
21. Gompertz, B.: On the nature of the function expressive of the human mortality and on a new mode of determining the value of life contingencies. Philos. Transduct. R. Soc. Lond. **115**, 513–585 (1825)
22. Gupta, R.D., Kundu, D.: Generalized exponential distributions. Aust. N. Z. J. Stat. **41**, 173–188 (1999)
23. Houchens, R.L.: Record Value Theory and Inference. ProQuest LLC, Ann Arbor (1984)
24. Huang, J.S., Kotz, S.: Modifications of the Farlie–Gumbel–Morgenstern distributions. A tough hill to climb. Metrika **49**, 135–145 (1999)
25. Mokhlis, N.A., Khames, S.K.: Concomitants of record values from a general Farlie–Gumbel–Morgenstern distributoion. J. Adv. Math. **7**(3), 1328–1340 (2014a)
26. Mokhlis, N.A., Khames, S.K.: On concomitants of record values from generalized Farlie–Gumbel–Morgenstern distributoion. J. Stat. Sci. Appl. **2**, 175–192 (2014b)
27. Tahmasebi, S., Behboodian, J.: Shannon information for concomitants of generalized order statistics in Farlie–Gumbel–Morgenstern (FGM) family. Bull. Malays. Math. Sci. Soc. **35**(4), 975–981 (2012)
28. Tahmasebi, S., Jafari, A.A.: Estimators for the parameter mean of Morgenstern type bivariate generalized exponential distribution using ranked set sampling. SORT **38**(2), 161–180 (2014)
29. Tahmasebi, S., Jafari, A.A.: Concomitants of order statistics and record values from Morgenstern type bivariate-generalized exponential distribution. Bull. Malays. Math. Sci. Soc. **38**, 1411–1423 (2015)
30. Tahmasebi, S., Jafari, A.A., Afshari, M.: Concomitants of dual generalized order statistics from Morgenstern type bivariate generalized exponential distribution. J. Stat. Theory Appl. **14**(1), 1–12 (2015)
31. Tahmasebi, S., Jafari, A.A., Ahsanullah, M.: Properties on concomitants of generalized order statistics from a bivariate Rayleigh distribution. Bull. Malays. Math. Sci. Soc (2016). doi:10.1007/s40840-015-0297-8
32. Verhulstt, P.F.: Notice sur la loi la population suit dans son accroissement. Correspondence mathematique et physique publiee L. A. J. Quetelet **10**, 112–121 (1838)
33. Verhulst, P.F.: Recherches matheatiques sur la loi d'accorissement de la population. Nouvelles Mem. Ser. (2J) **18**, 38 (1845)
34. Verhulst, P.F.: Deuxieme memoire sur la loi d'accorissement de la population. Memoires de L'Academie Royale des Sciences des Lettres et des beaux-Arts de Belgique Ser. 2 **20**, 32 (1847)