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Independent Rainbow Domination of Graphs

Zehui Shao[1,2] · Zepeng Li[3] · Aljoša Peperko[4,5] · Jiafu Wan[6] · Janez Žerovnik[4,5]

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Abstract Given a positive integer t and a graph F, the goal is to assign a subset of the color set $\{1, 2, \ldots, t\}$ to every vertex of F such that every vertex with the empty set assigned has all t colors in its neighborhood. Such an assignment is called the t-rainbow dominating function (tRDF) of the graph F. A tRDF is independent (ItRDF) if vertices assigned with non-empty sets are pairwise non-adjacent. The weight of a tRDF g of a graph F is the value $w(g) = \sum_{v \in V(F)} |g(v)|$. The independent t-rainbow domination number $i_{rt}(F)$ is the minimum weight over all ItRDFs of F. In this article, it is proved that the independent t-rainbow domination problem is **NP**-complete even

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✉ Janez Žerovnik
janez.zerovnik@fs.uni-lj.si

1 School of Computer Science and Educational Software, Guangzhou University, Guangzhou 510006, China

2 School of Information Science and Engineering, Chengdu University, Chengdu 610106, China

3 School of Electronic Engineering and Computer Science, Peking University, Beijing 100871, China

4 FME, University of Ljubljana, Aškerčeva 6, 1000 Ljubljana, Slovenia

5 Institute of Mathematics, Physics and Mechanics, Jadranska 19, 1000 Ljubljana, Slovenia

6 School of Mechanical and Automotive Engineering, South China University of Technology, Guangzhou, China

if the input graph is restricted to a bipartite graph or a planar graph, and the results of the study provide some bounds for the independent t-rainbow domination number of any graph for a positive integer t. Moreover, the exact values and bounds of the independent t-rainbow domination numbers of some Petersen graphs and torus graphs are given.

Keywords Rainbow domination · Domination number · Independent rainbow domination · NP-complete

Mathematics Subject Classification 05C69 · 05C15 · 05C76 · 05C85

1 Introduction

As a combinatorial optimization issue, ordinary domination consists of determining the minimum number of places in which to keep a resource such that every place either is adjacent to the place in which a resource exists or has a resource. In practical applications, some additional constraints or desires must be taken into account. For example [12], if we are given a large computer network which consists of some clients and servers with t distinct resources $s_1, s_2, \ldots, s_t$, we need to seek the minimum number of servers each one possessing a non-empty subset of these resources in order that any client can be connected directly to a subset of servers that together have each resource s_i $(1 \leq i \leq t)$. On the assumption that all resources have an identical cost, the goal is to seek the minimum value of the number of copies of such t resources. This application naturally can be modeled by the concept of t-rainbow domination. In addition, if a constraint prevents any pair of servers from occupying adjacent locations, then we have the independent t-rainbow domination problem.

For a graph F, $S \subseteq V(F)$ and $w \in V(F)$, let $N_S(w)$ denote the open neighborhood of w in S, i.e., $\{u|uw \in E(F), u \in S\}$, and let $N_S[w]$ denote the closed neighborhood of w, i.e., $N_S[w] = \{w\} \cup N_S(w)$. If $S = V(F)$ and no confusion can occur, $N_S(w)$ and $N_S[w]$ will be denoted shortly by $N(w)$ and $N[w]$, respectively. If $S' \subseteq V(F)$, then the definition $N(S') = \cup_{x \in S'} N(x)$ is applied. The degree of a vertex w is the total number of edges incident to w, and in this paper, $\Delta(F)$ denotes the maximum degree of a vertex in the graph F.

Inspired by several facility location problems, Brešar, Henning and Rall [1–3] initiated the study of the k-rainbow domination problem, and such a problem is proved to be NP-complete even if the input graph is a chordal graph or a bipartite graph (see Chang [4]). This problem has attracted considerable attention (see [19,21,25]), and many other types of domination are widely applied to real-world scenarios, see, for example, [5,7,9,10,13,14,22,24,26].

An independent set S of a graph F is a subset of $V(F)$ for which vertices are pairwise non-adjacent. The independence number of F, denoted as $\alpha(F)$, is the maximum size of an independent set in F. Given a positive integer t and a graph F, the goal is to assign a subset of the color set $\{1, 2, \ldots, t\}$ to every vertex of F such that every vertex with the empty set assigned has all t colors in its neighborhood. Such an assignment is called the t-rainbow dominating function (tRDF) of the graph F. A tRDF is independent

(ItRDF) if vertices assigned non-empty sets are pairwise non-adjacent. The weight of a tRDF g of a graph F is the value $w(g) = \sum_{v \in V(F)} |g(v)|$. If H is a vertex-induced subgraph of $V(F)$, the weight restricted to H is $w_H(g) = \sum_{v \in H} |g(v)|$. The *independent t-rainbow domination number* $i_{rt}(F)$ is the minimum weight of an ItRDFs in F. The *upper t-rainbow domination number* of F, denoted by $I_{rt}(F)$, is the maximum weight of a minimal t-rainbow dominating function. (In other words, there is a dominating function f of weight $I_{rt}(F)$ on F such that no proper restriction of f is dominating.) We use $i(F)$ to denote the independent domination number, i.e., the size of a smallest independent dominating set, of a graph F.

The rest of the paper is organized as follows: In Sect. 2, the independent k-rainbow domination problem is proved to be NP-complete. Section 3 provides some general bounds for the independent t-rainbow domination number. In Sect. 4, the independent 2-rainbow domination problem is studied on generalized Petersen graphs and tori. Closed expressions for infinite families are provided on the basis of constructions for which we conjecture that they are optimal. In Sect. 5, the independent 3-rainbow domination of trees is considered and a tight upper bound for $i_{r3}(T)$ of a tree T is given. Conclusions are summarized in the last section.

2 Complexity of Independent Rainbow Domination

It is well known that determining whether F has an independent dominating set with at most k vertices (where k is a positive integer) is **NP**-complete for a graph F even when F is a bipartite graph or a planar graph, or belong to some other classes of graphs; see [8]. In this section, we will prove that it is **NP**-complete to determine whether F has an independent t-rainbow dominating function of weight at most k for a positive integer k and a given graph F even if F is bipartite or planar.

Formally, the problem can be stated as

PROBLEM: INDEPENDENT t-RAINBOW DOMINATION

INSTANCE: Graph $G = (V, E)$. A natural number B.

QUESTION: Decide whether the independent t-rainbow domination number $i_{rt}(G)$ is at most B.

Theorem 1 *Given a positive integer t, the independent t-rainbow domination problem is* ***NP****-complete for general graphs.*

Proof Theorem 1 can be proved via reducing to the independent t-rainbow domination problem from the independent domination problem. With a graph F on n vertices, the graph F' with the vertex set $V(F') = V(F) \cup \{v_2, v_3, \ldots, v_t : v \in V(F)\} \cup \{v_x, v_y : v \in V(F)\}$ and edge set $E(F') = E(F) \cup \{vv_i, v_i v_x : v \in V(F), 2 \le i \le t\} \cup \{v_x v_y : v \in V(F)\}$ is considered (see Fig. 1). We claim that F has an independent dominating set I_D with $|I_D| \le s$ if and only if F' has an ItRDF f of weight $w(f) \le s + nt$.

Assume F contains an independent dominating set I_D of cardinality at most s. Consider the following function f from $V(F')$ to $\mathcal{P}(\{1, 2, \ldots, t\})$ by setting

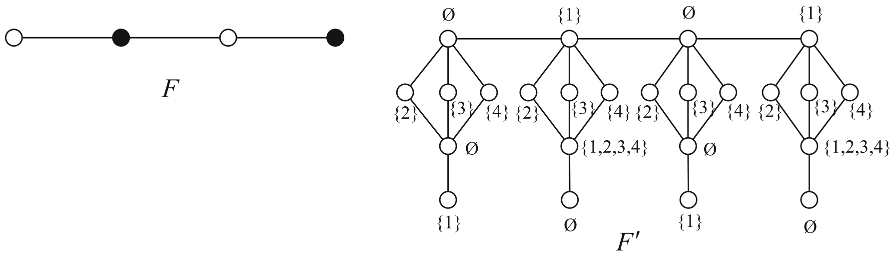


Fig. 1 The graphs F and F' from the proof of Theorem 1, an example

$$f(u) = \begin{cases} \{1\}, & \text{if } u \in I_D \text{ or} \\ & u = v_y \text{ for some } v \in V(F)\backslash I_D, \\ \emptyset, & \text{if } u \in V(G)\backslash I_D \\ & u = v_i \text{ for some } v \in I_D \text{ and } 2 \le i \le t \text{ or} \\ & u = v_y \text{ for some } v \in I_D \text{ or} \\ & u = v_x \text{ for some } v \in V(F)\backslash I_D, \\ \{i\}, & \text{if } u = v_i \text{ for some } v \in V(F)\backslash I_D \text{ and } 2 \le i \le t, \\ \{1, 2, \ldots, t\}, & \text{if } u = v_x \text{ for some } v \in I_D. \end{cases}$$

If $f(u) = \emptyset$, the following four cases are considered:
Case 1 $u \in V(F)\backslash I_D$.
From the definition of f, we obtain $f(u_i) = i$ for $2 \le i \le t$. Moreover, u has a neighbor $v \in I_D$. Hence, $f(v) = \{1\}$, and therefore, $f(N_{F'}(u)) = \{1, 2, \ldots, t\}$.
Case 2 $u = v_i$ for some $v \in I_D$ and $2 \le i \le t$.
From the definition of f, we obtain $f(v_x) = \{1, 2, \ldots, t\}$. Since v_x is a neighbor of v_i, we have $f(N_{F'}(u)) = \{1, 2, \ldots, t\}$.
Case 3 $u = v_y$ for some $v \in I_D$.
From the definition of f, we obtain $f(v_x) = \{1, 2, \ldots, t\}$. Since v_x is a neighbor of v_y, we have $f(N_{F'}(u)) = \{1, 2, \ldots, t\}$.
Case 4 $u = v_x$ for some $v \in V(F)\backslash I_D$.
From the definition of f, we obtain $f(v_i) = \{i\}$ for any $2 \le i \le t$ and $f(v_y) = \{1\}$. Since $v_i (2 \le i \le t)$ and v_y are the neighborhood of v_x, we have $f(N_{F'}(u)) = \{1, 2, \ldots, t\}$.

Since any two vertices $u_1, u_2 \in V(F')$ (with $f(u_i) \neq \emptyset$ for $i = 1, 2$) are not adjacent in F', it follows that f is an ItRDF of F'. Also the weight of f is $|I_D| + nt \le s + nt$.

Suppose F' has an ItRDF f of weight at most $s + nt$. Since $v_x v_y \in E(F')$ for some $v \in V(F)$, together with the definition of independent rainbow dominating function, we have $f(v_x) = \emptyset$ or $f(v_y) = \emptyset$. Therefore, $|f(v_y)| + |f(v_x)| + \sum_{i=2}^{t} |f(v_i)| \ge t$. Let $I_D = \{v \in V(F): f(v) \neq \emptyset\}$. Then we obtain $\sum_{v \in V(F')} = |I_D| + \sum_{v \in V(F)} (|f(v_y)| + |f(v_x)| + \sum_{i=2}^{t} |f(v_i)|) \le s + nt$. Therefore, $|I_D| \le s$, as needed. □

Noting that F' is bipartite or planar if F is a bipartite or planar graph, together with the fact that the independent domination problem is **NP**-complete for bipartite or planar graphs, we have

Corollary 1 *The independent t-rainbow domination problem is **NP**-complete for bipartite graphs or planar graphs for any positive integer t.*

3 Bounds for the Independent t-Rainbow Domination Number

Theorem 2 *For any graph F with maximum degree Δ, we have $\lceil \frac{|V(F)|t}{\Delta+t} \rceil \leq \gamma_{rt}(F) \leq i_{rt}(F) \leq t\alpha(F) \leq I_{rt}(F)$.*

Proof The result $\lceil \frac{|V(F)|t}{\Delta+t} \rceil \leq \gamma_{rt}(F)$ is proved in [19]. Since any $ItRDF$ of a graph F is a tRDF of F, we have $\gamma_{rt}(F) \leq i_{rt}(F)$. Let I be a maximal independent set of F with $|I| = \alpha(F)$. Let f be a function from $V(F)$ to $\mathcal{P}(\{1, 2, \ldots, t\})$ defined by $f(v) = \emptyset$ if $v \notin I$ and $f(v) = \{1, 2, \ldots, t\}$ if $v \in I$. Then f is an $ItRDF$ of F with weight $w(f) = t|I| = t\alpha(F)$. Therefore, $i_{rt}(F) \leq t\alpha(F)$. By the definition of the upper t-rainbow domination number, we have $t\alpha(F) = w(f) \leq I_{rt}(F)$. □

Similarly to [19, Theorem 1 and Corollary 1], we have the following two results. The proof of Theorem 3 is left to the reader.

Theorem 3 *For positive integers $t' \geq t$ and a connected graph F, we have*

i) $i_{rt'}(F) \leq i_{rt}(F) + (t' - t) \left\lfloor \frac{i_{rt}(F)}{t} \right\rfloor$;
ii) $i_{rt'}(F) \leq t' \frac{i_{rt}(F)}{t}$.

Theorem 4 *Let F be a non-trivial graph. Then* $\max\{i(F), t\} \leq i_{rt}(F) \leq ti(F)$*, and these bounds are sharp.*

Proof On the assumption that f is an ItRDF of F, and $D = \{u \in V(F) : f(u) \neq \emptyset\}$. Then D is an independent dominating set of F. So $i(F) \leq |D| \leq i_{rt}(F)$. Note that F is a non-trivial graph, so F has at least one vertex v for which $f(v) = \emptyset$. Hence, $t \leq i_{rt}(F)$ and the inequality $\max\{i(F), t\} \leq i_{rt}(F)$ holds. Moreover, for any independent domination set D' of F, we can obtain an ItRDF f' of F with $f'(u) = \{1, 2, \ldots, t\}$ if $u \in D'$ and $f'(u) = \emptyset$ otherwise. Hence, the inequality $i_{rt}(F) \leq ti(F)$ holds.

Furthermore, it can be seen that $\max\{i(F), t\} = i_{rt}(F)$ for any complete bipartite graph F and $i_{rt}(F) = ti(F)$ for any star. Therefore, these bounds are sharp. □

4 The Independent 2-Rainbow Domination Numbers of Some Classes of Graphs

Let $n > 0, b \geq 0, 0 \leq k < n$ be integers, we denote by $\mathbb{Z}_n^{k,b} = \{x | x \in \mathbb{Z}, x \geq b, x \equiv k \pmod{n}\}$

4.1 Generalized Petersen Graphs

For positive integers $n \geq 3$ and $k \in \{1, 2, \ldots, n-1\}$, the generalized Petersen graph $P(n, k)$ is defined to be a graph with the vertex set $\{h_i^1, h_i^2 \mid i \in \{0, 1, \ldots, n-1\}\}$ and the edge set $\{h_i^1 h_i^2, h_i^1 h_{i+k}^1, h_i^2 h_{i+1}^2 \mid i \in \{0, 1, \ldots, n-1\}\}$, in which the subscripts are computed modulo n (see [6,23]).

An ItRDF f is given by a pattern of two lines, where the values of the upper line are values of $\{f(h_0^2), f(h_1^2), \ldots, f(h_{n-1}^2)\}$, and the values of the bottom line are values of $\{f(h_0^1), f(h_1^1), \ldots, f(h_{n-1}^1)\}$, i.e., h_i^2 lies exactly above h_i^1 for each i. Moreover, the sets Ø, {1}, {2} and {1, 2} are encoded with 0, 1, 2 and 3, respectively.

Let f be an I2RDF of $P(n, k)$, we define the *weight sequence* $S_f = s_1 s_2 \ldots s_n$ of $P(n, k)$ under f the following way: $s_i = |f(h_i^1)| + |f(h_i^2)|$ for each i. For example, let $f\left(\begin{bmatrix} h_0^2 & h_1^2 & h_2^2 & h_3^2 & h_4^2 \\ h_0^1 & h_1^1 & h_2^1 & h_3^1 & h_4^1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 3 & 0 & 1 \\ 3 & 0 & 0 & 1 & 0 \end{bmatrix}$ be an I2RDF of $P(5, 1)$, then $S_f = 20211$.

Theorem 5 *Let* $n \geq 4$.

$$i_{r2}(P(n, 1)) = \begin{cases} n, & \text{if } n \equiv 0 \pmod 2 \\ n+1, & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

Proof First the upper bounds are proved by giving constructions of I2RDF of $P(n, 1)$ with the weight n if $n \equiv 0 \pmod 2$ and $n + 1$ if $n \equiv 1 \pmod 2$ as follows.

1. $n \in \mathbb{Z}_4^{0,4}$:

```
0202...0202
1010...1010
```

2. $n \in \mathbb{Z}_4^{1,9}$:

```
0202...0202 00301
1010...1010 30020
```

3. $n \in \mathbb{Z}_4^{2,6}$:

```
0202...0202 02
1010...1010 10
```

4. $n \in \mathbb{Z}_4^{3,7}$:

```
0030...0030 003
3000...3000 300
```

The pattern $\begin{bmatrix} 0 & 0 & 3 & 0 & 1 \\ 3 & 0 & 0 & 1 & 0 \end{bmatrix}$ yields an I2RDF of $P(5, 1)$ of the weight 6. Thus, $i_{r2}(P(5, 1)) \leq 6$ and so all the upper bounds are established. Note that $P(n, 1)$ is the ladder graph, which is the Cartesian product $P_2 \square P_n$.

Now, we will show the lower bounds. Let f be an I2RDF of $P(n, 1)$ with minimum weight. For $i = 0, 1, 2, 3, 4$, let L_i denote a set of columns of weight i and let $|L_i| = n_i$. Since f is independent, we have $n_3 = n_4 = 0$. We have

$$w(f) = n_1 + 2n_2, \text{ and } n = n_0 + n_1 + n_2.$$

□

Claim 1 *If* $f(h_i^1) = f(h_i^2) = \emptyset$, *then* $|f(h_{i-1}^1)| + |f(h_{i-1}^2)| = 2$ *and* $|f(h_{i+1}^1)| + |f(h_{i+1}^2)| = 2$.

From Claim 1, we have $n_2 \geq n_0$. Hence,

$$w(f) - n = n_2 - n_0 \geq 0.$$

Therefore, the theorem holds for $n \equiv 0 \pmod 2$.

We now consider $n \equiv 1 \pmod 2$. Let S be a segment of a weight sequence of $P(n, 1)$. We define three types of subsequences (segments) in the following way:

Type 1: $S = a_1, a_2, \ldots, a_{2\ell+1}$ with $a_i = 2$ for odd i, and $a_i = 0$ for even i.
Type 2: $S = a_1, a_2, \ldots, a_\ell$ with $a_i = 1$ for each i.
Type 3: $S = a_1, a_2, \cdots, a_\ell$ with $a_i = 2$ for each i.

From Claim 1, it can be seen that the weight sequence S_f of $P(n, 1)$ under f can be decomposed into disjoint segments of Type 1, Type 2 and Type 3. Let p_i be the number of maximal segments of Type i for $i \in \{1, 2, 3\}$. Then

$$w(f) - n = n_2 - n_0 = p_1 + p_3.$$

Since n is odd and f is independent, then there exists an i such that $a_i = 0$. So we have $p_1 > 0$ and $w(f) > n$, and so the proof is completed.

Theorem 6 *Let* $n \geq 7$.

$$i_{r2}(P(n,2)) \leq \begin{cases} \left\lceil \frac{4n}{5} \right\rceil, & n \equiv 0 \bmod 10 \\ \left\lceil \frac{4n}{5} \right\rceil + 1, & n \equiv 9 \bmod 10 \\ \left\lceil \frac{4n}{5} \right\rceil + 2, & n \equiv 2, 3, 4, 5, 7, 8 \bmod 10 \\ \left\lceil \frac{4n}{5} \right\rceil + 3, & n \equiv 1, 6 \bmod 10 \end{cases}$$

Proof First the upper bounds are proved by giving constructions of I2RDF of $P(n, 2)$ of the desired weight as follows.

1. $n \in \mathbb{Z}_{10}^{0,10}$:

```
2010010200...2010010200
0002200011...0002200011
```

2. $n \in \mathbb{Z}_{10}^{1,21}$:

```
0200201001...0200201001 02000301001
0011000220...0011000220 00330000320
```

3. $n \in \mathbb{Z}_{10}^{2,22}$:

```
2010010200...2010010200 201001030200
0002200011...0002200011 000230000031
```

4. $n \in \mathbb{Z}_{10}^{3,23}$:

```
0010200201...0010200201 0030030200201
2200011000...2200011000 2300200011000
```

5. $n \in \mathbb{Z}_{10}^{4,14}$:

```
0102002010...0102002010 0030
2000110002...2000110002 3003
```

6. $n \in \mathbb{Z}_{10}^{5,15}$:

```
0100102002...0100102002 01002
0022000110...0022000110 00330
```

7. $n \in \mathbb{Z}_{10}^{6,16}$:

```
0201001020...0201001020 030030
1000220001...1000220001 300003
```

8. $n \in \mathbb{Z}_{10}^{7,17}$:

```
0201001020...0201001020 0203020
1000220001...1000220001 3000003
```

9. $n \in \mathbb{Z}_{10}^{8,18}$:

```
0200201001...0200201001 02000303
0011000220...0011000220 00130020
```

10. $n \in \mathbb{Z}_{10}^{9,19}$:

```
1001020020...1001020020 100030020
0220001100...0220001100 023001100
```

Let $P_7 = \begin{bmatrix} 0\,3\,0\,1\,0\,0\,2 \\ 0\,0\,0\,0\,3\,3\,0 \end{bmatrix}$, $P_8 = \begin{bmatrix} 3\,0\,3\,0\,0\,0\,2\,0 \\ 0\,2\,0\,0\,3\,1\,0\,0 \end{bmatrix}$,
$P_{13} = \begin{bmatrix} 0\,0\,1\,0\,2\,0\,0\,2\,0\,3\,0\,0\,3 \\ 3\,2\,0\,0\,0\,1\,1\,0\,0\,0\,2\,0\,0 \end{bmatrix}$,
$P_9 = \begin{bmatrix} 1\,0\,3\,0\,3\,0\,1\,0\,0 \\ 0\,0\,0\,1\,0\,0\,0\,2\,2 \end{bmatrix}$, $P_{11} = \begin{bmatrix} 0\,0\,3\,0\,0\,1\,0\,2\,0\,0\,3 \\ 0\,0\,0\,3\,2\,0\,0\,0\,1\,3\,0 \end{bmatrix}$,
$P_{12} = \begin{bmatrix} 3\,0\,1\,0\,0\,1\,0\,2\,0\,0\,2\,0 \\ 0\,0\,0\,3\,2\,0\,0\,0\,1\,3\,0\,0 \end{bmatrix}$. Then the pattern P_n yields an I2RDF of $P(n, 2)$ with the desired weight for each $n \in \{7, 8, 9, 11, 12, 13\}$, and so all the upper bounds are established. □

Theorem 7 *Let* $n \geq 7$.

$$i_{r2}(P(n,3)) \leq \begin{cases} \lceil \frac{7n}{8} \rceil, & n \equiv 0, 2, 4, 14 \bmod 16 \\ \lceil \frac{7n}{8} \rceil + 1, & n \equiv 5, 7, 8, 10, 12, 13, 15 \bmod 16 \\ \lceil \frac{7n}{8} \rceil + 2, & n \equiv 1, 3, 6, 9, 11 \bmod 16 \end{cases}$$

Proof First the upper bounds are proved by giving constructions of I2RDF of $P(n, 3)$ of the desired weight.

1. $n \in \mathbb{Z}_{16}^{0,16}$:
```
300202003001010 0...300202003001010 0
```
2. $n \in \mathbb{Z}_{16}^{1,33}$:

```
0300202003001010...0300202003001010 03002020203001010
0001010100020202...0001010100020202 00010101000030302
```

3. $n \in \mathbb{Z}_{16}^{2,18}$:

```
0101003002020030...0101003002020030 01
2020200010101000...2020200010101000 20
```

4. $n \in \mathbb{Z}_{16}^{3,35}$:

```
0100300202003001...0100300202003001 0100302000201010101
2020001010100020...2020001010100020 3020000131000202020
```

5. $n \in \mathbb{Z}_{16}^{4,20}$:

```
0101003002020030...0101003002020030 0101
2020200010101000...2020200010101000 2020
```

6. $n \in \mathbb{Z}_{16}^{5,37}$:

```
2020030010100300...2020030010100300 202010200102010100300
0101000202020001...0101000202020001 010100033000202020001
```

7. $n \in \mathbb{Z}_{16}^{6,22}$:

```
0300202003001010...0300202003001010 101010
0001010100020202...0001010100020202 020202
```

8. $n \in \mathbb{Z}_{16}^{7,39}$:

```
3002020030010100...3002020030010100 30020202010001030010100
0010101000202020...0010101000202020 00101010002320000203020
```

9. $n \in \mathbb{Z}_{16}^{8,24}$:

```
0300101003002020...0300101003002020 20202020
0002020200010101...0002020200010101 01010101
```

10. $n \in \mathbb{Z}_{16}^{9,25}$:

```
0300101003002020...0300101003002020 203002020
0002020200010101...0002020200010101 000030301
```

11. $n \in \mathbb{Z}_{16}^{10,26}$:

```
1003002020030010...1003002020030010 1020102010
0200010101000202...0200010101000202 0201010202
```

12. $n \in \mathbb{Z}_{16}^{11,27}$:

```
1003002020030010...1003002020030010 10101030010
0200010101000202...0200010101000202 02020000303
```

13. $n \in \mathbb{Z}_{16}^{12,28}$:

```
0200300101003002...0200300101003002 010201010202
1010002020200010...1010002020200010 101020202010
```

14. $n \in \mathbb{Z}_{16}^{13,29}$:

```
0100300202003001...0100300202003001 0102010010201
2020001010100020...2020001010100020 2020003300020
```

15. $n \in \mathbb{Z}_{16}^{14,30}$:

```
0010100300202003...0010100300202003 00101003003003
0202020001010100...0202020001010100 02020200010100
```

16. $n \in \mathbb{Z}_{16}^{15,31}$:

```
1003002020030010...1003002020030010 101020002030010
0200010101000202...0200010101000202 020001310000203
```

Let $P_7 = \begin{bmatrix} 3\,0\,0\,3\,0\,0\,0 \\ 0\,0\,0\,0\,2\,3\,2 \end{bmatrix}$, $P_8 = \begin{bmatrix} 0\,0\,1\,0\,1\,0\,0\,3 \\ 0\,2\,0\,3\,0\,2\,0\,0 \end{bmatrix}$, $P_9 = \begin{bmatrix} 0\,0\,3\,0\,0\,3\,0\,0\,3 \\ 0\,0\,0\,0\,0\,0\,3\,3\,0 \end{bmatrix}$,
$P_{11} = \begin{bmatrix} 3\,0\,0\,0\,3\,0\,0\,0\,3\,0\,0 \\ 0\,1\,3\,0\,0\,0\,3\,2\,0\,0\,0 \end{bmatrix}$, $P_{12} = \begin{bmatrix} 0\,0\,1\,0\,2\,0\,3\,0\,1\,0\,0\,3 \\ 0\,2\,0\,1\,0\,2\,0\,1\,0\,2\,0\,0 \end{bmatrix}$,
$P_{13} = \begin{bmatrix} 0\,2\,0\,1\,0\,0\,1\,0\,2\,0\,1\,0\,1 \\ 2\,0\,0\,0\,3\,3\,0\,0\,0\,2\,0\,2\,0 \end{bmatrix}$, $P_{14} = \begin{bmatrix} 2\,0\,1\,0\,2\,0\,2\,0\,0\,3\,0\,0\,1\,0 \\ 0\,2\,0\,1\,0\,1\,0\,1\,0\,0\,0\,2\,0\,2 \end{bmatrix}$,
$P_{15} = \begin{bmatrix} 0\,3\,0\,1\,0\,0\,0\,1\,0\,2\,0\,2\,0\,2\,0 \\ 0\,0\,0\,0\,2\,3\,2\,0\,0\,0\,1\,0\,3\,0\,1 \end{bmatrix}$, $P_{17} = \begin{bmatrix} 0\,3\,0\,0\,2\,0\,2\,0\,1\,0\,2\,0\,1\,0\,0\,0\,1 \\ 0\,0\,0\,1\,0\,3\,0\,1\,0\,2\,0\,0\,0\,2\,3\,2\,0 \end{bmatrix}$,
$P_{19} = \begin{bmatrix} 0\,2\,0\,2\,0\,0\,3\,0\,1\,0\,0\,0\,1\,0\,2\,0\,2\,0\,2 \\ 1\,0\,3\,0\,1\,0\,0\,0\,0\,2\,3\,2\,0\,0\,0\,1\,0\,1\,0 \end{bmatrix}$,
$P_{21} = \begin{bmatrix} 0\,1\,0\,1\,0\,0\,3\,0\,0\,2\,0\,2\,0\,1\,0\,2\,0\,0\,1\,0\,2 \\ 2\,0\,2\,0\,2\,0\,0\,0\,1\,0\,1\,0\,1\,0\,0\,0\,3\,3\,0\,0\,0 \end{bmatrix}$,
$P_{23} = \begin{bmatrix} 0\,2\,0\,1\,0\,1\,0\,1\,0\,0\,3\,0\,0\,2\,0\,2\,0\,1\,0\,2\,0\,0\,1 \\ 0\,0\,2\,0\,2\,0\,2\,0\,2\,0\,0\,0\,1\,0\,1\,0\,1\,0\,0\,0\,3\,3\,0 \end{bmatrix}$. Then the pattern P_n yields an I2RDF of $P(n, 3)$ with the desired weight for each $n \in \{7, 8, 9, 11, 12, 13, 14, 15, 17, 19, 21, 23\}$, and this completes the proof. □

Remark We believe that the bounds given in Theorems 6 and 7 are the exact values. A method for the proofs may follow the path algebra approach [11,15–17,27]. As the corresponding matrices are rather large already for $P(n, 2)$ and $P(n, 3)$, we do not go in more detail here and leave it for future work.

In [20], it was shown that

Theorem 8 *Let $m > 3$ and k, ℓ relatively prime to m with $\ell k \equiv 1 \bmod m$. Then $P(m, k)$ is isomorphic to $P(m, \ell)$.*

By Theorem 8, we have $P(2k+1, k+1) \cong P(2k+1, 2)$ and $P(3k+2, k+1) \cong P(3k+2, 3)$ for any $k \geq 2$. Therefore, we have

Corollary 2 *Let* $k \geq 2$.

$$i_{r2}(P(2k+1, k+1)) \leq \begin{cases} \lceil \frac{8k+4}{5} \rceil + 1, & k \equiv 4 \bmod 5 \\ \lceil \frac{8k+4}{5} \rceil + 2, & k \equiv 1, 2, 3 \bmod 5 \\ \lceil \frac{8k+4}{5} \rceil + 3, & k \equiv 0 \bmod 5 \end{cases}$$

Corollary 3 *Let* $k \geq 2$.

$$i_{r2}(P(3k+2, k+1)) \leq \begin{cases} \lceil \frac{21k+14}{8} \rceil, & 3k \equiv 0, 2, 12, 14 \bmod 16 \\ \lceil \frac{21k+14}{8} \rceil + 1, & 3k \equiv 1, 3, 4, 6, 8, 9, 11 \bmod 16 \\ \lceil \frac{21k+14}{8} \rceil + 2, & 3k \equiv 5, 7, 10, 13, 15 \bmod 16 \end{cases}$$

4.2 Torus

The *Cartesian product* of graphs F_1 and F_2 is defined to be the graph $F_1 \square F_2$ with vertex set $F_1 \times F_2 = \{xy | x \in V(F_1) \text{ and } y \in V(F_2)\}$ such that $(x_1, y_1)(x_2, y_2) \in E(F_1 \square F_2)$ if $x_1x_2 \in E(F_1)$ and $y_1 = y_2$, or $y_1y_2 \in E(F_2)$ and $x_1 = x_2$ (see [15]). The torus is the graph $T_{m,n} = C_m \square C_n$, which can be viewed as a kind of grid graph with n columns and m rows.

Theorem 9 *Let* $n \geq 3$.

$$i_{r2}(T_{3,n})) = \begin{cases} n, & n \equiv \ (0 \bmod 6) \\ n+1, & n \equiv \ (3 \bmod 6) \\ n+2, & n \equiv 1, 2, 4, 5 \ (\bmod 6) \end{cases}$$

Proof First the upper bounds are proved by giving constructions of I2RDF of $T_{3,n}$ of the desired weight.

1. $n \in \mathbb{Z}_6^{0,6}$:

```
010020...010020
200100...200100
002001...002001
```

2. $n \in \mathbb{Z}_6^{1,13}$:

```
002001...002001 0001003
200100...200100 2020010
010020...010020 0300200
```

3. $n \in \mathbb{Z}_6^{2,14}$:

```
100200...100200 10100300
001002...001002 00020002
020010...020010 03001010
```

4. $n \in \mathbb{Z}_6^{3,15}$:

```
100200...100200 300
020010...020010 010
001002...001002 002
```

5. $n \in \mathbb{Z}_6^{4,10}$:

```
100200...100200 1020
020010...020010 0300
001002...001002 0003
```

6. $n \in \mathbb{Z}_6^{5,11}$:

```
010020...010020 00020
200100...200100 30100
002001...002001 03001
```

Let $P_3 = \begin{bmatrix} 0\,0\,3 \\ 0\,1\,0 \\ 2\,0\,0 \end{bmatrix}$, $P_4 = \begin{bmatrix} 0\,0\,0\,3 \\ 0\,3\,0\,0 \\ 1\,0\,1\,0 \end{bmatrix}$, $P_5 = \begin{bmatrix} 2\,0\,3\,0\,0 \\ 0\,3\,0\,2\,0 \\ 0\,0\,0\,0\,1 \end{bmatrix}$, $P_7 = \begin{bmatrix} 0\,1\,0\,0\,2\,0\,1 \\ 3\,0\,0\,1\,0\,0\,0 \\ 0\,0\,2\,0\,0\,3\,0 \end{bmatrix}$, $P_8 = \begin{bmatrix} 0\,0\,2\,0\,0\,0\,0\,1 \\ 0\,1\,0\,0\,3\,0\,2\,0 \\ 2\,0\,0\,1\,0\,3\,0\,0 \end{bmatrix}$, $P_9 = \begin{bmatrix} 0\,2\,0\,0\,3\,0\,0\,1\,0 \\ 1\,0\,0\,2\,0\,0\,2\,0\,0 \\ 0\,0\,1\,0\,0\,1\,0\,0\,2 \end{bmatrix}$. Then the pattern P_n yields an I2RDF of $T_{3,n}$ with the desired weight for each $n \in \{3,4,5,7,8,9\}$, and all the upper bounds are established. The lower bounds can be determined by the dynamic algorithm used in [18], and the implementation is left to the reader. □

Theorem 10 *If m and n are positive integers, then $i_{r2}(T_{3m,6n}) = 6mn$.*

Proof By Theorem 2, we have $i_{r2}(T_{3m,6n}) \geq 6mn$. Let

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{bmatrix}, \; Q = [\overbrace{P, \ldots, P}^{n \text{ times}}], \text{ and } R = \begin{bmatrix} Q \\ \vdots \\ Q \end{bmatrix},$$

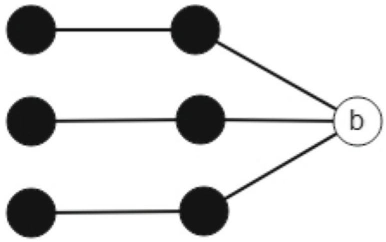
where Q is repeated m times in R. Then R yields an I2RDF of $T_{3m,6n}$ with weight $6mn$, which completes the proof. □

5 The Independent 3-Rainbow Domination Numbers of Trees

The following result for paths and stars is easily proved:

Proposition 1 *If $n \geq 3$, then*

Fig. 2 The spider $S(2, 2, 2)$



1) $i_{r3}(P_n) = \begin{cases} \lfloor \frac{3n}{4} \rfloor + 1, & n=2 \ (mod\ 4) \\ \lfloor \frac{3n}{4} \rfloor + 2, & n=0,1,3 \ (mod\ 4) \end{cases}$

2) $i_{r3}(S_n) = 3$.

We first define the spider graph as follows. A spider $S = S(a_1, \ldots, a_r)$ is a tree formed by joining $r \geq 1$ vertex-disjoint paths of orders $a_1, \ldots, a_r$ as pendent paths to a single vertex b, which is called the anchor of S, e.g., Fig. 2 is the spider $S(2, 2, 2)$. We denote by $H(k) = S(2, 2, \ldots, 2)$ with a total of k 2s.

Lemma 1 *Let integers $t, k \geq 1$ and T belong to one of the following trees with n vertices:*

(i) a vertex u connected to every anchor of t spiders $H(k)$.
(ii) a vertex u connected to another vertex and each anchor of t spiders $H(k)$.
(iii) a vertex u connected to a pendent $P_2 = v_1v_2$ and each anchor of t spiders $H(k)$.
Then T has an I3RDF g with $w_T(g) \leq n$ such that $g(u) = \emptyset$.

Proof i) Assume T is a tree which is a vertex z connected to each anchor y_i $(1 \leq i \leq s)$ of s spiders $H(k)$. Define a function $g : V(T) \to \mathcal{P}(\{3, 2, 1\})$ as follows. If $k \geq 2$, let $g(z) = \emptyset$, $g(y_1) = \{1, 2\}$. The vertices of P_2 attached to y_1 is assigned with $\emptyset$ and $\{3\}$, respectively. For $i \geq 2$, $g(y_i) = \{2, 3\}$. The vertices of P_2 attached to y_i are assigned with $\emptyset$ and $\{1\}$, respectively. Then it is easy to see that $w_T(g) \leq n$. If $k = 1$, let $g(z) = \emptyset$, $g(y_1) = \{3, 2, 1\}$. The vertices of P_2 attached to y_1 are assigned with $\emptyset$ and $\{3\}$, respectively. Then $w_T(g) \leq n$.

ii) Consider a function $g : V(T) \to \mathcal{P}(\{3, 2, 1\})$ by letting the same color to the vertices as case i except v and let $g(v) = 1$. Then we obtain $w_T(g) \leq n$.

iii) Consider a function $g : V(T) \to \mathcal{P}(\{3, 2, 1\})$ by letting the same color to the vertices as case i except v_1, v_2. Assume v_2 be the leaf, and let $g(v_1) = \{3, 2, 1\}$ and $g(v_2) = \emptyset$. Then we obtain $w_T(g) \leq n$. □

Remark Figures 3, 4 and 5 show examples of I3RDFs of graphs of such three cases, where we use 0 to denote the empty set $\emptyset$.

Theorem 11 *If T is an n-vertex tree with at least three vertices, then $i_{r3}(T) \leq n$.*

Proof Assume that there exists an n-vertex tree T with minimum n such that $i_{r3}(T) > n$, i.e., $i_{r3}(T^R) \leq n'$ for each n'-vertex tree T^R if $n' < n$. Then we have □

Claim 2 *T has no vertex which is adjacent to at least two leaves.*

Fig. 3 A tree in case i

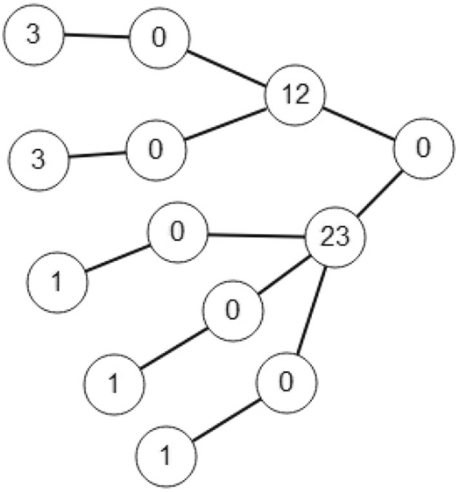
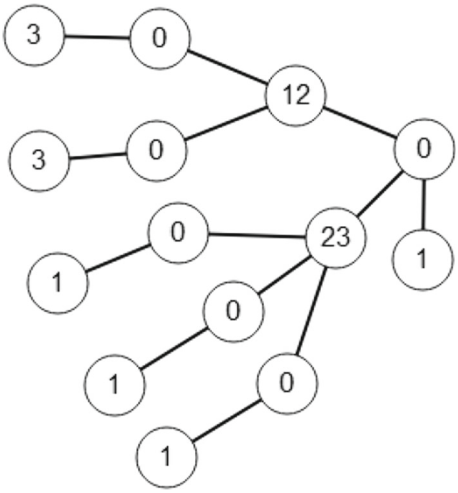


Fig. 4 A tree in case ii

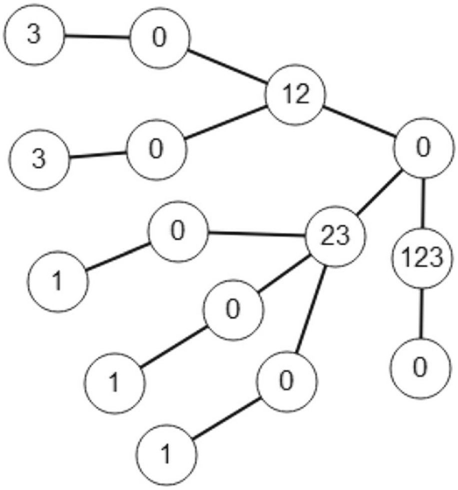


Proof of Claim 1. Assume that $u \in V(T)$ and u_1 and u_2 are two leaves adjacent to u. Let $T^R = T - u_1$ and f' be an optimal I3RDF of T^R. Then we have $w(f') = i_{r3}(T^R) \leq n-1$. If $f'(u) = \emptyset$, then $f'(N_{T^R}(u)) = \{1, 2, 3\}$. Extend f' to g by putting $g(u_1) = \{1\}$ and $g(x) = f'(x)$ for $x \in V(T)\setminus\{u_1\}$. Then g is an I3RDF of T with $w(g) \leq n$, a contradiction. If $f'(u) \neq \emptyset$, then $f'(u_2) = \emptyset$ and so $f'(u) = \{1, 2, 3\}$. Extend f' to g by putting $g(u_1) = \emptyset$ and $g(x) = f'(x)$ for $x \in V(T)\setminus\{u_1\}$. Then g is an I3RDF of T with $w(g) \leq n - 1$, a contradiction. □

Claim 3 *T has no vertex u which is adjacent to both a leaf and a pendent P_2.*

Proof of Claim 2. Suppose to the contrary that $u \in V(T)$ and there exist vertices $u_1, u_2, u_3 \in V(T)$ such that $\{uu_1, uu_2, u_2u_3\} \in E(T)$, u_1, u_3 are leaves and $d_T(u_2) = 2$. Let $T^R = T - u_1$ and f' be an optimal I3RDF of T^R. Then we have $w(f') =$

Fig. 5 A tree in case iii



$i_{r3}(T^R) \leq n-1$. If $f'(u) = \emptyset$, then $f'(N_{T^R}(u)) = \{1,2,3\}$. Extend f' to g by putting $g(u_1) = \{1\}$ and $g(x) = f'(x)$ for $x \in V(T)\backslash\{u_1\}$. Then g is an I3RDF of T with $w(g) \leq n$, a contradiction. If $f'(u) \neq \emptyset$, then $f'(u_1) = f'(u_2) = \emptyset$ and $f'(\{u, u_3\}) = \{1,2,3\}$. We may w.l.o.g assume that $f'(u) = \{1,2\}$. Extend f' to g by putting $g(u_1) = \emptyset$, $g(u) = \{1,2,3\}$ and $g(x) = f'(x)$ for $x \in V(T)\backslash\{u_1, u\}$. Then we have $w(g) \leq n$, a contradiction. □

Claim 4 *T has no vertex u which is adjacent to a pendent* P_3.

Proof of Claim 3. Suppose to the contrary that $u \in V(T)$ and u is adjacent to a pendent $P_3 = u_1u_2u_3$, say $uu_1, u_1u_2, u_2u_3 \in E(T)$. Let $T^R = T - \{u_1, u_2, u_3\}$ and f' be an optimal I3RDF of T^R. Then we have $w(f') = i_{r3}(T^R) \leq n-3$. Extend f' to g by putting $g(u_1) = g(u_3) = \emptyset$, $g(2) = \{1,2,3\}$ and $g(x) = f'(x)$ for $x \in V(T)\backslash\{u_1, u_2, u_3\}$. Then g is an I3RDF of T with $w(g) \leq n$, a contradiction. □

Let $P = x_1, x_2, \ldots, x_m$ be a longest path in T. By Claims 2, 3 and 4 , we have that x_1 is a leaf and $d(x_2) = 2$, x_3 is adjacent to at least two pendent P_2s and x_4 is adjacent to some anchors of pendent spiders $H(k)$. Let T_1 and T_2 be two trees obtained by deleting the edge x_4x_5 from T and T_1 be the tree rooted at x_4. Let f' be an optimal I3RDF of T_2.

We now consider the following possible cases:

Case 1 There exists no leaf or pendent P_2 adjacent to x_4 in T_1 (see, e.g., Fig. 6).

Let f'' be an I3RDF of T_1 described in case i of Lemma 1. Then we obtain an I3RDF of T by putting $g(x) = f''(x)$ if $x \in V(T_1)$ and $g(x) = f'(x)$ if $x \in V(T_2)$. By Lemma 1, $g(x_4) = \emptyset$ and so g is independent. Therefore, $w_T(g) \leq w_{T_1}(g) + w_{T_2}(g) \leq n$, a contradiction.

Case 2 There exists exactly one leaf z adjacent to x_4 in T_1 (see, e.g., Fig. 7).

Let f'' be an I3RDF of T_1 described in case ii of Lemma 1. Then we obtain an I3RDF of T by putting $g(x) = f''(x)$ if $x \in V(T_1)$ and $g(x) = f'(x)$ if $x \in V(T_2)$. By

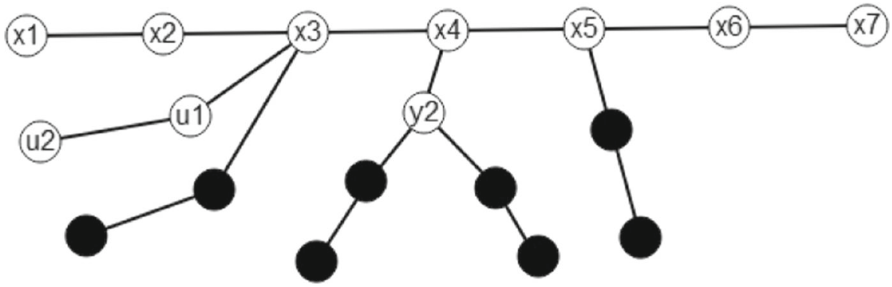


Fig. 6 x_4 adjacent to no leaf or pendent P_2

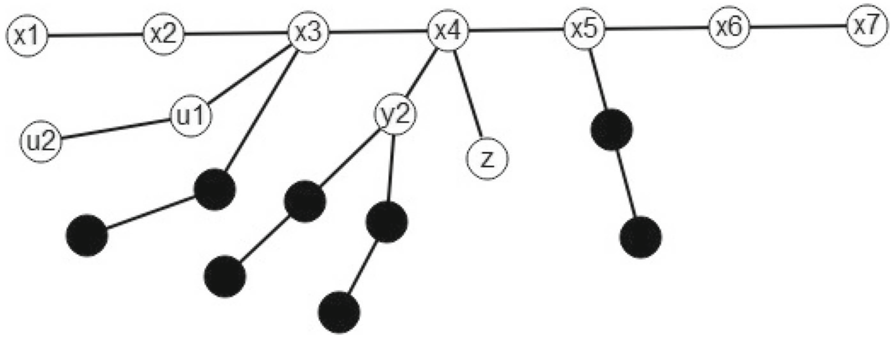


Fig. 7 x_4 adjacent to one pendent vertex z

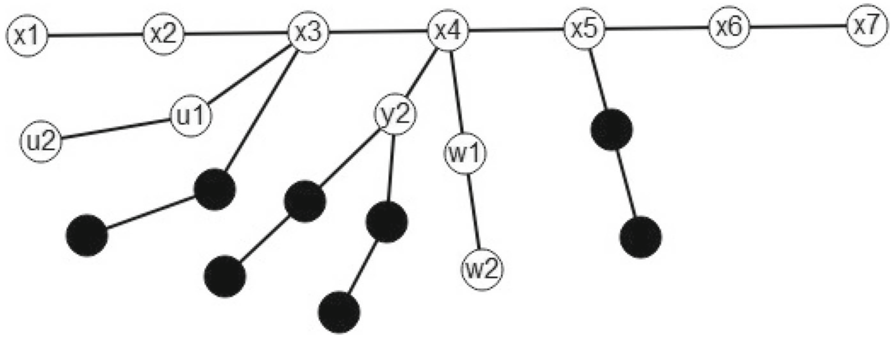


Fig. 8 x_4 adjacent to one pendent vertex z

Lemma 1, $g(x_4) = \emptyset$ and so g is independent. Therefore, $w_T(g) \leq w_{T_1}(g) + w_{T_2}(g) \leq n$, a contradiction.

Case 3 There exists exactly one pendent $P_2 = w_1 w_2$ adjacent to x_4 (see, e.g., Fig. 8).

Let f'' be an I3RDF of T_1 described in case iii of Lemma 1. Then we obtain an I3RDF of T by putting $g(x) = f''(x)$ if $x \in V(T_1)$ and $g(x) = f'(x)$ if $x \in V(T_2)$. By Lemma 1, $g(x_4) = \emptyset$ and so g is independent. Therefore, $w_T(g) \leq w_{T_1}(g) + w_{T_2}(g) \leq n$, a contradiction.

Case 4 There exist at least two pendent P_2s adjacent to x_4 (see, e.g., Fig. 9).

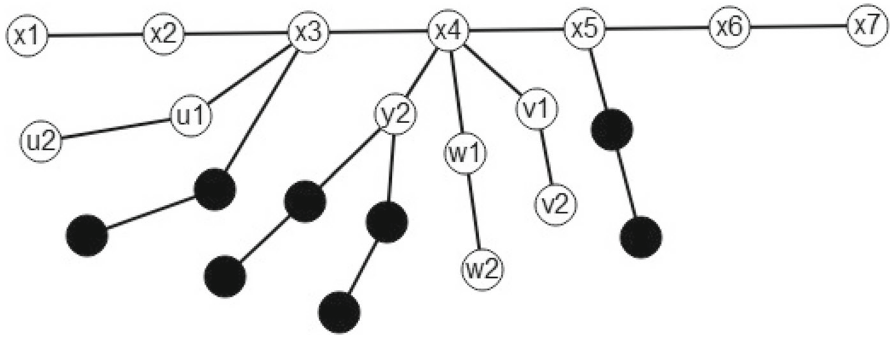


Fig. 9 x_4 adjacent to at least two pendent P_2s

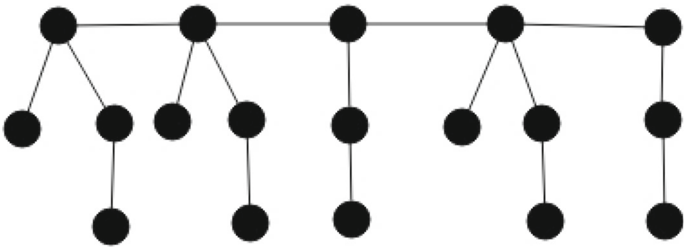


Fig. 10 A tree T in $\mathcal{L}_5$ with $i_{r3}(T) = n$

Let u_1u_2 be a P_2 adjacent to x_3, w_1w_2 be a P_2 adjacent to x_4 and $T^R = T - \{u_1, u_2, w_1, w_2\}$. Let f_1 be an optimal I3RDF of T^R. Then it is impossible that $f_1(x_3) = f_1(x_4) = \emptyset$ (otherwise, recolor x_3 and the pendent P_2s adjacent to x_3 as described in Lemma 1 to obtain an I3RDF with smaller weight). Assume that $f_1(x_3) = \emptyset$ and $f_1(x_4) \neq \emptyset$. If $|f_1(x_4)| = 1$, we can adjust the coloring of T_1 as described in Lemma 1 such that $|f_1(x_4)| = 2$. Let u_2 and w_2 be leaves. We can assume w.l.o.g. that $f_1(x_4) = \{2, 3\}$. Extending f_1 to g by putting $g(u_1) = \{1, 2, 3\}$, $g(u_2) = g(w_1) = \emptyset$, $g(w_2) = \{1\}$ and $g(x) = f_1(x)$ for $x \in V(T)\backslash\{u_1, u_2, w_1, w_2\}$. Therefore, $w_T(g) \leq w_{T^R}(g) + 4 \leq n$, a contradiction. The case when $f_1(x_3) \neq \emptyset$ and $f_1(x_4) = \emptyset$ is similar.

Case 5 $d(x_4) = 2$.

In this case, we have that x_3 is adjacent to at least two pendent P_2s by Claim 4. We define a function $f'': V(T_1) \to \mathcal{P}(\{1, 2, 3\})$ as follows. $f''(x_4) = \emptyset$, $f''(x_3) = \{1, 2, 3\}$. For each pendent $P_2 = v_1v_2$ (v_2 is a leaf) adjacent to x_3, let $f''(v_2) = \{1\}$ and $f''(v_1) = \emptyset$. Then we obtain an I3RDF of T by putting $g(x) = f''(x)$ if $x \in V(T_1)$ and $g(x) = f'(x)$ if $x \in V(T_2)$. Therefore, $w_T(g) \leq w_{T_1}(g) + w_{T_2}(g) \leq n$, a contradiction.

Theorem 12 *If $n \geq 6$, there exists an n-vertex tree T for which $i_{r3}(T) = n$.*

Proof Let P_4 be a path with $V(P_4) = \{v_1, x, v_3, v_4\}$ and $E(P_4) = \{v_1x, xv_3, v_3v_4\}$, and P_3 be a path with $V(P_3) = \{x, v_2, v_3\}$ and $E(P_3) = \{xv_2, v_2v_3\}$. Let $\mathcal{L}_k$ be the set of graphs each of which consists of the disjoint union of k copies of P_4 or P_3 plus a path through the vertex x of these copies. Figure 10 shows a graph in $\mathcal{L}_5$.

It can be seen that each tree T in $\mathcal{L}_k$ satisfies that $i_{r3}(T) = n$. By adding some P_3s or P_4s to a path, we can obtain such a tree of any order n for $n \geq 6$. □

6 Conclusion

In this paper, we investigate the complexity of the independent t-rainbow domination problem. More precisely, we prove that the independent t-rainbow domination problem is **NP**-complete if the input is a bipartite graph or planar graph. Moreover, considering generalized Petersen graphs and tori, the exact values of $i_{r2}(P(n, 1))$ and $i_{r2}(T_{3,n})$ are determined, and upper bounds for $i_{r2}(P(n, 2))$ and $i_{r2}(P(n, 3))$ are given. Proving closed expressions of independent t-rainbow domination of the torus $T_{n,m}$ and the Petersen graph $P(n, k)$ for larger t, n and k remain a challenge for further work.

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