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Zero Point Problem of Accretive Operators in Banach Spaces

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Abstract Splitting methods have recently received much attention due to the fact that many nonlinear problems arising in applied areas such as image recovery, signal processing and machine learning are mathematically modeled as a nonlinear operator equation and this operator is decomposed as the sum of two (possibly simpler) nonlinear operators. Most of the investigation on splitting methods is however carried out in the framework of Hilbert spaces. In this paper, we consider these methods in the setting of Banach spaces. We shall introduce a viscosity iterative forward–backward splitting method with errors to find zeros of the sum of two accretive operators in Banach spaces. We shall prove the strong convergence of the method under mild conditions. We also discuss applications of these methods to monotone variational inequalities, convex minimization problem and convexly constrained linear inverse problem.

Keywords Accretive operator · Maximal monotone operator · Banach space · Splitting method · Forward–backward algorithm

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1 Introduction

Let X be a real Banach space. We study the following zero point problem: find $x^* \in X$ such that

$$0 \in Ax^* + Bx^*, \tag{1}$$

where $A : X \to X$ is an operator and $B : X \to 2^X$ is a set-valued operator. This problem includes, as special cases, convex programming, variational inequalities, split feasibility problem and minimization problem. To be more precise, some concrete problems in machine learning, image processing and linear inverse problem can be modeled mathematically as this form (1). For example:

Example 1 A stationary solution to the initial value problem of the evolution equation

$$0 \in \frac{\partial u}{\partial t} + Fu, \quad u(0) = u_0 \tag{2}$$

can be rewritten as (1) when the governing maximal monotone F is of the form $F = A + B$.

Example 2 In optimization, it often needs to solve a minimization problem of the form

$$\min_{x \in H}\{f(x) + g(Tx)\} \tag{3}$$

where H is a real Hilbert space, and f, g are proper lower-semicontinuous and convex functions from H to $(-\infty, \infty]$ and T is a bounded linear operator on H.

Indeed, (3) is equivalent to (1) if f and $g \circ T$ have a common point of continuity with $A := \partial f$ and $B =: T^* \circ \partial g \circ T$. Here T^* is the adjoint of T, and ∂f is the subdifferential operator of f. It is known [1,6,19] that the minimization problem (3) is widely used in image recovery, signal processing and machine learning.

Example 3 If $B = \partial \phi : H \to 2^H$, where $\phi : H \to (-\infty, \infty]$ is a proper convex and lower semicontinuous, and $\partial \phi$ is the subdifferential of ϕ, then problem (1) is equivalent to find $x^* \in H$ such that

$$\langle Ax^*, \; v - x^* \rangle + \phi(v) - \phi(x^*) \geq 0, \; \forall v \in H \tag{4}$$

which is said to be the mixed quasi-variational inequality.

Example 4 In Example 3, if ϕ is the *indicator function of* C, i.e.,

$$\phi(x) = \begin{cases} 0, & if \; x \in C, \\ +\infty, & if \; x \notin C, \end{cases}$$

then problem (4) is equivalent to the classical variational inequality problem, denoted by $VI(C; A)$, i.e., to find $x^* \in C$ such that

$$\langle Ax^*,\ v - x^*\rangle \geq 0,\ \forall v \in C. \tag{5}$$

It is easy to see that (5) is equivalent to finding a point $x^* \in C$ such that

$$0 \in (A + B)x^*,$$

where B is the subdifferential of the indicator of C.

A classical method for solving problem (1) is the *forward–backward splitting method* [6,10,14,21] which is defined by the following manner: for any fixed $x_1 \in X$ and for $r > 0$,

$$x_{n+1} = (I + rB)^{-1}(x_n - rAx_n),\ \forall n \geq 1. \tag{6}$$

We see that each step of the iteration involves only with A as the forward step and B as the backward step, but not the sum of B. In fact, this method includes, in particular, the proximal point algorithm [2,7,17] and the gradient method.

In 2012, Takashashi et al. [20] proved some strong convergence theorems of the Halpern-type iteration in a Hilbert space H, which is defined by the following manner: for any $x_1 \in H$,

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u + (1 - \alpha_n)J_{r_n}^B(x_n - r_n Ax_n)),\ \forall n \geq 1, \tag{7}$$

where $u \in H$ is a fixed point and A is an α-inverse strongly monotone mapping on H and B is an maximal monotone operator on H. Under suitable conditions, they proved that the sequence $\{x_n\}$ generated by (7) converges strongly to a zero point of $A + B$.

Recently, Lόpez et al. [11] introduced the following Halpern-type forward–backward method: for any $x_1 \in X$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(J_{r_n}^B(x_n - r_n(Ax_n + a_n)) + b_n) \tag{8}$$

where $u \in X$, A is an α-inverse strongly accretive mapping on X and B is an m-accretive operator on X, $\{r_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1]$ and $\{a_n\},\ \{b_n\}$ are the error sequences in X. They proved that the sequence $\{x_n\}$ generated by (8) strongly converges to a zero point of the sum of A and B under some appropriate conditions.

Very recently there have many works concerning the problem of finding zero points of the sum of two monotone operators (in Hilbert spaces) and accretive operators (in Banach spaces). For more details, see, e.g., [5,18,20,21,23–26] and the references therein.

In this paper, we introduce and consider a viscosity iterative forward–backward splitting method with errors to find zeros of the sum of two accretive operators in the setting of Banach spaces. We shall prove the strong convergence of the method under mild conditions. We also discuss applications of these methods to variational inequalities, convex minimization problem and convexly constrained linear inverse problem.

2 Preliminaries

In order to prove the main results of the paper, we need the following basic concepts, notations and lemmas.

In what follows, we always assume that X is a uniformly convex and q-uniformly smooth Banach space for some $q \in (1, 2]$ (the definitions and properties, see, for example [4]).

Recall that *the generalized duality mapping* $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{j_q(x) \in X^* : \langle j_q(x), x\rangle = ||x||^q, \quad ||j_q(x)|| = ||x||^{q-1}\},$$

and the following subdifferential inequality holds: for any $x, y \in X$,

$$||x + y||^q \le ||x||^q + q\langle y, j_q(x + y)\rangle, \; j_q(x + y) \in J_q(x + y). \tag{9}$$

Recall that [11] if X is q-uniformly smooth, $q \in (1, 2]$, then there exists a constant $\kappa_q > 0$ such that

$$||x + y||^q \le ||x||^q + q\langle y, \; j_q(x)\rangle + \kappa_q||y||^q, \; x, y \in X. \tag{10}$$

The best constant κ_q satisfying (10) will be called the q-uniform smoothness coefficient of X.

Proposition 1 *([4]). Let $1 < q \le 2$. Then the following conclusions hold:*

(1) Banach space X is smooth if and only if the duality mapping J_q is single valued.
(2) Banach space X is uniformly smooth if and only if the duality mapping J_q is single valued and norm-to-norm uniformly continuous on bounded sets of X.

Recall that a set-valued operator $A : X \to 2^X$ with the domain $D(A)$ and the range $R(A)$ is said to be *accretive* if, for each $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, \; j(x - y)\rangle \ge 0, \; \forall u \in Ax \text{ and } v \in Ay. \tag{11}$$

An accretive operator A is said to be m-accretive if the range $R(I+\lambda A) = X, \; \forall\lambda > 0$.

For any $\alpha > 0$ and $q \in (1, 2]$, we say that an accretive operator A is *α-inverse strongly accretive* (shortly, α-isa) of order q, if for each $x, y \in D(A)$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle u - v, \; j_q(x - y)\rangle \ge \alpha||u - v||^q, \; \forall u \in Ax \text{ and } v \in Ay. \tag{12}$$

Let C be a nonempty closed and convex subset of a real Banach space X and K be a nonempty subset of C. A mapping $T : C \to K$ is called a *retraction of C onto K* if $Tx = x$ for all $x \in K$. We say that T is *sunny* if, for each $x \in C$ and $t \ge 0$,

$$T(tx + (1 - t)Tx) = Tx, \tag{13}$$

whenever $tx + (1 - t)Tx \in C$. A sunny nonexpansive retraction is a sunny retraction which is also nonexpansive.

Proposition 2 *([15,28]). Let X be a uniformly smooth Banach space, $T : C \to C$ be a nonexpansive mapping with a fixed point and $f : C \to C$ be a contraction mapping. For each $t \in (0, 1)$ the unique fixed point $x_t \in C$ of the contractive mapping, $tf + (1 - t)T : C \to C$, converges strongly as $t \to 0$ to the unique fixed point z of T with $z = Qf(z)$, where $Q : C \to Fix(T)$ is the unique sunny nonexpansive retraction from C onto $Fix(T)$.*

Lemma 1 *([12, Lemma 3.1]). Let $\{a_n\}, \{c_n\} \subset R^+$, $\{\alpha_n\} \subset (0, 1)$ and $\{b_n\} \subset R$ be the sequences such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + b_n + c_n \forall n \geq 1.$$

Assume that $\sum_{n=1}^{\infty} c_n < \infty$. Then the following results hold:

(1) If $b_n \leq \alpha_n M$, where $M \geq 0$, then $\{a_n\}$ is bounded.
(2) If $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\alpha_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2 *([8]). Let $\{s_n\}$ be a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1 - \gamma_n)s_n + \gamma_n \tau_n$$

and

$$s_{n+1} \leq s_n - \eta_n + \rho_n, \forall n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in (0, 1), $\{\eta_n\}$ is a sequence of nonnegative real numbers, $\{\tau_n\}$ and $\{\rho_n\}$ are real sequences such that

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(b) $\lim_{n\to\infty} \rho_n = 0$;
(c) $\lim_{k\to\infty} \eta_{n_k} = 0$ implies $\limsup_{k\to\infty} \tau_{n_k} \leq 0$ for any subsequence $\{n_k\} \subset \{n\}$.

Then $\lim_{n\to\infty} s_n = 0$.

It is easy to prove the following conclusion holds.

Lemma 3 *For any $r > 0$, if*

$$T_r := J_r^B(I - rA) = (I + rB)^{-1}(I - rA),$$

then $Fix(T_r) = (A + B)^{-1}(0)$.

Lemma 4 *([11, Lemma 3.2]). For any $s \in (0, r]$ and $x \in X$, we have*

$$||x - T_s x|| \leq 2||x - T_r x||.$$

Lemma 5 *([11, Lemma 3.3]). Let X be a uniformly convex and q-uniformly smooth Banach space with $q \in (1, 2]$. Assume that A is a single-valued α-isa of order q on X.*

Then, for any $r > 0$, *there exists a continuous, strictly increasing and convex function* $\phi_q : R^+ \to R^+$ *with* $\phi_q(0) = 0$ *such that for all* $x, y \in B_r$,

$$||T_r x - T_r y||^q \leq ||x - y||^q - r(\alpha q - r^{q-1}\kappa_q)||Ax - Ay||^q \\ -\phi_q(||(I - J_r^B)(I - rA)x - (I - J_r^B)(I - rA)y||), \quad (14)$$

where κ_q *is the* q*-uniform smoothness coefficient of* X.

It is easy to prove that the following inequality holds.

Proposition 3 *Let* $1 < q \leq 2$ *and let* X *be a real smooth Banach space with the generalized duality mapping* j_q. *Let* m *be a fixed positive integer. Let* $\{x_i\}_{i=1}^m \subset X$ *and* $t_i \geq 0$ *for all* $i = 1, 2, ..., m$ *with* $\sum_{i=1}^m t_i \leq 1$. *Then we have*

$$||\sum_{i=1}^m t_i x_i||^q \leq \sum_{i=1}^m t_i ||x_i||^q. \quad (15)$$

3 Main Results

We are now in a position to give the following main results.

Theorem 1 *Let* X *be a uniformly convex and* q*-uniformly smooth Banach space,* $q \in (1, 2]$. *Let* $A : X \to X$ *be an* α*-isa of order* q *and* $B : X \to 2^X$ *be an* m*-accretive operator such that* $\Gamma := (A + B)^{-1}(0) \neq \emptyset$. *Let* $\{e_n\}$ *be a sequence in* X *and* $f : X \to X$ *be a contractive mapping with contractive constant* $\xi \in (0, 1)$. *Let* $\{x_n\}$ *be a sequence generated by* $x_1 \in X$ *and*

$$x_{n+1} = \alpha_n f(x_n) + \lambda_n x_n + \delta_n J_{r_n}^B(x_n - r_n Ax_n) + e_n, \ n \geq 1, \quad (16)$$

where $J_{r_n}^B = (I + r_n B)^{-1}$, κ_q *is the* q*-uniform smoothness coefficient of* X, $0 < r_n \leq (\frac{\alpha q}{\kappa_q})^{1/(q-1)}$ *and* $\{\alpha_n\}$, $\{\lambda_n\}$ *and* $\{\delta_n\}$ *are sequences in [0, 1] with* $\alpha_n + \lambda_n + \delta_n = 1$. *If* $\sum_{n=1}^\infty ||e_n|| < \infty$, *or* $\lim_{n\to\infty} ||e_n||/\alpha_n = 0$, *then* $\{x_n\}$ *is bounded.*

Proof For each $n \geq 1$, we put $T_n = J_{r_n}^B(I - r_n A)$ and let the sequence y_n be defined by

$$y_{n+1} = \alpha_n f(y_n) + \lambda_n y_n + \delta_n T_n y_n. \quad (17)$$

By the condition $0 < r_n \leq (\frac{\alpha q}{\kappa_q})^{1/(q-1)}$ and Lemma 5, we know that T_n is a nonexpansive mapping. Hence we have

$$\begin{aligned} ||x_{n+1} - y_{n+1}|| &= ||\lambda_n(f(x_n) - f(y_n)) + \lambda_n(x_n - y_n) + \delta_n(T_n x_n - T_n y_n) + e_n|| \\ &\leq \lambda_n \xi ||x_n - y_n|| + \lambda_n||x_n - y_n|| + \delta_n||x_n - y_n|| + ||e_n|| \\ &= (1 - \alpha_n(1 - \xi))||x_n - y_n|| + ||e_n||. \end{aligned}$$

By Lemma 1 (2), we conclude that $\lim_{n\to\infty} ||x_n - y_n|| = 0$.

Next we show that $\{y_n\}$ is bounded. Indeed, let $z \in \Gamma$. By Lemma 3 this implies that $z \in (A+B)^{-1}(0) = Fix(T_n), \forall n \geq 1$. Hence we have

$$\begin{aligned} ||y_{n+1} - z|| &= ||\alpha_n(f(y_n) - z) + \lambda_n(y_n - z) + \delta_n(T_n y_n - z)|| \\ &\leq \alpha_n ||f(y_n) - f(z)|| + \alpha_n ||f(z) - z|| + \lambda_n ||y_n - z|| + \delta_n ||y_n - z|| \\ &\leq (1 - \alpha_n(1 - \xi)) ||y_n - z|| + \alpha_n ||f(z) - z||. \end{aligned} \tag{18}$$

By Lemma 1 (1), $\{y_n\}$ is bounded, so is $\{x_n\}$. This completes the proof of Theorem 1. □

Theorem 2 *Let $X, A, B, f, q, \kappa_q, \{e_n\}, \Gamma$ and $\{x_n\}$ be the same as in Theorem 1. If $\Gamma \neq \emptyset$ and the following conditions are satisfied:*

(i) $\{\alpha_n\}$, $\{\lambda_n\}$ *and* $\{\delta_n\}$ *are sequences in [0, 1] with* $\alpha_n + \lambda_n + \delta_n = 1$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n \leq (\alpha q/\kappa_q)^{1/(q-1)}$;
(iv) $\lim inf_{n\to\infty} \delta_n > 0$, *and* $\sum_{n=1}^{\infty} ||e_n|| < \infty$ *or* $\lim_{n\to\infty} ||e_n||/\alpha_n = 0$,

then $\{x_n\}$ converges strongly to $z = Qf(z)$, where Q is a sunny nonexpansive retraction of X onto Γ.

Proof In Theorem 1 we have proved that $\lim_{n\to\infty} ||x_n - y_n|| = 0$. In order to prove the conclusion, it suffices to show that $\lim_{n\to\infty} y_n = z = Qf(z)$. In fact, from (9), we have

$$\begin{aligned} ||y_{n+1} - z||^q &= ||\alpha_n(f(y_n) - z) + \lambda_n(y_n - z) + \delta_n(T_n y_n - z)||^q \\ &\leq ||\lambda_n(y_n - z) + \delta_n(T_n y_n - z)||^q \\ &\quad + q\alpha_n \langle f(y_n) - z, \ j_q(y_{n+1} - z) \rangle. \end{aligned} \tag{19}$$

Since $z = Qf(z) \in \Gamma = Fix(T_n), \ \forall n \geq 1$, from Proposition 3 and Lemma 5 we have

$$\begin{aligned} &||\lambda_n(y_n - z) + \delta_n(T_n y_n - z)||^q \\ &\leq \lambda_n ||y_n - z||^q + \delta_n ||T_n y_n - z||^q \\ &\leq \lambda_n ||y_n - z||^q + \delta_n ||T_n y_n - T_n z||^q \\ &\leq \lambda_n ||y_n - z||^q + \delta_n \big\{ ||y_n - z||^q - r_n(\alpha q - r_n^{q-1} \kappa_q) ||A y_n - A z||^q \\ &\quad - \phi_q(||y_n - r_n A y_n - T_n y_n + r_n A z||) \big\} \\ &= (1 - \alpha_n) ||y_n - z||^q - \delta_n r_n(\alpha q - r_n^{q-1} \kappa_q) ||A y_n - A z||^q. \end{aligned} \tag{20}$$

Substituting (20) into (19) we have

$$\begin{aligned} ||y_{n+1} - z||^q \leq (1 - \alpha_n) ||y_n - z||^q &- \delta_n r_n (\alpha q - r_n^{q-1} \kappa_q) ||A y_n - A z||^q \\ &- \delta_n \phi_q(||y_n - r_n A y_n - T_n y_n + r_n A z||) \\ &+ q\alpha_n \langle f(y_n) - z, \ j_q(y_{n+1} - z) \rangle. \end{aligned} \tag{21}$$

Since $\alpha q - r_n^{q-1}\kappa_q > 0$, we have

$$||y_{n+1} - z||^q \leq (1-\alpha_n)||y_n - z||^q + q\alpha_n\langle f(y_n) - z,\ j_q(y_{n+1} - z)\rangle \tag{22}$$

and

$$\begin{aligned}||y_{n+1} - z||^q \leq\ & ||y_n - z||^q - \delta_n r_n(\alpha q - r_n^{q-1}\kappa_q)||Ay_n - Az||^q \\ & - \delta_n\phi_q(||y_n - r_n Ay_n - T_n y_n + r_n Az||) \\ & + q\alpha_n\langle f(y_n) - z,\ j_q(y_{n+1} - z)\rangle.\end{aligned} \tag{23}$$

For each $n \geq 1$, let

$$\begin{aligned} s_n &= ||y_n - z||^q; \quad \gamma_n = \alpha_n; \\ \rho_n &= q\alpha_n\langle f(y_n) - z,\ j_q(y_{n+1} - z)\rangle; \\ \tau_n &= q\langle f(y_n) - z,\ j_q(y_{n+1} - z)\rangle; \\ \eta_n &= \delta_n r_n(\alpha q - r_n^{q-1}\kappa_q)||Ay_n - Az||^q + \delta_n\phi_q(||y_n - r_n Ay_n - T_n y_n + r_n Az||).\end{aligned}$$

Then (22) and (23) can be written as:

$$s_{n+1} \leq (1-\gamma_n)s_n + \gamma_n\tau_n \tag{24}$$

and

$$s_{n+1} \leq s_n - \eta_n + \rho_n. \tag{25}$$

Since $\alpha_n \in (0, 1)$, $\alpha_n \to 0$ and $\Sigma_{n=1}^{\infty}\alpha_n = \infty$. It follows that $\gamma_n \in (0, 1)$, $\sum_{n=1}^{\infty}\gamma_n = \infty$ and $\lim_{n\to\infty}\rho_n = 0$. In order to prove $s_n \to 0$, by Lemma 2, it is sufficient to prove that for any subsequence $\{n_k\} \subset \{n\}$, if $\lim_{k\to\infty}\eta_{n_k} = 0$, then $\limsup_{k\to\infty}\tau_{n_k} \leq 0$.

Indeed, if $\{n_k\}$ is a subsequence of $\{n\}$ such that $\lim_{k\to\infty}\eta_{n_k} = 0$, then by the assumptions and the property of ϕ_q , we can deduce that

$$\begin{cases} \lim_{k\to\infty}||Ay_{n_k} - Az|| = 0; \\ \lim_{k\to\infty}||y_{n_k} - r_{n_k}Ay_{n_k} - T_{n_k}y_{n_k} + r_{n_k}Az|| = 0. \end{cases} \tag{26}$$

This implies, by the triangle inequality, that

$$\lim_{k\to\infty}||T_{n_k}y_{n_k} - y_{n_k}|| = 0. \tag{27}$$

Since $\liminf_{n\to\infty} r_n > 0$, there is $r > 0$ such that $r_n \geq r$ for all $n \geq 1$. In particular, $r_{n_k} \geq r$ for all $k \geq 1$. It follows from Lemma 4 and (27) that

$$\limsup_{k\to\infty}||T_r y_{n_k} - y_{n_k}|| \leq 2\limsup_{k\to\infty}||T_{n_k}y_{n_k} - y_{n_k}|| = 0, \tag{28}$$

which implies that

$$\lim_{k\to\infty} ||T_r y_{n_k} - y_{n_k}|| = 0. \tag{29}$$

Put

$$z_t = tf(z_t) + (1-t)T_r z_t, \ t \in (0,1).$$

By Proposition 2, z_t converges strongly as $t \to 0$ to the unique fixed point $z = Qf(z) \in Fix(T_r) = (A+B)^{-1}(0)$, where $Q : X \to Fix(T_r)$ is the unique sunny nonexpansive retraction from X onto $Fix(T_r) = (A+B)^{-1}(0)$. So we obtain

$$\begin{aligned} ||z_t - y_{n_k}||^q &= ||t(f(z_t) - y_{n_k}) + (1-t)(T_r z_t - y_{n_k})||^q \\ &\le (1-t)^q ||T_r z_t - y_{n_k}||^q + qt\langle f(z_t) - z_t, \ j_q(z_t - y_{n_k})\rangle \\ &\quad + qt\langle z_t - y_{n_k}, \ j_q(z_t - y_{n_k})\rangle \\ &\le (1-t)^q \{||T_r z_t - T_r y_{n_k}|| + ||T_r y_{n_k} - y_{n_k}||\}^q \\ &\quad + qt\langle f(z_t) - z_t, \ j_q(z_t - y_{n_k})\rangle + qt||z_t - y_{n_k}||^q \\ &\le (1-t)^q \{||z_t - y_{n_k}|| + ||T_r y_{n_k} - y_{n_k}||\}^q \\ &\quad + qt\langle f(z_t) - z_t, \ j_q(z_t - y_{n_k})\rangle + qt||z_t - y_{n_k}||^q. \end{aligned}$$

After simplifying we have

$$\begin{aligned} &\langle z_t - f(z_t), \ j_q(z_t - y_{n_k})\rangle \\ &\quad \le \frac{1}{qt}\{(1-t)^q(||z_t - y_{n_k}|| + ||T_r y_{n_k} - y_{n_k}||)^q + (qt-1)||z_t - y_{n_k}||^q\}. \end{aligned} \tag{30}$$

It follows from (29) and (30) that

$$\limsup_{k\to\infty}\langle z_t - f(z_t), \ j_q(z_t - y_{n_k})\rangle \le \frac{1}{qt}[(1-t)^q + (qt-1)]M^q, \tag{31}$$

where $M = \sup_{k\ge 1, \ t\in(0,1)} ||z_t - y_{n_k}||$. Since $\lim_{t\to 0} \frac{1}{qt}[(1-t)^q + (qt-1)] = 0$, $z_t \to z = Qfz$ as $t \to 0$ and by Proposition 1 (2) j_q is norm-to-norm uniformly continuous on bounded subsets of X, we have

$$||j_q(z_t - y_{n_k}) - j_q(z - y_{n_k})|| \to 0 \ (as \ t \to 0). \tag{32}$$

Observe that

$$\begin{aligned} &|\langle z_t - f(y_{n_k}), \ j_q(z_t - y_{n_k})\rangle - \langle z - f(y_{n_k}), \ j_q(z - y_{n_k})\rangle| \\ &\quad s \le |\langle z_t - z, \ j_q(z_t - y_{n_k})| + |\langle z - f(y_{n_k}), \ j_q(z_t - y_{n_k}) - j_q(z - y_{n_k})\rangle| \\ &\quad \le ||z_t - z||||z_t - y_{n_k}||^{q-1} + ||z - f(y_{n_k})||||j_q(z_t - y_{n_k}) - j_q(z - y_{n_k})||. \end{aligned} \tag{33}$$

This together with (32) shows that

$$\begin{aligned}
&\limsup_{k\to\infty}\langle z - f(y_{n_k}),\ j_q(z - y_{n_k})\rangle \\
&\quad = \limsup_{k\to\infty}\limsup_{t\to 0}\langle z_t - f(y_{n_k}),\ j_q(z_t - y_{n_k})\rangle \\
&\quad = \limsup_{k\to\infty}\limsup_{t\to 0}\langle z_t - f(z_t) + f(z_t) - f(y_{n_k}),\ j_q(z_t - y_{n_k})\rangle \\
&\quad = \limsup_{k\to\infty}\limsup_{t\to 0}\langle f(z_t) - f(y_{n_k}),\ j_q(z_t - y_{n_k})\rangle (by\ (31)) \\
&\quad = \limsup_{k\to\infty}\langle f(z) - f(y_{n_k}),\ j_q(z - y_{n_k})\rangle \\
&\quad = 0.
\end{aligned} \tag{34}$$

On the other hand, by (17) and (27), we see that

$$||y_{n_k+1} - y_{n_k}|| \le \alpha_{n_k}||f(y_{n_k})) - y_{n_k}|| + \delta_{n_k}||T_{n_k}y_{n_k} - y_{n_k}|| \to 0\ (as\ k \to \infty). \tag{35}$$

Combining (34) and (35), we get that

$$\limsup_{k\to\infty}\langle z - f(y_{n_k}),\ j_q(z - y_{n_k+1})\rangle \le 0.$$

This implies that $\limsup_{k\to\infty}\tau_{n_k} \le 0$. By Lemma 2, $y_n \to z$ (as $n \to \infty$). And so $x_n \to z$ (as $n \to \infty$). This completes the proof of Theorem 2. □

As well known, if X is a real Hilbert space, then it is a uniformly convex and 2-uniformly smooth Banach space, with the 2-uniform smoothness coefficient $\kappa_2 = 1$. And note that in this case the concept of monotonicity coincides with the concept of accretivity. Hence from Theorem 2 we can obtain the following result.

Theorem 3 *Let X be a real Hilbert space, $A : X \to X$ be an α-inverse strongly monotone operator of order 2 and $B : X \to 2^X$ be a maximal monotone operator such that $\Gamma := (A+B)^{-1}(0) \neq \emptyset$. Let f, $\{e_n\}$ and $\{x_n\}$ be the same as in Theorem 1. If the following conditions are satisfied:*

(i) $\{\alpha_n\}$, $\{\lambda_n\}$ *and* $\{\delta_n\}$ *are sequences in [0, 1] with* $\alpha_n + \lambda_n + \delta_n = 1$;
(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} r_n \le \limsup_{n\to\infty} r_n \le 2\alpha$;
(iv) $\liminf_{n\to\infty}\delta_n > 0$, *and* $\sum_{n=1}^{\infty}||e_n|| < \infty$ *or* $\lim_{n\to\infty}||e_n||/\alpha_n = 0$,

then $\{x_n\}$ converges strongly to $z = Qf(z)$, where Q is a sunny nonexpansive retraction of X onto Γ.

In Theorem 2, if $f(x) = u,\ \forall x \in X$, where u is a fixed point in X, then from Theorem 2 we have the following result.

Theorem 4 *Let $X, q, A, B, \{e_n\}$and Γ be the same as in Theorem 2. Let $\{x_n\}$ be the sequence generated by $x_1 \in X$ and*

$$x_{n+1} = \alpha_n u + \lambda_n x_n + \delta_n J^B_{r_n}(x_n - r_n Ax_n) + e_n,\ n \ge 1. \tag{36}$$

If $\Gamma \neq \emptyset$ and the following conditions are satisfied:

(i) $\{\alpha_n\}$, $\{\lambda_n\}$, *and* $\{\delta_n\}$ *are sequences in [0, 1] with* $\alpha_n + \lambda_n + \delta_n = 1$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n \leq (\frac{\alpha q}{\kappa_q})^{1/(q-1)}$;
(iv) $\liminf_{n\to\infty} \delta_n > 0$, *and* $\sum_{n=1}^{\infty} ||e_n|| < \infty$ *or* $\lim_{n\to\infty} ||e_n||/\alpha_n = 0$,

then $\{x_n\}$ converges strongly to $z = Qu$, where Q is a sunny nonexpansive retraction of X onto Γ.

Remark 1 Theorem 2 is an improvement of [3], and it is also a generalization of [9,13,22,27] from Hilbert spaces to Banach spaces.

4 Applications

In this section we shall utilize the forward–backward methods mentioned above to study monotone variational inequalities, convex minimization problem and convexly constrained linear inverse problem.

Throughout this section, let C be a nonempty closed and convex subset of a real Hilbert space H. Note that in this case the concept of monotonicity coincides with the concept of accretivity.

4.1 Application to Monotone Variational Inequality Problems

A monotone variational inequality problem (VIP) is formulated as the problem of finding a point $x^* \in C$ such that:

$$\langle Ax, y - x\rangle \geq 0 \quad \forall y \in C, \tag{37}$$

where $A : C \to H$ is a nonlinear monotone operator. We shall denote by Γ the solution set of (37) and assume $\Gamma \neq \emptyset$. In Example 4, we have pointed out that $VI(C; A)$ (37) is equivalent to finding a point x^* so that

$$0 \in (A + B)x^*, \tag{38}$$

where $B : C \to H$ is the subdifferential of the indicator of C, and it is a maximal monotone operator. By [16, Theorem 3] in this case, the resolvent of B is nothing but the projection operator P_C. Therefore the following result can be obtained from Theorem 3 immediately.

Corollary 1 *Let $A : C \to H$ be an α-inverse strongly monotone operator of order 2 and let f, $\{e_n\}$ be the same as in Theorem 1. Let $\{x_n\}$ be the sequence generated by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + \lambda_n x_n + \delta_n P_C(x_n - r_n Ax_n) + e_n, n \geq 1. \tag{39}$$

If the following conditions are satisfied:

(i) $\{\alpha_n\}$, $\{\lambda_n\}$ *and* $\{\delta_n\}$ *are sequences in [0, 1] with* $\alpha_n + \lambda_n + \delta_n = 1$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n \leq 2\alpha$;
(iv) $\liminf_{n\to\infty} \delta_n > 0$, *and* $\sum_{n=1}^{\infty} ||e_n|| < \infty$ *or* $\lim_{n\to\infty} ||e_n||/\alpha_n = 0$,

then $\{x_n\}$ *converges strongly to a solution* z *of monotone variational inequality (37).*

4.2 Application to the Convex Minimization Problems

Let $\psi : H \to R$ be a convex smooth function and $\phi : H \to R$ be a proper convex and lower-semicontinuous function. We consider the following convex minimization problem of finding $x^* \in H$ such that

$$\psi(x^*) + \phi(x^*) = \min_{x\in H}\{\psi(x) + \phi(x)\}. \tag{40}$$

This problem (40) is equivalent, by Fermat's rule, to the problem of finding $x^* \in H$ such that

$$0 \in \nabla\psi(x^*) + \partial\phi(x^*), \tag{41}$$

where $\nabla\psi$ is a gradient of ψ and $\partial\phi$ is a subdifferential of ϕ. Set $A = \nabla\psi$ and $B = \partial\phi$ in Theorem 3. If $\nabla\psi$ is (1/L)-Lipschitz continuous, then it is L-inverse strongly monotone. Moreover, $\partial\phi$ is maximal monotone. Hence from Theorem 3 we have the following result.

Theorem 5 *Let* $\psi : H \to R$ *be a convex and differentiable function with (1/L)-Lipschitz continuous gradient* $\nabla\psi$ *and* $\phi : H \to R$ *be a proper convex and lower-semicontinuous function such that* $\psi + \phi$ *attains a minimizer. Let* $f : H \to H$ *be a contractive mapping with a contractive coefficient* $\xi \in (0, 1)$, *and* $\{e_n\}$ *be a sequence in* H. *Let* $\{x_n\}$ *be the sequence generated by* $x_1 \in H$ *and*

$$x_{n+1} = \alpha_n f(x_n) + \lambda_n x_n + \delta_n J_{r_n}(x_n - r_n\nabla\psi(x_n)) + e_n, \forall n \geq 1, \tag{42}$$

where $J_{r_n} = (I + r_n\partial\phi)^{-1}$. *If the following conditions are satisfied:*

(i) $\{\alpha_n\}$, $\{\lambda_n\}$ *and* $\{\delta_n\}$ *are sequences in [0, 1] with* $\alpha_n + \lambda_n + \delta_n = 1$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n \leq 2\alpha$;
(iv) $\liminf_{n\to\infty} \delta_n > 0$, *and* $\sum_{n=1}^{\infty} ||e_n|| < \infty$ *or* $\lim_{n\to\infty} ||e_n||/\alpha_n = 0$,

then $\{x_n\}$ *strongly converges to a minimizer of* $\varphi + \psi$.

4.3 Application to the Convexly Constrained Linear Inverse Problem

Let $K : H \to C$ be a bounded linear operator and $b \in C$. The constrained linear system

$$Kx = b, \ x \in C \tag{43}$$

is called *convexly constrained linear inverse problem*. Define $\psi(x) : H \to R^+$ by

$$\psi(x) = \frac{1}{2}||Kx - b||^2, \ x \in H. \tag{44}$$

We have $\nabla\psi(x) = K^*(Kx - b)$, and $\nabla\psi$ is L-Lipschitzian, where $L = ||K||^2$, i.e., $\nabla\psi$ is $1/L$-inverse strongly monotone. It is easy to know that $x^* \in C$ is a solution of (43) if and only if $0 \in \nabla\psi(x^*) = K^*(Kx^* - b)$. Taking $A = \nabla\psi$ and $B = 0$ in Theorem 3 we have the following result.

Theorem 6 *If problem (43) is consistent and the following conditions are satisfied*

(i) $\{\alpha_n\}$, $\{\lambda_n\}$ *and* $\{\delta_n\}$ *are sequences in [0, 1] with* $\alpha_n + \lambda_n + \delta_n = 1$*;*
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n \leq 2/L$*;*
(iv) $\liminf_{n\to\infty} \delta_n > 0$*, and* $\sum_{n=1}^{\infty} ||e_n|| < \infty$ *or* $\lim_{n\to\infty} ||e_n||/\alpha_n = 0$*,*

then for any given contractive mapping $f : H \to C$*, the sequence* $\{x_n\}$ *generated by* $x_1 \in H$ *and*

$$x_{n+1} = \alpha_n f(x_n) + \lambda_n x_n + \delta_n P_C(x_n - r_n K^*(Kx_n - b)), \forall n \geq 1, \tag{45}$$

converges strongly to a solution of problem (43).

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