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Acyclic Edge Coloring of 4-Regular Graphs Without 3-Cycles

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Abstract A proper edge coloring is called acyclic if no bichromatic cycles are produced. It was conjectured that every simple graph G with maximum degree Δ is acyclically edge-$(\Delta + 2)$-colorable. Basavaraju and Chandran (J Graph Theory 61:192–209, 2009) confirmed the conjecture for non-regular graphs G with $\Delta = 4$. In this paper, we extend this result by showing that every 4-regular graph G without 3-cycles is acyclically edge-6-colorable.

Keywords Acyclic edge coloring · 4-Regular graph · Cycle

Mathematics Subject Classification 05C15

1 Introduction

Only simple graphs are considered in this paper. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A *proper edge-k-coloring* is a mapping $c : E(G) \to \{1, 2, \ldots, k\}$

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such that any two adjacent edges receive different colors. The *chromatic index*, denoted $\chi'(G)$, of G is the smallest integer k such that G is edge-k-colorable. A proper edge coloring of G is called *acyclic* if there are no bichromatic cycles in G, i.e., the union of any two color classes induces a subgraph of G that is a forest. The *acyclic chromatic index* of G, denoted $a'(G)$, is the smallest integer k such that G is acyclically edge-k-colorable.

Let $\Delta(G)$ (for short, Δ) denote the maximum degree of a graph G. By Vizing's Theorem [19], $\Delta \leq \chi'(G) \leq \Delta+1$. Obviously it holds trivially that $a'(G) \geq \chi'(G) \geq \Delta$. Fiamčik [10], and later Alon et al. [1] made the following conjecture.

Conjecture 1 *For any graph G, $a'(G) \leq \Delta + 2$.*

Using probabilistic method, Alon et al. [2] proved that $a'(G) \leq 64\Delta$ for any graph G. This bound has been recently improved to that $a'(G) \leq 16\Delta$ in [13], that $a'(G) \leq \lceil 9.62(\Delta - 1) \rceil$ in [14], and that $a'(G) \leq 4\Delta$ in [9]. The acyclic edge coloring of some classical classes of graphs has been extensively investigated, including cubic graphs [3,4,18], outerplanar graphs [12], K_4-minor free graphs [20], 2-degenerate graphs [6], and planar graphs [7,8,11,15,21,23]. In particular, Conjecture 1 was confirmed for planar graphs without i-cycles for any fixed $i \in \{3, 4, 5, 6\}$ (see [16,17,22,24]). It was shown in [7] that every planar graph G has $a'(G) \leq \Delta + 12$. This bound was furthermore improved to $\Delta + 7$ [23], and then to $\Delta + 6$ [25].

Basavaraju and Chandran [5] showed that if G is a graph with $\Delta = 4$ which is not regular, then $a'(G) \leq 6$. In this paper, we are going to prove that every 4-regular graph G without triangles has $a'(G) \leq 6$, which extends, to some extent, the result of [5]. Here a *triangle* is synonymous with a 3-cycle.

2 Main Results

This section is devoted to investigate the acyclic edge coloring of triangle-free 4-regular graphs. Before establishing our main result, we need to introduce some notation.

Assume that c is a partial acyclic edge-k-coloring of a graph G using the color set $C = \{1, 2, \ldots, k\}$. For a vertex $v \in V(G)$, we use $C(v)$ to denote the set of colors assigned to the edges incident to v under c. If the edges of a cycle $ux \cdots vu$ are alternately colored with the colors i and j, then we call it an $(i, j)_{(u,v)}$-*cycle*. If the edges of a path $ux \cdots v$ are alternately colored with the colors i and j, then we call it an $(i, j)_{(u,v)}$-*path*. For simplicity, we use $\{e_1, e_2, \ldots, e_m\} \to a$ to express that all the edges $e_1, e_2, \ldots, e_m$ are colored or recolored with same color a. In particular, if $m = 1$, we write simply $e_1 \to a$. Moreover, we use $(e_1, e_2, \ldots, e_m)_c = (a_1, a_2, \ldots, a_m)$ to denote that $c(e_i) = a_i$ for $i = 1, 2, \ldots, m$. Let $(e_1, e_2, \ldots, e_n) \to (b_1, b_2, \ldots, b_n)$ denote that e_i is colored or recolored with the color b_i for $i = 1, 2, \ldots, n$.

Now we need to establish a useful lemma, which will be frequently used in the following.

Lemma 1 *Suppose that a graph G has an edge-6-coloring c. Let $P = uv_1v_2 \cdots v_kv_{k+1}$ be a maximal $(a, b)_{(u,v_{k+1})}$-path in G with $c(uv_1) = a$ and $b \notin C(u)$. If $w \notin V(P)$, then there does not exist a $(a, b)_{(u,w)}$-path in G under c.*

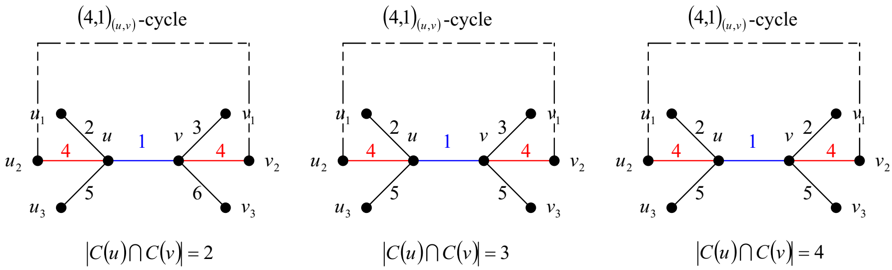

Fig. 1 Three cases on $|C(u) \cap C(v)|$

Proof Otherwise, assume that there is an $(a, b)_{(u,w)}$-path $Q = uw_1 \cdots w_m w$ in G with $m \geq 1$. Note that some w_i may be identical to some v_j, $1 \leq i \leq m$, $1 \leq j \leq k+1$. Since $b \notin C(u)$ and $w \notin V(P)$, it is easy to see that there exist two vertices v_t and w_s such that $v_t v_{t+1}$ and $v_t w_s$ have the same color a or b, which contradicts the fact that c is a proper edge coloring. □

Theorem 1 *If G is a triangle-free* 4*-regular graph, then* $a'(G) \leq 6$.

Proof By Vizing's theorem [19], G admits a proper edge-6-coloring c using the color set $C = \{1, 2, \ldots, 6\}$. Let $\tau(c)$ denote the number of bichromatic cycles in G with respect to the coloring c. If $\tau(c) = 0$, then c is an acyclic edge-6-coloring of G. Otherwise, $\tau(c) > 0$. Let B be a bichromatic cycle of G. We are going to show that there is a proper edge-6-coloring c', formed from c by recoloring suitably some edges of G, such that B is no longer bichromatic and no new bichromatic cycles are produced. Namely, $\tau(c') < \tau(c)$. By repeating this process, we finally obtain an acyclic edge-6-coloring of G.

To arrive at our conclusion, assume, w.l.o.g., that B is a $(4, 1)_{(u,v)}$-cycle with $c(uv) = 1$. Let u_1, u_2, u_3 be the neighbors of u different from v, and v_1, v_2, v_3 be the neighbors of v different from u. Since G contains no triangles, $u_1, u_2, u_3, v_1, v_2, v_3$ are pairwise distinct. Moreover, we may assume that $c(uu_2) = c(vv_2) = 4$. Since $2 \leq |C(u) \cap C(v)| \leq 4$, three cases are needed to be considered, as shown in Fig. 1. It should be explained that, in the following figures, solid points are pairwise distinct, whereas the others may be identical.

Case 1 $|C(u) \cap C(v)| = 2$, say $(uu_1, uu_3)_c = (2, 5)$ and $(vv_1, vv_3)_c = (3, 6)$.

By symmetry, we may assume that $C(u_2) \in \{\{1, 4, 3, 6\}, \{1, 4, 2, 6\}, \{1, 4, 2, 5\}\}$.

Case 1.1 $C(u_2) = \{1, 4, 3, 6\}$.

Assume that $3 \notin C(u_1) \cup C(u_3)$. If G contains no $(4, 3)_{(u_1,u_2)}$-path, let $(uu_1, uv) \to (3, 2)$. Otherwise, G contains a $(4, 3)_{(u_1,u_2)}$-path. Let $(uu_3, uv) \to (3, 5)$ and thus no $(4, 3)_{(u,u_3)}$-cycles exist by Lemma 1. So assume that $3, 6 \in C(u_1) \cup C(u_3)$.

Assume that $1 \notin C(u_1)$. Let $uu_1 \to 1$. If G contains no $(1, 5)_{(u_1,u_3)}$-path, let $uv \to 2$. If $2 \notin C(u_3)$, let $(uu_3, uv) \to (2, 5)$. Assume that $4 \notin C(u_1)$. Let $uu_1 \to 4$. If G contains no $(4, 5)_{(u_1,u_3)}$-path, let $uu_2 \to 2$. If $2 \notin C(u_3)$, let $(uu_2, uu_3) \to (5, 2)$. By Lemma 1, no new bichromatic cycles are produced. Otherwise, we derive the following.

Claim 1 (a) $\{1,4,3,6\} \subseteq C(u_1) \cup C(u_3)$.
(b) If $\{1,4\} \backslash C(u_1) \neq \emptyset$, then G contains a $(5,i)_{(u_1,u_3)}$-path for each $i \in \{1,4\} \backslash C(u_1)$, $5 \in C(u_1)$, $2 \in C(u_3)$, and $\{1,4\} \backslash C(u_1) \subseteq C(u_3)$.
(c) If $\{1,4\} \backslash C(u_3) \neq \emptyset$, then G contains a $(2,i)_{(u_1,u_3)}$-path for each $i \in \{1,4\} \backslash C(u_3)$, $5 \in C(u_1)$, $2 \in C(u_3)$, and $\{1,4\} \backslash C(u_3) \subseteq C(u_1)$.

It follows that $5 \in C(u_1)$ and $2 \in C(u_3)$ if $i \notin C(u_1) \cap C(u_3)$ for some $i \in \{1,4\}$. Suppose $1 \in C(u_1)$ and let x_1, x_2, x_3 ($\neq u$) be the other neighbors of u_1 with $c(u_1x_1) = 1$. So it suffices to consider the following three subcases.

Case 1.1.1 $C(u_1) = \{2,1,4,3\}$ and $C(u_3) = \{5,1,4,6\}$.

If $1 \notin C(v_1)$, let $(uu_2, uu_3) \to (5,3)$. If $1 \notin C(v_3)$, let $(uu_2, uu_1) \to (2,6)$. If G contains no $(4,6)_{(u_2,u_1)}$-path, let $(uu_1, uv) \to (6,2)$. If G contains no $(4,3)_{(u_2,u_3)}$-path, let $(uu_3, uv) \to (3,5)$. Otherwise, we may further assume that $1 \in C(v_1) \cap C(v_3)$, and G contains a $(4,6)_{(u_2,u_1)}$-path, and a $(4,3)_{(u_2,u_3)}$-path that do not pass through v_2.

Suppose that we can recolor some edges in $\{vv_1, vv_2, vv_3\}$ such that $C(u) \cap C(v) \in \{\{1,2\},\{1,5\},\{1,4\},\{1,2,4\},\{1,5,4\}\}$ with $c(vv_2) \neq 4$, and no new bichromatic cycles are yielded in $G-uv$. By Lemma 1, B is no longer bichromatic even if $4 \in C(v)$. If G contains a $(1,i)_{(u,v)}$-cycle for some $i \in \{2,5\}$, then let $(uu_1, uu_3) \to (5,2)$ and thus no new bichromatic cycles are produced. Notice that the following assertion holds automatically.

($*_1$) *If some of vv_1, vv_2, vv_3 can be recolored such that $C(u) \cap C(v) \in \{\{1,2\},\{1,5\},\{1,4\},\{1,2,4\},\{1,5,4\}\}$ with $c(vv_2) \neq 4$, and no new bichromatic cycles are produced in $G - uv$, then the proof is complete.*

- Assume that $C(v_2) = \{4,1,2,5\}$.

By the previous discussion, it suffices to assume that $4,2,5 \in C(v_1) \cup C(v_3)$. Hence, $\{3,6\} \backslash ((C(v_1) \backslash \{c(vv_1)\}) \cup (C(v_3) \backslash \{c(vv_3)\})) \neq \emptyset$ and, w.l.o.g., assume that $3 \notin C(v_3)$. Let $(vv_3, vv_2) \to (i,6)$, where $i \in \{4,2,5\} \backslash C(v_3)$, and so we are done by ($*_1$).

- Assume that $\{2,5\} \backslash C(v_2) \neq \emptyset$ and, w.l.o.g., assume that $2 \notin C(v_2)$.

If G contains neither $(2,3)_{(v_1,v_2)}$-path nor $(2,6)_{(v_2,v_3)}$-path, let $vv_2 \to 2$ and so we are done by ($*_1$). Otherwise, G contains a $(2,3)_{(v_1,v_2)}$-path or a $(2,6)_{(v_2,v_3)}$-path. If $2 \notin C(v_1)$, let $(vv_1, uv) \to (2,3)$; if $2 \notin C(v_3)$, let $(vv_3, uv) \to (2,6)$. Since G contains a $(4,6)_{(u_2,u_1)}$-path and a $(4,3)_{(u_2,u_3)}$-path, we are done by Lemma 1. Thus, $2 \in C(v_1) \cap C(v_3)$ and $5 \in C(v_1) \cap C(v_3)$ if $5 \notin C(v_2)$. By symmetry, we have to consider the following three possibilities.

(i) $C(v_2) = \{4,1,3,6\}$. It follows that $2,5 \in C(v_1) \cap C(v_3)$, i.e., $C(v_1) = \{3,1,2,5\}$ and $C(v_3) = \{6,1,2,5\}$. If G contains no $(2,3)_{(v_1,v_2)}$-path, let $(vv_2, vv_3) \to (2,4)$. Otherwise, let $(vv_1, vv_2, vv_3) \to (4,2,3)$ and we are done by ($*_1$).

(ii) $C(v_2) = \{4,1,5,3\}$. Then G contains a $(2,3)_{(v_1,v_2)}$-path, and $2 \in C(v_1) \cap C(v_3)$. If $4 \notin C(v_1)$ and G contains no $(4,6)_{(v_1,v_3)}$-path, let $(vv_1, vv_2) \to (4,2)$. Otherwise, $4 \in C(v_1)$ or G contains a $(4,6)_{(v_1,v_3)}$-path. If $C(v_1) = \{3,1,2,4\}$, then let $vv_2 \to 6$, and let $vv_3 \to 5$ if $5 \notin C(v_3)$, or $vv_3 \to 4$ if $C(v_3) = \{6,1,2,5\}$. Otherwise, $4 \notin C(v_1)$ and G contains a $(4,6)_{(v_1,v_3)}$-path, i.e., $C(v_1) = \{3,1,2,6\}$ and $C(v_3) = \{6,1,2,4\}$. Let $(vv_2, uv) \to (2,3)$. If G contains no $(5,2)_{(v_1,v_2)}$-path, then let $vv_1 \to 5$; otherwise, $(vv_1, vv_3) \to (4,5)$.

(iii) $C(v_2) = \{4, 1, 5, 6\}$. The discussion is similar to (ii).

Case 1.1.2 $C(u_1) = \{2, 5, 1, 3\}$ and $C(u_3) = \{5, 2, 4, 6\}$ with $c(u_1x_2) = 3$.

Note that G contains a $(1, 2)_{(u_1,u_3)}$-path, and $2 \in C(x_1)$. If G contains no $(5, 6)_{(u_1,u_3)}$-path, let $(uu_1, uv) \to (6, 2)$. Otherwise, G contains a $(5, 6)_{(u_1,u_3)}$-path which does not pass through u_2. If G contains no $(3, 2)_{(x_2,u_3)}$-path, let $(uu_2, uu_3) \to (5, 3)$. Otherwise, G contains a $(3, 2)_{(x_2,u_3)}$-path which does not pass through v and $2 \in C(x_2)$. If $\{4, 6\}\backslash C(x_i) \neq \emptyset$ for some $i \in \{1, 2\}$, let $(uu_2, uu_3, uv) \to (5, c(u_1x_i), 2)$ and $uu_1 \to \alpha \in \{4, 6\}\backslash C(x_i)$. Otherwise, $C(x_1) = \{1, 2, 4, 6\}$, $C(x_2) = \{3, 2, 4, 6\}$, we let $(u_1x_1, u_1x_2, uu_3, uv) \to (3, 1, 1, 5)$. Obviously, no $(1, 2)_{(u,u_1)}$-cycle exists by Lemma 1.

Case 1.1.3 $C(u_1) = \{2, 5, 1, 4\}$ and $C(u_3) = \{5, 2, 3, 6\}$ with $c(u_1x_2) = 4$.

By Claim 1, we see that $2 \in C(x_1) \cap C(x_2)$, and G contains a $(1, 2)_{(u_1,u_3)}$-path and a $(4, 2)_{(u_1,u_3)}$-path. Assume that $3 \notin C(x_2)$. Let $(uu_1, uv) \to (3, 2)$. If G contains no $(3, 5)_{(u_1,u_3)}$-path, we are done. Otherwise, we let $(uu_2, uu_3) \to (5, 4)$. So assume that $3 \in C(x_2)$ and $6 \in C(x_2)$ similarly. Hence, $C(x_2) = \{4, 2, 3, 6\}$. Assume that $3 \notin C(x_1)$. Let $uu_1 \to 3$. If G contains no $(3, 4)_{(u_1,u_2)}$-path, let $(uu_3, uv) \to (1, 2)$; otherwise, let $(uu_2, uu_3) \to (2, 4)$. Now assume that $3, 6 \in C(x_1)$ and $C(x_1) = \{1, 2, 3, 6\}$. Let $(u_1x_1, u_1x_2, uu_3, uv) \to (4, 1, 1, 5)$. Clearly, no $(1, 2)_{(u,u_1)}$-cycle exists by Lemma 1.

Case 1.2 $C(u_2) = \{1, 4, 2, 6\}$.

If $C(v_2) = \{1, 4, 2, 5\}$, the proof can be reduced to Case 1.1. If G contains no $(i, 2)_{(u_1,u_3)}$-path for some $i \in \{1, 3\}\backslash C(u_3)$, then let $(uu_3, uv) \to (i, 5)$. Otherwise, we may do the following assumption.

Claim 2 (a) $\{2, 5\}\backslash C(v_2) \neq \emptyset$.
(b) $1, 3 \in C(u_3)$, or G contains a $(i, 2)_{(u_1,u_3)}$-path for any $i \in \{1, 3\}\backslash C(u_3)$.
(c) $1, 3 \in C(u_3) \cup C(u_1)$, and $2 \in C(u_3)$ if $\{1, 3\}\backslash C(u_3) \neq \emptyset$.

Case 1.2.1 G contains no $(2, 3)_{(u_1,u_2)}$-path.

If G contains no $(1, 3)_{(u_2,v_1)}$-path, let $uu_2 \to 3$. Otherwise, G contains a $(1, 3)_{(u_2,v_1)}$-path and $1 \in C(v_1)$. If $C(v_1) = \{3, 1, 2, 5\}$, then by Case 1.1, we can destroy that $(1, 3)_{(u,v)}$-cycle by setting $uu_2 \to 3$ and hence destroy B such that no new bichromatic cycles are produced. Thus, $\{2, 5\}\backslash C(v_1) \neq \emptyset$.

If there exists $a \in \{2, 5\}\backslash(C(v_1) \cup C(v_2))$, then let $(vv_1, uv) \to (a, 3)$ if G contains no $(6, a)_{(v_1,v_3)}$-path, and let $(vv_2, uu_2, uv) \to (a, 3, 4)$ otherwise. Thus, assume that $2, 5 \in C(v_1) \cup C(v_2)$. Recall that $\{2, 5\}\backslash C(v_i) \neq \emptyset$ for any $i \in \{1, 2\}$, say $2 \in C(v_1)\backslash C(v_2)$ and $5 \in C(v_2)\backslash C(v_1)$ (if $5 \in C(v_1)\backslash C(v_2)$ and $2 \in C(v_2)\backslash C(v_1)$, we let $uu_2 \to 3$ to give a similar discussion).

If G contains neither $(2, 3)_{(v_1,v_2)}$-path nor $(2, 6)_{(v_2,v_3)}$-path, let $(vv_2, uu_2, uv) \to (2, 3, 4)$. If G contains neither $(5, 4)_{(v_1,v_2)}$-path nor $(5, 6)_{(v_1,v_3)}$-path, let $(vv_1, uv) \to (5, 3)$. Otherwise, assume that G contains either a $(2, 3)_{(v_1,v_2)}$-path or a $(2, 6)_{(v_2,v_3)}$ path, and at the same time, either a $(5, 4)_{(v_1,v_2)}$-path or a $(5, 6)_{(v_1,v_3)}$-path.

- Assume that G contains a $(2, 3)_{(v_1,v_2)}$-path so that $C(v_2) = \{4, 1, 5, 3\}$.

If $\{1, 2\}\backslash C(v_3) \neq \emptyset$, let $uv \to 6$ and $vv_3 \to \alpha \in \{1, 2\}\backslash C(v_3)$. So $1, 2 \in C(v_3)$. Assume that $5 \notin C(v_3)$. Then G contains a $(5, 4)_{(v_1,v_2)}$-path. Let $(vv_3, uv) \to (5, 6)$

and we are done by Lemma 1. Otherwise, $C(v_3) = \{6, 1, 2, 5\}$. If $6 \notin C(v_1)$, let $(vv_1, vv_2, vv_3, uv) \to (6, 2, 4, 3)$. Otherwise, let $(vv_1, vv_3, uv) \to (5, 3, 6)$ and we are done since G contains a $(5, 6)_{(v_1,v_3)}$-path.

- Assume that G contains a $(2, 6)_{(v_2,v_3)}$-path so that $C(v_2) = \{4, 1, 5, 6\}$.

Then $2 \in C(v_3)$. Since G contains either a $(5, 4)_{(v_1,v_2)}$-path or a $(5, 6)_{(v_1,v_3)}$-path, we have to consider the following two possibilities:

(1) G contains a $(5, 6)_{(v_1,v_3)}$-path, i.e., $C(v_1) = \{3, 1, 2, 6\}$ and $5 \in C(v_3)$. If $4 \notin C(v_3)$, let $(vv_2, vv_3, uv) \to (2, 4, 6)$. Otherwise, $C(v_3) = \{6, 2, 5, 4\}$ and let $uv \to 6$. If G contains no $(4, 6)_{(u_2,v_2)}$-path, let $vv_3 \to 1$; Otherwise, let $(vv_1, vv_2, vv_3) \to (4, 2, 3)$.

(2) G contains a $(5, 4)_{(v_1,v_2)}$-path, i.e., $C(v_1) = \{3, 1, 2, 4\}$. If $4 \notin C(v_3)$, let $(vv_1, vv_2, vv_3) \to (6, 3, 4)$. If $5 \notin C(v_3)$, let $(vv_3, uu_2, uv) \to (5, 3, 6)$. Otherwise, $C(v_3) = \{6, 2, 4, 5\}$ and let $vv_3 \to 1$. Furthermore, if G contains no $(4, 6)_{(u_2,v_2)}$-path, let $uv \to 6$; otherwise, let $(vv_1, uv) \to (6, 3)$.

By Lemma 1, no new bichromatic cycles are produced.

Case 1.2.2 G contains a $(2, 3)_{(u_1,u_2)}$-path and $3 \in C(u_1)$.

If $3 \notin C(u_3)$, let $(uu_3, uu) \to (3, 5)$, so we are done by Lemma 1. Hence, $3 \in C(u_3)$.

- $2 \notin C(u_3)$.

Then $1 \in C(u_3)$ by Claim 2. If $4 \notin C(u_1)$, let $(uu_1, uu_2, uu_3) \to (4, 5, 2)$. Otherwise, $4 \in C(u_1)$. If $1 \notin C(u_1)$ and G contains no $(4, 2)_{(u_2,u_3)}$-path, let $(uu_1, uu_3, uv) \to (1, 2, 5)$. Otherwise, $1 \in C(u_1)$ or G contains a $(4, 2)_{(u_2,u_3)}$ path. Note that if G contains no $(4, 2)_{(u_2,u_3)}$-path, then it follows that $1 \in C(u_1)$. If $C(u_3) = \{5, 1, 3, 6\}$, then $C(u_1) = \{2, 3, 4, 1\}$ and let $(uu_2, uu_3) \to (5, 4)$. Otherwise, $C(u_3) = \{5, 1, 3, 4\}$. If G contains no $(4, 6)_{(u_2,u_3)}$-path, let $(uu_3, uv) \to (6, 5)$. Otherwise, G contains a $(4, 6)_{(u_2,u_3)}$-path. If $1, 5 \notin C(u_1)$, let $(uu_1, uv) \to (1, 2)$ and we are done because G contains a $(4, 2)_{(u_2,u_3)}$-path. Otherwise, $\{1, 5\} \cap C(u_1) \neq \emptyset$. Hence, $6 \notin C(u_1)$ and let $uu_1 \to 6$. If G contains no $(2, 4)_{(u_2,v_2)}$-path, let $uv \to 2$. Otherwise, G contains a $(2, 4)_{(u_2,v_2)}$-path and thus $1 \in C(u_1)$. Let $(uu_3, uv) \to (2, 5)$, and we are done by Lemma 1.

- $2 \in C(u_3)$.

Then $1 \in C(u_1) \cup C(u_3)$ by Claim 2. Assume that $6 \notin C(u_1) \cup C(u_3)$. If G contains no $(4, 6)_{(u_2,u_3)}$-path, let $(uu_3, uv) \to (6, 5)$; otherwise, G contains a $(4, 6)_{(u_2,u_3)}$-path and $C(u_3) = \{5, 2, 3, 4\}$. If G contains no $(5, 6)_{(u_3,v_1)}$-path, let $(uu_1, uu_3, uv) \to (6, 1, 5)$; otherwise, let $(uu_1, uu_2, uu_3, uv) \to (6, 5, 1, 2)$. Thus, $6 \in C(u_1) \cup C(u_3)$.

If $4 \notin C(u_1)$ and G contains no $(4, 5)_{(u_1,u_3)}$-path, let $(uu_1, uu_2, uv) \to (4, 3, 2)$. Otherwise, $4 \in C(u_1)$, or G contains a $(4, 5)_{(u_1,u_3)}$-path. Since $1, 6 \in C(u_1) \cup C(u_3)$, we derive that $4 \in C(u_1)$, and the proof splits into the following two subcases:

(1) $C(u_3) = \{5, 2, 3, 1\}$ and $C(u_1) = \{2, 3, 4, 6\}$. First, let $uv \to 2$. Next, if G contains no $(4, 2)_{(u_2,v_2)}$-path, then let $uu_1 \to 1$; otherwise $(uu_1, uu_2, uu_3) \to (5, 3, 4)$.

(2) $C(u_3) = \{5, 2, 3, 6\}$ and $C(u_1) = \{2, 3, 4, 1\}$. Let $(uu_1, uv) \to (5, 2)$. If G contains no $(4, 2)_{(u_2,v_2)}$-path, then let $uu_3 \to 1$; otherwise, let $(uu_2, uu_3) \to (3, 4)$.

Case 1.3 $C(u_2) = \{1, 4, 2, 5\}$.

By Cases 1.1 and 1.2, assume that $C(v_2) = \{1, 4, 3, 6\}$. If G contains no $(5, i)_{(u_1,u_3)}$-path for some $i \in \{1, 3, 6\}\backslash C(u_1)$, let $(uu_1, uv) \to (i, 2)$. If G contains no $(2, i)_{(u_1,u_3)}$-path for some $i \in \{1, 3, 6\}\backslash C(u_3)$, let $(uu_3, uv) \to (i, 5)$. Otherwise, we give the following assumption.

Claim 3 (a) $1, 3, 6 \in C(u_1)$, or G contains a $(5, i)_{(u_1,u_3)}$-path for each $i \in \{1, 3, 6\}\backslash C(u_1)$.
(b) $1, 3, 6 \in C(u_3)$, or G contains a $(2, i)_{(u_1,u_3)}$-path for each $i \in \{1, 3, 6\}\backslash C(u_3)$.
(c) $1, 2, 5 \in C(v_3)$, or G contains a $(3, i)_{(v_1,v_3)}$-path for each $i \in \{1, 2, 5\}\backslash C(v_3)$.
(d) $1, 2, 5 \in C(v_1)$ or G contains a $(6, i)_{(v_1,v_3)}$-path for each $i \in \{1, 2, 5\}\backslash C(v_1)$.
(e) $1, 3, 6 \in C(u_1) \cup C(u_3)$ and $1, 2, 5 \in C(v_1) \cup C(v_3)$.
(f) $5 \in C(u_1)$ if $\{1, 3, 6\}\backslash C(u_1) \neq \emptyset$, $2 \in C(u_3)$ if $\{1, 3, 6\}\backslash C(u_3) \neq \emptyset$, $6 \in C(v_1)$ if $\{1, 2, 5\}\backslash C(v_1) \neq \emptyset$, and $3 \in C(v_3)$ if $\{1, 2, 5\}\backslash C(v_3) \neq \emptyset$.

Case 1.3.1 $5 \notin C(u_1)$.

It is easy to see that $C(u_1) = \{2, 1, 3, 6\}$ by Claim 3. If $\{1, 3, 6\}\backslash C(u_3) \neq \emptyset$, let $(uu_1, uv) \to (5, 2)$, $uu_3 \to \alpha \in \{1, 3, 6\}\backslash C(u_3)$, so the proof is complete by Claim 3 and Lemma 1. Otherwise, $C(u_3) = \{5, 1, 3, 6\}$. If G contains neither $(2, 3)_{(v_1,v_2)}$-path nor $(2, 6)_{(v_2,v_3)}$-path, let $vv_2 \to 2$. If G contains no $(1, 2)_{(u_1,v_2)}$-path, we are done. Otherwise, let $(uu_1, uu_3) \to (5, 2)$. Or else, assume, w.l.o.g., that G contains a $(2, 3)_{(v_1,v_2)}$-path. Let $(uu_2, uv) \to (3, 2)$. If G contain no $(3, 5)_{(u_2,u_3)}$-path, let $uu_1 \to 4$; otherwise, let $(uu_1, uu_3) \to (5, 4)$.

Case 1.3.2 $5 \in C(u_1)$.

Furthermore, assume that $2 \in C(u_3)$, $3 \in C(v_3)$, and $6 \in C(v_1)$. Since $1, 3, 6 \in C(u_1) \cup C(u_3)$ by Claim 3, we may suppose that $1 \in C(u_1)$, $6 \notin C(u_1)$, and it suffices to consider the following two subcases. Note that G contains a $(5, 6)_{(u_1,u_3)}$-path in this case.

- $C(u_1) = \{2, 5, 1, 3\}$, $6 \in C(u_3)$ and $4 \notin C(u_3)$.

Let $uu_2 \to 6$. If $\{1, 2\}\backslash C(v_3) \neq \emptyset$, let $uu_1 \to 4$ and $uv \to \alpha \in \{1, 2\}\backslash C(v_3)$. Otherwise, $C(v_3) = \{3, 6, 1, 2\}$ and let $(uu_3, uv) \to (4, 5)$.

- $C(u_1) = \{2, 5, 1, 3\}$ and $C(u_3) = \{5, 2, 6, 4\}$; or $C(u_1) = \{2, 5, 1, 4\}$ and $C(u_3) = \{5, 2, 3, 6\}$. By Claim 3, G contains a $(2, 3)_{(u_1,u_3)}$-path if $C(u_1) = \{2, 5, 1, 3\}$ and a $(5, 3)_{(u_1,u_3)}$-path if $C(u_1) = \{2, 5, 1, 4\}$. If $1 \notin C(v_1)$, let $uu_2 \to 3$. If $1 \notin C(v_3)$, let $uu_2 \to 6$. Otherwise, it follows that $1 \in C(v_1) \cap C(v_3)$. Since $2, 5 \in C(v_1) \cup C(v_3)$, we have to handle two possibilities: If $C(v_1) = \{3, 6, 1, 2\}$ and $C(v_3) = \{3, 6, 1, 5\}$, let $(vv_2, vv_3) \to (5, 4)$. Otherwise, $C(v_1) = \{3, 6, 1, 5\}$, $C(v_3) = \{3, 6, 1, 2\}$, and let $(vv_1, vv_2) \to (4, 5)$.

Case 2 $|C(u) \cap C(v)| = 3$, say $(uu_1, uu_3)_c = (2, 5)$ and $(vv_1, vv_3)_c = (3, 5)$.

If $C(u_2) = \{1, 4, 3, 6\}$ or $C(v_2) = \{1, 4, 2, 6\}$, then the proof can be reduced to Case 1. If G contains neither $(4, 6)_{(u_2,v_2)}$-path nor $(5, 6)_{(u_3,v_3)}$-path, let $uv \to 6$. Suppose that we can recolor some edges such that $C(u) \cap C(v) = \{i, j\}$ and no new bichromatic cycles are produced in $G - uv$. If G contains an $(i, j)_{(u,v)}$-cycle, then by Case 1, we can destroy this $(i, j)_{(u,v)}$-cycle as well as B such that no new bichromatic cycles are produced. Thus, we have the following claim.

Claim 4 (a) $C(u_2) \neq \{1,4,3,6\}$ and $C(v_2) \neq \{1,4,2,6\}$.
(b) G contains either $(4,6)_{(u_2,v_2)}$-path or a $(5,6)_{(u_3,v_3)}$-path.
(c) If we can recolor some edges of G such that $|C(u) \cap C(v)| = 2$, and no new bichromatic cycles are produced in $G - uv$, then we are done.

If $6 \notin C(u_3)$ and G contains no $(2,6)_{(u_1,u_3)}$-path, let $uu_3 \to 6$; if $1 \notin C(u_3)$ and G contains no $(1,2)_{(u_1,u_3)}$-path, let $(uu_3, uv) \to (1,6)$. Clearly, $|C(u) \cap C(v)| = 2$ and no new bichromatic cycles are produced in $G - uv$. By Claim 4, we are done. Assume that $6 \notin C(u_2)$. If G contains no $(2,6)_{(u_1,u_2)}$-path, let $uu_2 \to 6$; If $4 \notin C(u_1)$ and G contains no $(4,5)_{(u_1,u_3)}$-path, let $(uu_1, uu_2) \to (4,6)$ and we are done by Lemma 1. Hence, we have the following:

Claim 5 (a) $1,6 \in C(u_3)$, or G contains a $(2,i)_{(u_1,u_3)}$-path for $i \in \{1,6\}\backslash C(u_3)$, $2 \in C(u_3)$.
(b) $1,6 \in C(u_1) \cup C(u_3)$ and if $2 \notin C(u_3)$, then $1,6 \in C(u_3)$.
(c) $1,6 \in C(v_3)$, or G contains a $(3,i)_{(v_1,v_3)}$-path for $i \in \{1,6\}\backslash C(v_3)$, $3 \in C(v_3)$.
(d) $1,6 \in C(v_1) \cup C(v_3)$ and if $3 \notin C(v_3)$, then $1,6 \in C(v_3)$.
(e) If $6 \notin C(u_2) \cap C(v_2)$, then G contains a $(5,6)_{(u_3,v_3)}$-path and $6 \in C(u_3) \cap C(v_3)$.
(f) $\{2,6\} \cap C(u_2) \neq \emptyset$, and $6 \in C(u_2)$, or (i) G contains a $(2,6)_{(u_1,u_2)}$-path, $2 \in C(u_2)$, $6 \in C(u_1)$ and (ii) $4 \in C(u_1)$, or G contains a $(4,5)_{(u_1,u_3)}$-path and $5 \in C(u_1)$, $4 \in C(u_3)$.
(g) $\{3,6\} \cap C(v_2) \neq \emptyset$, and $6 \in C(v_2)$, or (i) G contains a $(3,6)_{(v_1,v_2)}$-path, $3 \in C(v_2)$, $6 \in C(v_1)$ and (ii) $4 \in C(v_1)$, or G contains a $(4,5)_{(v_1,v_3)}$-path and $5 \in C(v_1)$, $4 \in C(v_3)$.

We only need to suppose that $C(u_2)\backslash\{1,4\} \in \{\{2,3\},\{2,5\},\{2,6\},\{5,6\}\}$ and $C(v_2)\backslash\{1,4\} \in \{\{2,3\},\{3,5\},\{3,6\},\{5,6\}\}$.

Case 2.1 $C(u_2)\backslash\{1,4\} \in \{\{2,3\},\{2,5\}\}$, and $2 \notin C(v_2)$ if $C(u_2) = \{4,1,2,5\}$.

Note that $6 \notin C(u_2)$. By Claim 5, G contains a $(5,6)_{(u_3,v_3)}$-path with $6 \in C(u_3) \cap C(v_3)$, and G contains a $(2,6)_{(u_1,u_2)}$-path with $6 \in C(u_1)$, and $4 \in C(u_1)$, or G contains a $(4,5)_{(u_1,u_3)}$-path with $5 \in C(u_1)$, $4 \in C(u_3)$. Thus, we need to consider the following two subcases.

Case 2.1.1 $4 \notin C(u_1)$, and $\{2,5,6\} \subseteq C(u_1)$, $\{5,4,6\} \subseteq C(u_3)$.

If $2 \notin C(u_3)$, let $(uu_1, uu_2, uu_3) \to (4,6,2)$. Otherwise, $C(u_3) = \{5,6,4,2\}$ and then $C(u_1) = \{2,6,5,1\}$ since $1 \in C(u_1) \cup C(u_3)$ by Claim 5. If $2 \notin C(v_2)$, let $(uu_1, uu_3, uv) \to (3,1,2)$. Otherwise, $2 \in C(v_2)$ and $6 \notin C(v_2)$. Recall that G contains a $(3,6)_{(v_1,v_2)}$-path by Claim 5. Let $(uu_1, uu_3, uv) \to (3,1,6)$ and then we are done by Lemma 1.

Case 2.1.2 $4 \in C(u_1)$ and $\{2,6,4\} \subseteq C(u_1)$, $\{5,6\} \subseteq C(u_3)$

Assume that $1 \notin C(u_3)$. Then $C(u_1) = \{2,6,1,4\}$ by Claim 5. If $5 \notin C(u_2)$, let $(uu_1, uu_3, uv) \to (5,1,6)$ and no $(5,6)_{(u,u_1)}$-cycle exists by Claim 5 and Lemma 1. Otherwise, $C(u_2) = \{4,1,2,5\}$ and $2 \notin C(v_2)$. If G contains no $(1,3)_{(u_1,u_3)}$ path, let $(uu_1, uu_3, uv) \to (3,1,2)$; otherwise, G contains a $(1,3)_{(u_1,u_3)}$-path and let $(uu_1, uu_2, uu_3, uv) \to (5,3,1,6)$. Now assume that $1 \in C(u_3)$ and $5,6,1 \in C(u_3)$.

- Assume that $C(u_2) = \{4,1,2,3\}$.

If $2 \notin C(u_3)$ and G contains no $(2,4)_{(u_2,u_3)}$-path, let $(uu_3, uv) \to (2,6)$ and $uu_1 \to \alpha \in \{1,5\}\backslash C(u_1)$. By Claim 5 and Lemma 1, no new bichromatic cycles are produced. Otherwise, $2 \in C(u_3)$ or G contains a $(2,4)_{(u_2,u_3)}$-path and $4 \in C(u_3)$. First assume that $C(u_3) = \{5,1,6,4\}$. Then G contains a $(2,4)_{(u_2,u_3)}$-path. If $1, 5 \notin C(u_1)$, let $(uu_1, uv) \to (1,2)$. Otherwise, $\{1,5\} \cap C(u_1) \neq \emptyset$ and $3 \notin C(u_1)$. Then let $uv \to 2$, and let $uu_1 \to 3$ if G contains no $(4,3)_{(u_1,u_2)}$-path, or let $uu_3 \to 3$, $uu_1 \to \beta \in \{1,5\} \backslash C(u_1)$ otherwise.

Next assume that $C(u_3) = \{5,1,6,2\}$. If $C(u_1) = \{2,6,4,3\}$, let $(uu_1, uu_2, uu_3) \to (5,6,4)$. Otherwise, $3 \notin C(u_1)$. If G contains no $(3,6)_{(u_3,v_1)}$-path, let $(uu_3, uv) \to (3,6)$. Otherwise, G contains a $(3,6)_{(u_3,v_1)}$-path. Note that G contains a $(3,6)_{(v_1,v_2)}$ path if $6 \notin C(v_2)$ by Claim 5. Hence, $6 \in C(v_2)$ and then $2 \notin C(v_2)$ by Claim 4. If $C(u_1) = \{2,6,4,1\}$, let $(uu_1, uu_2, uu_3, uv) \to (5,6,4,2)$. Otherwise, $C(u_1) = \{2,6,4,5\}$. If G contains no $(2,3)_{(u_3,v_1)}$-path, let $(uu_1, uu_3, uv) \to (1,3,2)$. Otherwise, G contains a $(2,3)_{(u_3,v_1)}$-path and $2 \in C(v_1)$. If G contains no $(2,5)_{(v_2,v_3)}$-path, let $vv_2 \to 2$. Otherwise, G contains a $(2,5)_{(v_2,v_3)}$-path and $2 \in C(v_3)$, $C(v_2) = \{4,1,6,5\}$. By Claim 5, we have $C(v_3) = \{5,6,2,1\}$, or $C(v_3) = \{5,6,2,3\}$ and $C(v_1) = \{3,2,6,1\}$. Then let $(vv_2, vv_3) \to (2,4)$ and we are done.

- Assume that $C(u_2) = \{4,1,2,5\}$ and $2 \notin C(v_2)$.

First assume that $3 \notin C(u_1)$. If G contains no $(3,5)_{(u_1,u_3)}$-path and $2 \notin C(u_3)$, let $(uu_1, uv) \to (3,2)$. Otherwise, G contains a $(3,5)_{(u_1,u_3)}$-path or $2 \in C(u_3)$. If $C(u_3) = \{5,6,1,2\}$, let $(uu_1, uu_2, uu_3, uv) \to (3,6,4,2)$. Otherwise, $C(u_1) = \{2,6,4,5\}$, $C(u_3) = \{5,6,1,3\}$ and let $(uu_1, uu_3, uv) \to (1,2,6)$. Next assume that $C(u_1) = \{2,6,4,3\}$.

If G contains no $(2,5)_{(u_3,v_3)}$-path, let $(uu_1, uv) \to (1,2)$. So G contains a $(2,5)_{(u_3,v_3)}$-path and $2 \in C(u_3) \cap C(v_3)$. If G contains no $(2,3)_{(v_1,v_2)}$-path, let $vv_2 \to 2$. Thus, G contains a $(2,3)_{(v_1,v_2)}$-path and $2 \in C(v_1)$, $3 \in C(v_2)$. Note that $\{1,3\} \cap C(v_3) \neq \emptyset$ by Claim 5 and hence $4 \notin C(v_3)$. If $4 \notin C(v_1)$, let $(vv_1, vv_2) \to (4,2)$. Otherwise, $4 \in C(v_1)$. If $1 \notin C(v_1)$, then $C(v_3) = \{5,6,2,1\}$ and let $(vv_1, vv_3, uv) \to (1,3,6)$. Now assume that $C(v_1) = \{3,2,4,1\}$. If $6 \notin C(v_2)$, let $vv_2 \to 6$. Otherwise, $C(v_2) = \{4,1,3,6\}$. Then let $(vv_1, uv) \to (5,6)$, $vv_3 \to \alpha \in \{1,3\} \backslash C(v_3)$ and we are done since $4 \notin C(v_3)$.

Case 2.2 $C(u_2) \backslash \{1,4\} \in \{\{2,6\},\{5,6\}\}$ and $C(v_2) \backslash \{1,4\} \in \{\{3,6\},\{5,6\}\}$.

Case 2.2.1 G contains no $(2,3)_{(u_1,u_2)}$-path and $(5,3)_{(u_2,u_3)}$-path.

First, we let $uu_2 \to 3$. If G contains no $(1,3)_{(u_2,v_1)}$-path, then we are done. Otherwise, G contains a $(1,3)_{(u_2,v_1)}$-path and $1 \in C(v_3)$ by Claim 5. If $6 \notin C(v_3)$, then G contains a $(4,6)_{(u_2,v_2)}$-path and a $(3,6)_{(v_1,v_3)}$-path by Claims 4 and 5, and let $uv \to 6$. Otherwise, $6 \in C(v_3)$. If $C(v_1) \backslash \{1,3\} \notin \{\{4,6\},\{5,6\}\}$, then by Case 1, Case 2.1 or Case 2.2, we can destroy this $(1,3)_{(u,v)}$-cycle as well as B so that no new bichromatic cycles are produced. Thus, assume that $C(v_1) \backslash \{1,3\} \in \{\{4,6\},\{5,6\}\}$ and then $2 \notin C(v_1) \cup C(v_2)$. If G contains no $(5,2)_{(v_2,v_3)}$-path and $4 \notin C(v_3)$, let $(vv_2, uv) \to (2,4)$. Otherwise, G contains a $(5,2)_{(v_2,v_3)}$-path or $C(v_3) = \{5,1,6,4\}$. Let $(uu_2, vv_1, uv) \to (4,2,3)$ so that $(5,2)_{(v_1,v_3)}$ exists by Lemma 1 even if $2 \in C(v_3)$.

Case 2.2.2 G contains a $(2,3)_{(u_1,u_2)}$-path or a $(5,3)_{(u_2,u_3)}$-path.

(2.2.2.1) G contains a $(2,3)_{(u_1,u_2)}$-path and $C(u_2) = \{4,1,2,6\}$, $3 \in C(u_1)$.

Suppose that we can recolor some edges in $\{vv_1, vv_2, vv_3\}$ such that $C(v) = \{2, 1, 3, 5\}$, or $C(v) = \{2, 1, 3, 4\}$ with $c(vv_2) \neq 4$, and no new bichromatic cycles are produced in $G - uv$. By Lemma 1, B does not exist even if $4 \in C(v)$. If G contains a $(1, 2)_{(u,v)}$-cycle, then we can destroy this $(1, 2)_{(u,v)}$-cycle by Cases 1 or 2.1 since $3 \in C(u_1)$. Next, we may give the following assertion.
$(*_2)$ *If we can recolor some edges in* $\{vv_1, vv_2, vv_3\}$ *such that* $C(v) = \{2, 1, 3, 5\}$, *or* $C(v) = \{2, 1, 3, 4\}$ *with* $c(vv_2) \neq 4$, *and no new bichromatic cycles, other than* $(1, 2)_{(u,v)}$*-cycle, are produced, then we are done.*

• $C(v_2) = \{4, 1, 5, 6\}$.

By Case 2.2.1, we may further assume that G contains a $(2, 5)_{(v_2,v_3)}$-path and $2 \in C(v_3)$. Assume that $2 \notin C(v_1)$. Let $vv_1 \to 2$. If G contains no $(5, 3)_{(u_3,v_3)}$-path, let $uv \to 3$; otherwise, let $vv_2 \to 3$ and we are done by $(*_2)$. So assume that $2 \in C(v_1)$. If $4 \notin C(v_3)$ and G contains no $(3, 4)_{(v_1,v_3)}$-path, let $(vv_3, vv_2) \to (4, 2)$ and we are done by $(*_2)$. Otherwise, $4 \in C(v_3)$ or G contains a $(3, 4)_{(v_1,v_3)}$-path. Together with Claim 5, we have $3 \in C(v_3)$, $1, 6, 4 \in C(v_1) \cup C(v_3)$ and then $\{1, 6\} \backslash C(v_3) \neq \emptyset$, $5 \notin C(v_1)$. Choose $\alpha \in \{1, 6\} \backslash C(v_3)$ and $\beta \in \{1, 6\} \backslash \{\alpha\}$. Then let $(vv_1, vv_2, vv_3, uv) \to (5, 3, \alpha, \beta)$ and we are done by Claims 4 and 5 and Lemma 1.

• $C(v_2) = \{4, 1, 3, 6\}$.

By Case 2.2.1, assume that G contains a $(2, 3)_{(v_1,v_2)}$-path and $2 \in C(v_1)$. If $3 \notin C(u_3)$ and $2 \notin C(v_3)$, let $(uu_3, vv_3, uv) \to (3, 2, 5)$. Otherwise, $3 \in C(u_3)$ or $2 \in C(v_3)$, and assume, w.l.o.g., that $2 \in C(v_3)$. If $4 \notin C(v_3)$ and $5 \notin C(v_1)$, let $(vv_1, vv_2, vv_3, uv) \to (5, 2, 4, 3)$. Otherwise, $4 \in C(v_3)$ or $5 \in C(v_1)$. Then there are the following two possibilities:

(1) $4 \in C(v_3)$. By Claim 5, we may assume that $C(v_3) = \{5, 2, 4, 3\}$ and $C(v_1) = \{3, 2, 1, 6\}$. It suffices to let $(vv_1, vv_3, uv) \to (5, 1, 3)$.
(2) $4 \notin C(v_3)$ and $5 \in C(v_1)$. If $4 \notin C(v_1)$ and G contains no $(5, 3)_{(u_3,v_3)}$-path, let $(vv_1, vv_2, uv) \to (4, 2, 3)$. Otherwise, $4 \in C(v_1)$ or G contains a $(5, 3)_{(u_3,v_3)}$-path. If $4 \in C(v_1)$, then $C(v_1) = \{3, 2, 5, 4\}$, $C(v_3) = \{5, 2, 1, 6\}$ and let $(vv_1, vv_3, uv) \to (1, 3, 6)$. Otherwise, $4 \notin C(v_1)$ and G contains a $(5, 3)_{(u_3,v_3)}$-path. Then let $(vv_1, vv_2, uv) \to (4, 5, 3)$ and $vv_3 \to \alpha \in \{1, 6\} \backslash C(v_3)$. By Claims 4 and 5 and Lemma 1, no bichromatic cycles are produced.

(2.2.2.2) G contains a $(3, 5)_{(u_2,u_3)}$-path and $C(u_2) = \{4, 1, 5, 6\}$, $3 \in C(u_3)$.

By the previous discussion, we may further assume that $C(v_2) = \{4, 1, 5, 6\}$ and G contains a $(2, 5)_{(v_2,v_3)}$-path with $2 \in C(v_3)$. If $2 \notin C(v_1)$, let $(vv_1, uv) \to (2, 3)$. Otherwise, $2 \in C(v_1)$. Note that G contains a $(3, i)_{(v_1,v_3)}$-path for any $i \in \{1, 6\} \backslash C(v_3)$ and $1, 6 \in C(v_1) \cup C(v_3)$ by Claim 5. First assume that $3 \notin C(v_3)$ and thus $C(v_3) = \{5, 2, 1, 6\}$. If $4 \notin C(v_1)$, let $(vv_1, vv_2) \to (4, 3)$; if $1 \notin C(v_1)$, let $(vv_1, vv_3, uv) \to (1, 3, 6)$; otherwise, $C(v_1) = \{3, 2, 4, 1\}$ and let $(vv_1, vv_2) \to (6, 3)$. Next assume that $3 \in C(v_3)$. Choose $\alpha \in \{1, 6\} \backslash C(v_3)$ and $\beta \in \{1, 6\} \backslash \{\alpha\}$, and then let $(vv_3, vv_2, uv) \to (\alpha, 3, \beta)$. If $5 \notin C(v_1)$, let $vv_1 \to 5$. Otherwise, $5 \in C(v_1)$ and it follows that $4 \notin C(v_1) \cup C(v_3)$ from $1, 6 \in C(v_1) \cup C(v_3)$. Let $vv_1 \to 4$. By Claims 4 and 5 and Lemma 1, we are done in each step.

Case 3 $|C(u) \cap C(v)| = 4$, say $(uu_1, uu_3)_c = (2, 5)$ and $(vv_1, vv_3)_c = (2, 5)$.

By Cases 1 and 2, we may assume that $C(u_2) = C(v_2) = \{1, 2, 4, 5\}$. If G contains neither $(2, j)_{(u,v)}$-path nor $(5, j)_{(u,v)}$-path for some $j \in \{3, 6\}$, let $uv \to j$. Otherwise, for any $j \in \{3, 6\}$, G contains an $(i, j)_{(u,v)}$-path for some $i \in \{2, 5\}$. W.l.o.g., assume that G contains a $(2, 3)_{(u,v)}$-path with $3 \in C(u_1)$. If G contains no $(3, 5)_{(u_2,u_3)}$-path, let $uu_2 \to 3$; if G contains no $(3, 5)_{(v_2,v_3)}$-path, let $vv_2 \to 3$. Otherwise, G contains a $(3, 5)_{(v_2,v_3)}$-path and a $(3, 5)_{(u_2,u_3)}$-path with $3 \in C(u_3)$. Similarly, assume that $6 \in C(u_1) \cap C(u_3)$. If $1 \notin C(u_1)$ and G contains no $(1, 5)_{(u_1,u_3)}$-path, let $(uu_1, uv) \to (1, 3)$. Otherwise, $1 \in C(u_1)$ or G contains a $(1, 5)_{(u_1,u_3)}$-path. Note that $1 \in C(u_3)$ and $5 \in C(u_1)$ if $1 \notin C(u_1)$. Thus, $\{1, 5\} \cap C(u_1) \neq \emptyset$ and $4 \notin C(u_1)$. Then if $4 \notin C(u_3)$, let $(uu_2, uu_3) \to (3, 4)$; otherwise, $C(u_3) = \{5, 3, 6, 4\}$, $C(u_1) = \{2, 3, 6, 1\}$, and let $(uu_1, uu_3, uv) \to (5, 1, 3)$. □

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