

## Leibniz Algebras Whose Semisimple Part is Related to $sl_2$

L. M. Camacho<sup>1</sup> · S. Gómez-Vidal<sup>1</sup> ·  
B. A. Omirov<sup>2</sup> · I. A. Karimjanov<sup>2</sup>

Received: 21 January 2014 / Revised: 1 September 2014 / Published online: 20 January 2017  
© Malaysian Mathematical Sciences Society and Penerbit Universiti Sains Malaysia 2017

**Abstract** In this paper we identify the structure of complex finite-dimensional Leibniz algebras with associated Lie algebras  $sl_2^1 \oplus sl_2^2 \oplus \cdots \oplus sl_2^s \oplus R$ , where  $R$  is a solvable radical. The classifications of such Leibniz algebras in the cases  $\dim R = 2, 3$  and  $\dim I \neq 3$  have been obtained. Moreover, we classify Leibniz algebras with  $L/I \cong sl_2^1 \oplus sl_2^2$  and some conditions on ideal  $I = \text{id} \langle [x, x] \mid x \in L \rangle$ .

**Keywords** Leibniz algebra · Simple algebra  $sl_2$  · Direct sum of algebras · Right module · Irreducible module

---

Communicated by V. Ravichandran.

---

This work has been funded by Mathematics Institute and V Research Plan of Sevilla University by IMU/CDC-program. The authors were supported by Ministerio de Economía y Competitividad (Spain), grant MTM2016-79661-P and Ministry of Education and Science of the Republic of Kazakhstan, grant No. 0828/GF4.

---

✉ L. M. Camacho  
lcamacho@us.es

S. Gómez-Vidal  
samuel.gomezvidal@gmail.com

B. A. Omirov  
omirovb@mail.ru

I. A. Karimjanov  
iqboli@gmail.com

<sup>1</sup> Dpto. Matemática Aplicada I, Universidad de Sevilla, Avda. Reina Mercedes, 41012 Sevilla, Spain

<sup>2</sup> Institute of Mathematics and Information Technologies of Academy of Uzbekistan, 29, Do'rmon Yo'li Street, Tashkent, Uzbekistan 100125

**Mathematics Subject Classification** 17A32 · 17A60 · 17B10 · 17B20

## 1 Introduction

The notion of Leibniz algebra was first introduced by Loday in [7,8] as a non-antisymmetric generalization of Lie algebra. During the last 20 years, the theory of Leibniz algebras has been actively studied and many results of the theory of Lie algebras have been extended to Leibniz algebras. A lot of papers have been devoted to the study of finite-dimensional Leibniz algebras [1,2,11], etc. However, just a few works are related to the semisimple part of Leibniz algebras [3,5,10].

We know that an arbitrary Lie algebra can be decomposed into a semidirect sum of the solvable radical and its semisimple subalgebra (Levi's theorem [6]). According to the Cartan–Killing theory, a semisimple Lie algebra can be represented as a direct sum of simple ideals, which are completely classified [6].

In a recent study, Barnes has proved an analogue of Levi's theorem for the case of Leibniz algebras [3]. Namely, a Leibniz algebra is decomposed into a semidirect sum of its solvable radical and a semisimple Lie algebra.

The inherent properties of non-Lie Leibniz algebras imply that the subspace spanned by squares of elements of the algebra is a non-trivial ideal (denoted by  $I$ ). Moreover, the ideal  $I$  is abelian, and hence, it is contained in the solvable radical. Thanks to result of Barnes in order to describe Leibniz algebras it is enough to investigate the relationship between products of a semisimple Lie algebra and the radical (see [5,9,10]).

The present work is devoted to describing the structure of Leibniz algebras with the associated Lie algebras isomorphic to  $sl_2^1 \oplus sl_2^2 \oplus \cdots \oplus sl_2^s \oplus R$  with  $I$  an irreducible right  $sl_2^k$ -module for some  $k$ . Since the description of such Leibniz algebras is very complicated, we have to focus on Leibniz algebras with semisimple part  $sl_2^1 \oplus sl_2^2$  under some conditions on the ideal  $I$ .

In order to achieve our goal we organize the paper as follows. In Sect. 2, we give some necessary notions and preliminary results about Leibniz algebras with associated Lie algebra  $sl_2 \dot{+} R$ . Section 3 is devoted to the study of the structure of Leibniz algebras, whose semisimple part is a direct sum of copies of  $sl_2$  algebra and with some conditions on the ideal  $I$ . In Sect. 4, we classify Leibniz algebras whose semisimple part is a direct sum  $sl_2^1 \oplus sl_2^2$  and  $I$  is decomposed into a direct sum of two irreducible modules  $I_{1,1}, I_{1,2}$  over  $sl_2^1$  such that  $\dim I_{1,1} = \dim I_{1,2}$ . The description of the structure of Leibniz algebras with associated Lie algebra  $sl_n$  and  $\dim I = 1, 2$  is obtained in Sect. 5.

Throughout the work, vector spaces and algebras are finite dimensional over the field of complex numbers. Moreover, in the table of multiplication of an algebra the omitted products are assumed to be zero. We shall use the following symbols:  $+$ ,  $\oplus$  and  $\dot{+}$  for notations of the direct sum of the vector spaces, the direct and semidirect sums of algebras, respectively.

## 2 Preliminaries

In this section we give some necessary definitions and preliminary results.

**Definition 2.1** [7] An algebra  $(L, [\cdot, \cdot])$  over a field  $F$  is called a Leibniz algebra if for any  $x, y, z \in L$  the so-called Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

holds.

Let  $L$  be a Leibniz algebra, and let  $I = \text{ideal} \langle [x, x] \mid x \in L \rangle$  be an ideal of  $L$  generated by all squares. The natural epimorphism  $\varphi : L \rightarrow L/I$  determines the associated Lie algebra  $L/I$  of the Leibniz algebra  $L$ . It is clear that the ideal  $I$  is the minimal ideal with respect to the property that the quotient algebra by this ideal is a Lie algebra.

It was noted in [3] that the ideal  $I$  coincides with the space spanned by squares of elements of an algebra.

According to [6] there exists a unique (up to isomorphism) three-dimensional simple Lie algebra with the following table of multiplications:

$$sl_2 : [e, h] = -[h, e] = 2e, \quad [h, f] = -[f, h] = 2f, \quad [e, f] = -[f, e] = h,$$

The basis  $\{e, f, h\}$  is called the *canonical basis*.

In [9], Leibniz algebras whose quotient Lie algebra is isomorphic to  $sl_2$  are described. Let us present a Leibniz algebra  $L$  with the table of multiplication in a basis  $\{e, f, h, x_0^1, \dots, x_{t_1}^1, x_0^2, \dots, x_{t_2}^2, \dots, x_0^p, \dots, x_{t_p}^p\}$  and the quotient algebra  $L/I$  is  $sl_2$  as follows:

$$\begin{aligned} [e, h] &= -[h, e] = 2e, & [h, f] &= -[f, h] = 2f, & [e, f] &= -[f, e] = h, \\ [x_k^j, h] &= (t_j - 2k)x_k^j, & & & & 0 \leq k \leq t_j, \\ [x_k^j, f] &= x_{k+1}^j, & & & & 0 \leq k \leq t_j - 1, \\ [x_k^j, e] &= -k(t_j + 1 - k)x_{k-1}^j, & & & & 1 \leq k \leq t_j. \end{aligned}$$

where  $L = sl_2 + I_1 + I_2 + \dots + I_p$  and  $I_j = \langle x_1^j, \dots, x_{t_j}^j \rangle$ ,  $1 \leq j \leq p$ .

The last three types of products of the above table of multiplication are characterized as an irreducible  $sl_2$ -module with the canonical basis of  $sl_2$  [6].

Now we give the notion of semisimplicity for Leibniz algebras.

**Definition 2.2** [5] A Leibniz algebra is called semisimple if its maximal solvable ideal is equal to  $I$ .

Since in the Lie algebras case the ideal  $I$  is equal to zero, this definition also agrees with the definition of semisimple Lie algebra.

Although Levi’s theorem is proved for the left Leibniz algebras [3], it is also true for right Leibniz algebras (here we consider the right Leibniz algebras).

**Theorem 2.3** [3] (Levi’s theorem) *Let  $L$  be a finite-dimensional Leibniz algebra over a field of characteristic zero and  $R$  be its solvable radical. Then there exists a semisimple subalgebra  $S$  of  $L$ , such that  $L = S \dot{+} R$ .*

A Leibniz algebra  $L$  is called *simple* if its only ideals are  $\{0\}$ ,  $I$ ,  $L$  and  $[L, L] \neq I$ . From the proof of Theorem 2.3, it is not difficult to see that  $S$  is a semisimple Lie algebra. Therefore, we have that a simple Leibniz algebra is a semidirect sum of simple Lie algebra  $S$  and the irreducible right module  $I$  over  $S$ , i.e.,  $L = S \dot{+} I$ . Hence, we obtain the description of the simple Leibniz algebras in terms of simple Lie algebras and ideal  $I$ .

**Definition 2.4** [6] A nonzero module  $M$  over a Lie algebra whose only submodules are the module itself and zero module is called *irreducible module*. A nonzero module  $M$  which admits decomposition into a direct sum of irreducible modules is said to be *completely reducible*.

Further, we shall use the following Weyl’s semisimplicity theorem.

**Theorem 2.5** [6] *Let  $G$  be a semisimple Lie algebra over a field of characteristic zero. Then every finite-dimensional module over  $G$  is completely reducible.*

Now we present results on classification of Leibniz algebras with the conditions  $L/I \cong sl_2 \oplus R$ ,  $\dim R = 2, 3$ , where  $I$  is an irreducible right module over  $sl_2$  ( $\dim I \neq 3$ ).

**Theorem 2.6** [4] *Let  $L$  be a Leibniz algebra whose quotient  $L/I \cong sl_2 \oplus R$ , where  $R$  is a two-dimensional solvable ideal and  $I$  is a right irreducible module over  $sl_2$  ( $\dim I \neq 3$ ). Then there exists a basis  $\{e, h, f, x_0, x_1, \dots, x_m, y_1, y_2\}$  of the algebra  $L$  such that the table of multiplication in  $L$  has the following form:*

$$\left\{ \begin{array}{ll} [e, h] = -[h, e] = 2e, & [h, f] = -[f, h] = 2f, [e, f] = -[f, e] = h, \\ [y_1, y_2] = -[y_2, y_1] = y_1, & [x_k, y_2] = ax_k, \quad 0 \leq k \leq m, a \in \mathbb{C}, \\ [x_k, h] = (m - 2k)x_k & 0 \leq k \leq m, \\ [x_k, f] = x_{k+1}, & 0 \leq k \leq m - 1, \\ [x_k, e] = -k(m + 1 - k)x_{k-1}, & 1 \leq k \leq m. \end{array} \right.$$

The following theorem extends Theorem 2.6 for  $\dim R = 3$ .

**Theorem 2.7** [10] *Let  $L$  be a Leibniz algebra whose quotient  $L/I \cong sl_2 \oplus R$ , where  $R$  is a three-dimensional solvable ideal and  $I$  is an irreducible right module over  $sl_2$  ( $\dim I \neq 3$ ). Then there exists a basis  $\{e, h, f, x_0, x_1, \dots, x_m, y_1, y_2, y_3\}$  of the algebra  $L$  such that the table of multiplication in  $L$  has one of the following two forms:*

$$\begin{aligned}
 L_1(\alpha, a) : & \begin{cases} [e, h] = -[h, e] = 2e, & [h, f] = -[f, h] = 2f, & [e, f] = -[f, e] = h, \\ [y_1, y_2] = -[y_2, y_1] = y_1, & [y_3, y_2] = -[y_2, y_3] = \alpha y_3, \\ [x_k, h] = (m - 2k)x_k, & 0 \leq k \leq m, \\ [x_k, f] = x_{k+1}, & 0 \leq k \leq m - 1, \\ [x_k, e] = -k(m + 1 - k)x_{k-1}, & 1 \leq k \leq m, \\ [x_i, y_2] = ax_i, & 0 \leq i \leq m. \end{cases} \\
 L_2(a) : & \begin{cases} [e, h] = -[h, e] = 2e, & [h, f] = -[f, h] = 2f, & [e, f] = -[f, e] = h, \\ [y_1, y_2] = -[y_2, y_1] = y_1 + y_3, & [y_3, y_2] = -[y_2, y_3] = y_3, \\ [x_k, h] = (m - 2k)x_k, & 0 \leq k \leq m, \\ [x_k, f] = x_{k+1}, & 0 \leq k \leq m - 1, \\ [x_k, e] = -k(m + 1 - k)x_{k-1}, & 1 \leq k \leq m, \\ [x_i, y_2] = ax_i, & 0 \leq i \leq m. \end{cases}
 \end{aligned}$$

For a semisimple Lie algebra  $S$  we consider a semisimple Leibniz algebra  $L$  such that  $L = (sl_2 \oplus S) \dot{+} I$ . We put  $I_1 = [I, sl_2]$ .

Let  $I_1$  be a reducible module over  $sl_2$ . Then by Theorem 2.5 we have a decomposition:

$$I_1 = I_{1,1} \oplus I_{1,2} \oplus \dots \oplus I_{1,p},$$

where  $I_{1,j}$  are irreducible modules over  $sl_2$  for every  $j, 1 \leq j \leq p$ .

**Theorem 2.8** [5] *Let  $\dim I_{1,j_1} = \dim I_{1,j_2} = \dots = \dim I_{1,j_s} = t + 1, 1 \leq s \leq p$ . Then there exist  $t + 1$  submodules  $I_{2,1}, I_{2,2}, \dots, I_{2,t+1}$  of dimension  $s$  of the module  $I_2 = [I, S]$  such that*

$$I_{2,1} + I_{2,2} + \dots + I_{2,t+1} = I_1 \cap I_2.$$

### 3 On Leibniz Algebras Whose Associated Lie Algebra is Isomorphic to $sl_2^1 \oplus sl_2^2 \oplus \dots \oplus sl_2^s \oplus R$

In this section, we will consider a Leibniz algebra satisfying the following conditions:

- (i) The quotient algebra  $L/I$  is isomorphic to the direct sum  $sl_2^1 \oplus sl_2^2 \oplus \dots \oplus sl_2^s \oplus R$ , where  $R$  is  $n$ -dimensional solvable Lie algebra;
- (ii) the ideal  $I$  is a right irreducible  $sl_2^k$ -module for some  $k \in \{1, \dots, s\}$ .

By reordering direct sums and changing indexes in the condition (ii) we can assume  $k = 1$ .

We consider the standard basis  $\{e_i, f_i, h_i\}$  of  $sl_2^1, I = \langle x_0, \dots, x_m \rangle$  and  $R = \langle y_1, \dots, y_n \rangle$ . Thus, a basis of  $L$  is  $\{e_1, f_1, h_1, \dots, e_s, f_s, h_s, x_0, \dots, x_m, y_1, \dots, y_n\}$ .

Then due to [6] we have

$$\begin{aligned}
 [e_1, h_1] &= -[h_1, e_1] = 2e_1, & [h_1, f_1] &= -[f_1, h_1] = 2f_1, & [e_1, f_1] &= -[f_1, e_1] = h_1, \\
 [x_i, h_1] &= (m - 2i)x_i, & & 0 \leq i \leq m, \\
 [x_i, f_1] &= x_{i+1}, & & 0 \leq i \leq m - 1, \\
 [x_i, e_1] &= -i(m + 1 - i)x_{i-1}, & & 1 \leq i \leq m.
 \end{aligned}$$

The validity of the next lemma follows from Lemma 3.3 in [5].

**Lemma 3.1** *Let  $L$  be a Leibniz algebra satisfying the conditions (i)–(ii). Then  $[I, sl_2^j] = 0$  for any  $j \in \{2, \dots, s\}$ .*

The following lemma is a direct consequence of Theorem 2.3.

**Lemma 3.2** *Let  $L$  be a Leibniz algebra satisfying the conditions (i)–(ii). Then  $[sl_2^t, sl_2^t] = sl_2^t$  for  $1 \leq t \leq s$  and  $[sl_2^i, sl_2^j] = 0$  for  $1 \leq i, j \leq s, i \neq j$ .*

The next lemma establishes that the solvable ideal  $R$  is contained in two-sided annihilator of each  $sl_2^i, 2 \leq i \leq s$ .

**Lemma 3.3** *Let  $L$  be a Leibniz algebra satisfying the conditions (i)–(ii). Then*

$$[R, sl_2^i] = [sl_2^i, R] = 0, \quad 2 \leq i \leq s.$$

*Proof* Applying Leibniz identity for the following triples

$$(y_s, e_1, a), (y_s, f_1, a), (a, y_s, e_1), (a, y_s, f_1)$$

leads  $[y_s, a] = 0, [a, y_s] = 0, 1 \leq s \leq n$  for an arbitrary element  $a \in sl_2^i, 2 \leq i \leq s$ . □

Summarizing the results of Lemmas 3.1–3.3, we obtain the following theorem.

**Theorem 3.4** *Let  $L$  be a finite-dimensional Leibniz algebra satisfying the conditions:*

- (i)  $L/I \cong sl_2^1 \oplus sl_2^2 \oplus \dots \oplus sl_2^s \oplus R$ , where  $R$  is an  $n$ -dimensional solvable Lie algebra;
- (ii) the ideal  $I$  is a right irreducible module over  $sl_2^1$ .

Then,  $L \cong ((sl_2^1 \oplus R) \dot{+} I) \oplus sl_2^2 \oplus \dots \oplus sl_2^s$ .

As a result of Theorems 2.6–2.7 and 3.4, we have the following corollaries.

**Corollary 3.5** *Let  $L/I \cong sl_2^1 \oplus sl_2^2 \oplus \dots \oplus sl_2^s \oplus R$  with  $\dim R = 2$  and  $\dim I \neq 3$ . Then  $L$  is isomorphic to the following algebra:*

$$\left\{ \begin{array}{ll} [e_j, h_j] = -[h_j, e_j] = 2e_j, & [h_j, f_j] = -[f_j, h_j] = 2f_j, \\ [e_j, f_j] = -[f_j, e_j] = h_j, & 1 \leq j \leq s, \\ [y_1, y_2] = -[y_2, y_1] = y_1, & \\ [x_k, h_1] = (m - 2k)x_k, & 0 \leq k \leq m, \\ [x_k, f_1] = x_{k+1}, & 0 \leq k \leq m - 1, \\ [x_k, e_1] = -k(m + 1 - k)x_{k-1}, & 1 \leq k \leq m, \\ [x_k, y_2] = ax_k, & 0 \leq k \leq m, a \in \mathbb{C}. \end{array} \right.$$

**Corollary 3.6** *Let  $L/I \cong sl_2^1 \oplus sl_2^2 \oplus \dots \oplus sl_2^s \oplus R$ , with  $\dim R = 3$  and  $\dim I \neq 3$ . Then  $L$  is isomorphic to the following non-isomorphic algebras:*

$$L_1(\alpha, a) : \begin{cases} [e_j, h_j] = -[h_j, e_j] = 2e_j, & [h_j, f_j] = -[f_j, h_j] = 2f_j, \\ [e_j, f_j] = -[f_j, e_j] = h_j, & 1 \leq j \leq s, \\ [y_1, y_2] = -[y_2, y_1] = y_1, & [y_3, y_2] = -[y_2, y_3] = \alpha y_3, \\ [x_k, h_1] = (m - 2k)x_k, & 0 \leq k \leq m, \\ [x_k, f_1] = x_{k+1}, & 0 \leq k \leq m - 1, \\ [x_k, e_1] = -k(m + 1 - k)x_{k-1}, & 1 \leq k \leq m, \\ [x_i, y_2] = ax_i, & 0 \leq i \leq m, \end{cases}$$

$$L_2(a) : \begin{cases} [e_j, h_j] = -[h_j, e_j] = 2e_j, & [h_j, f_j] = -[f_j, h_j] = 2f_j, \\ [e_j, f_j] = -[f_j, e_j] = h_j, & 1 \leq j \leq s, \\ [y_1, y_2] = -[y_2, y_1] = y_1 + y_3, & [y_3, y_2] = -[y_2, y_3] = y_3, \\ [x_k, h_1] = (m - 2k)x_k, & 0 \leq k \leq m, \\ [x_k, f_1] = x_{k+1}, & 0 \leq k \leq m - 1, \\ [x_k, e_1] = -k(m + 1 - k)x_{k-1}, & 1 \leq k \leq m, \\ [x_i, y_2] = ax_i, & 0 \leq i \leq m. \end{cases}$$

### 4 On Leibniz Algebras with Semisimple Part $sl_2^1 \oplus Sl_2^2$

Let the quotient Lie algebra  $L/I$  for a Leibniz algebra  $L$  be isomorphic to a direct sum of two copies of the  $sl_2$ , i.e.,  $L/I \cong sl_2 \oplus sl_2$ . In this section we shall investigate the case when the ideal  $I$  is reducible over each copy of  $sl_2$ . In order to distinguish copies of  $sl_2$  we shall denote them by  $sl_2^1$  and  $sl_2^2$ . One can assume that  $I$  is reducible over  $sl_2^1$ . Due to Theorem 2.5 we have the following decomposition:

$$I = I_{1,1} \oplus I_{1,2} \oplus \dots \oplus I_{1,s+1},$$

where  $I_{1,j}$ ,  $1 \leq j \leq s + 1$  are irreducible  $sl_2^1$ -modules.

We shall focus our study on the case when  $\dim I_{1,1} = \dim I_{1,2} = \dots = \dim I_{1,s+1} = m + 1$ .

Let us introduce notations as follows:

$$I_{1,j} = \langle x_0^j, x_1^j, \dots, x_m^j \rangle, \quad 1 \leq j \leq s + 1.$$

In the proof of Theorem 3.7 in the paper [5] it was proved that

$$[x_i^j, e_2] = \sum_{k=1}^{s+1} a_{i,j}^k x_i^k, \quad [x_i^j, f_2] = \sum_{k=1}^{s+1} b_{i,j}^k x_i^k, \quad [x_i^j, h_2] = \sum_{k=1}^{s+1} c_{i,j}^k x_i^k,$$

where  $0 \leq i \leq m$ ,  $1 \leq j \leq s + 1$ .

Without loss of generality one can assume that the products  $[I_{1,j}, sl_2^1]$ ,  $1 \leq j \leq s + 1$  are expressed as follows:

$$\begin{aligned} [x_i^j, e_1] &= -i(m + 1 - i)x_{i-1}^j, & 1 \leq i \leq m, \\ [x_i^j, f_1] &= x_{i+1}^j, & 0 \leq i \leq m - 1, \\ [x_i^j, h_1] &= (m - 2i)x_i^j, & 0 \leq i \leq m. \end{aligned}$$

**Proposition 4.1** *Let  $L/I$  be isomorphic to  $sl_2^1 \oplus sl_2^2$ , where  $I = I_{1,1} \oplus I_{1,2} \oplus \dots \oplus I_{1,s+1}$  with  $\dim I_{1,j} = m + 1$  and  $I_{1,j}$  be irreducible  $sl_2^1$ -modules for  $1 \leq j \leq s + 1$ . Then*

$$[x_i^j, e_2] = \sum_{k=1}^{s+1} a_j^k x_i^k, \quad [x_i^j, f_2] = \sum_{k=1}^{s+1} b_j^k x_i^k, \quad [x_i^j, h_2] = \sum_{k=1}^{s+1} c_j^k x_i^k,$$

where  $0 \leq i \leq m$ ,  $1 \leq j \leq s + 1$ .

*Proof* Applying the Leibniz identity for the following triples of elements:

$$(x_1^j, e_1, e_2), \quad 1 \leq j \leq s + 1$$

we derive the restrictions:

$$a_{1,j}^k = a_{0,j}^k, \quad 1 \leq k \leq s + 1.$$

Consequently, we obtain

$$[x_0^j, e_2] = \sum_{k=1}^{s+1} a_{0,j}^k x_0^k, \quad [x_1^j, e_2] = \sum_{k=1}^{s+1} a_{0,j}^k x_1^k, \quad 1 \leq j \leq s + 1.$$

By induction we shall prove the equality

$$[x_i^j, e_2] = \sum_{k=1}^{s+1} a_{0,j}^k x_i^k, \quad 0 \leq i \leq m. \tag{4.1}$$

Using the assumption of correctness of Equality 4.1 for  $i$  in the following chain of equalities:

$$\begin{aligned} 0 &= [x_{i+1}^j, [e_1, e_2]] = [[x_{i+1}^j, e_1], e_2] - [[x_{i+1}^j, e_2], e_1] = -[(i + 1)(m - i)x_i^j, e_2] \\ &\quad - \sum_{k=1}^{s+1} a_{i+1,j}^k [x_{i+1}^k, e_1] = -(i + 1)(m - i) \sum_{k=1}^{s+1} a_{0,j}^k x_i^k \\ &\quad + \sum_{k=1}^{s+1} a_{i+1,j}^k (i + 1)(m - i)x_i^k, \end{aligned}$$



we conclude that  $a_{i+1,j}^k = a_{0,j}^k$  for  $1 \leq k \leq s + 1$ , that is,  $[x_{i+1}^j, e_2] = \sum_{k=1}^{s+1} a_{0,j}^k x_{i+1}^k$  and Equality 4.1 is proved.

Putting  $a_j^k = a_{0,j}^k$ , we have  $[x_i^j, e_2] = \sum_{k=1}^{s+1} a_j^k x_i^k$ ,  $1 \leq j \leq s + 1$ ,  $0 \leq i \leq m$ .

Applying the Leibniz identity for the triples of elements:

$$(x_1^j, e_1, f_2), \quad 1 \leq j \leq s + 1,$$

we obtain

$$b_{1,j}^k = b_{0,j}^k, \quad 1 \leq k \leq s + 1.$$

Therefore, we obtain

$$[x_0^j, f_2] = \sum_{k=1}^{s+1} b_{0,j}^k x_0^k, \quad [x_1^j, f_2] = \sum_{k=1}^{s+1} b_{0,j}^k x_1^k, \quad 1 \leq j \leq s + 1.$$

Applying the induction and the following chain of equalities

$$\begin{aligned} 0 &= [x_{i+1}^j, [e_1, f_2]] = [[x_{i+1}^j, e_1], f_2] - [[x_{i+1}^j, f_2], e_1] = -[(i + 1)(m - i)x_i^j, f_2] \\ &\quad - \sum_{k=1}^{s+1} b_{i+1,j}^k [x_{i+1}^k, e_1] = -(i + 1)(m - i) \sum_{k=1}^{s+1} b_{0,j}^k x_i^k \\ &\quad + \sum_{k=1}^{s+1} b_{i+1,j}^k (i + 1)(m - i)x_i^k, \end{aligned}$$

we derive the equality

$$[x_i^j, f_2] = \sum_{k=1}^{s+1} b_{0,j}^k x_i^k, \quad 0 \leq i \leq m, \quad 1 \leq j \leq s + 1.$$

Setting  $b_j^k = b_{0,j}^k$ , we obtain  $[x_i^j, f_2] = \sum_{k=1}^{s+1} b_j^k x_i^k$ ,  $0 \leq i \leq m$ ,  $1 \leq j \leq s + 1$ .

Analogously, one can prove the equality  $[x_i^j, h_2] = \sum_{k=1}^{s+1} c_j^k x_i^k$  with  $1 \leq j \leq s + 1$ . □

Now we shall describe Leibniz algebras such that  $L/I \cong sl_2^1 \oplus sl_2^2$  and  $I = I_{1,1} \oplus I_{1,2}$ , where  $I_{1,1}, I_{1,2}$  are irreducible  $sl_2^1$ -modules. Without loss of generality we can suppose

$$\begin{aligned} [x_i^j, h_1] &= (m - 2i)x_i^j, & 0 \leq i \leq m, \\ [x_i^j, f_1] &= x_{i+1}^j, & 0 \leq i \leq m - 1, \\ [x_i^j, e_1] &= -i(m + 1 - i)x_{i-1}^j, & 1 \leq i \leq m. \end{aligned}$$

for  $j = 1, 2$ .

According to Proposition 4.1, one can assume that

$$\begin{aligned} [x_i^1, e_2] &= a_1x_i^1 + a_2x_i^2, & [x_i^2, e_2] &= a_3x_i^1 + a_4x_i^2, \\ [x_i^1, f_2] &= b_1x_i^1 + b_2x_i^2, & [x_i^2, f_2] &= b_3x_i^1 + b_4x_i^2, \\ [x_i^1, h_2] &= c_1x_i^1 + c_2x_i^2, & [x_i^2, h_2] &= c_3x_i^1 + c_4x_i^2, \end{aligned}$$

where  $0 \leq i \leq m$ .

From the following chains of equalities obtained applying the Leibniz identity

$$\begin{aligned} 2(a_1x_0^1 + a_2x_0^2) &= 2[x_0^1, e_2] = [x_0^1, [e_2, h_2]] \\ &= (a_2c_3 - c_2a_3)x_0^1 + (a_1c_2 + a_2c_4 - c_1a_2 - c_2a_4)x_0^2, \\ 2(a_3x_0^1 + a_4x_0^2) &= 2[x_0^2, e_2] = [x_0^2, [e_2, h_2]] \\ &= (a_3c_1 + a_4c_3 - c_3a_1 - c_4a_3)x_0^1 + (a_3c_2 - a_2c_3)x_0^2, \\ -2(b_1x_0^1 + b_2x_0^2) &= -2[x_0^1, f_2] = [x_0^1, [f_2, h_2]] = (b_2c_3 - c_2b_3)x_0^1 \\ &\quad + (b_1c_2 + b_2c_4 - c_1b_2 - c_2b_4)x_0^2, -2(b_3x_0^1 + b_4x_0^2) \\ &= 2[x_0^2, e_2] = [x_0^2, [f_2, h_2]] \\ &= (b_3c_1 + b_4c_3 - c_3b_1 - c_4b_3)x_0^1 + (b_3c_2 - b_2c_3)x_0^2, \\ -c_1x_1^1 - c_2x_1^2 &= -[x_1^1, h_2] = [x_1^1, [f_2, e_2]] \\ &= (a_3b_2 - a_2b_3)x_1^1 + 2(a_2b_1 - a_1b_2)x_1^2, \\ -c_3x_1^1 - c_4x_1^2 &= -[x_1^2, h_2] = [x_1^2, [f_2, e_2]] \\ &= 2(a_1b_3 - a_3b_1)x_1^1 + (a_2b_3 - a_3b_2)x_1^2. \end{aligned}$$

we derive:

$$\left\{ \begin{array}{l} 2a_1 = a_2c_3 - a_3c_2, \\ 2a_2 = a_1c_2 + a_2c_4 - c_1a_2 - c_2a_4, \\ 2a_3 = a_3c_1 + a_4c_3 - c_3a_1 - c_4a_3, \\ 2a_4 = a_3c_2 - a_2c_3, \\ -2b_1 = b_2c_3 - c_2b_3, \\ -2b_2 = b_1c_2 + b_2c_4 - c_1b_2 - c_2b_4, \\ -2b_3 = b_3c_1 + b_4c_3 - c_3b_1 - c_4b_3, \\ -2b_4 = b_3c_2 - b_2c_3, \\ c_1 = a_2b_3 - a_3b_2, \\ c_2 = 2(a_1b_2 - a_2b_1), \\ c_3 = 2(a_3b_1 - a_1b_3), \\ c_4 = a_3b_2 - a_2b_3. \end{array} \right. \tag{4.2}$$

It is easy to see that  $a_4 = -a_1, b_4 = -b_1$  and  $c_4 = -c_1$ .

Thus, we obtain the following products:

$$\begin{aligned} [x_i^1, e_2] &= a_1x_i^1 + a_2x_i^2, & [x_i^1, f_2] &= b_1x_i^1 + b_2x_i^2, \\ [x_i^2, e_2] &= a_3x_i^1 - a_1x_i^2, & [x_i^2, f_2] &= b_3x_i^1 - b_1x_i^2, \\ [x_i^1, h_2] &= (a_2b_3 - a_3b_2)x_i^1 + 2(a_1b_2 - a_2b_1)x_i^2, \\ [x_i^2, h_2] &= 2(a_3b_1 - a_1b_3)x_i^1 - (a_2b_3 - a_3b_2)x_i^2, \end{aligned} \tag{4.3}$$

where the structure constants  $a_1, a_2, a_3$  and  $b_1, b_2, b_3$  satisfy relations (4.2).

We present the classification of Leibniz algebras satisfying the following conditions:

- (a)  $L/I \cong sl_2^1 \oplus sl_2^2$ ;
- (b)  $I = I_{1,1} \oplus I_{1,2}$  such that  $I_{1,1}, I_{1,2}$  are irreducible  $sl_2^1$ -modules and  $\dim I_{1,1} = \dim I_{1,2}$ ;
- (c)  $I = I_{2,1} \oplus I_{2,2} \oplus \dots \oplus I_{2,m+1}$  such that  $I_{2,k}$  are irreducible  $sl_2^2$ -modules with  $1 \leq k \leq m + 1$ .

**Theorem 4.2** *An arbitrary Leibniz algebra satisfying the conditions (a)–(c) is isomorphic to the following algebra:*

$$\left\{ \begin{array}{l} [e_i, h_i] = -[h_i, e_i] = 2e_i, \\ [e_i, f_i] = -[f_i, e_i] = h_i, \\ [h_i, f_i] = -[f_i, h_i] = 2f_i, \\ [x_k^i, h_1] = (m - 2k)x_k^i, & 0 \leq k \leq m, \\ [x_k^i, f_1] = x_{k+1}^i, & 0 \leq k \leq m - 1, \\ [x_k^i, e_1] = -k(m + 1 - k)x_{k-1}^i, & 1 \leq k \leq m, \\ [x_j^1, e_2] = [x_j^2, h_2] = x_j^2, \\ [x_j^1, h_2] = [x_j^2, f_2] = -x_j^1, \end{array} \right.$$

with  $1 \leq i \leq 2$  and  $0 \leq j \leq m$ .

*Proof* We set  $\dim I_{1,1} = \dim I_{1,2} = m + 1$ . Then, according to Theorem 2.8, we obtain  $\dim I_{2,k} = 2$  for  $1 \leq k \leq m + 1$ .

Let  $\{x_0^1, x_1^1, \dots, x_m^1\}$ ,  $\{x_0^2, x_1^2, \dots, x_m^2\}$  and  $\{y_{i+1}^1, y_{i+1}^2\}$  be bases of  $I_{1,1}$ ,  $I_{1,2}$  and  $I_{2,i+1}$ ,  $0 \leq i \leq m$ , respectively. We set

$$y_{i+1}^1 = \alpha_i^1 x_i^1 + \alpha_i^2 x_i^2, \quad y_{i+1}^2 = \beta_i^1 x_i^1 + \beta_i^2 x_i^2, \quad 0 \leq i \leq m. \tag{4.4}$$

Taking into account the products (4.3) for  $0 \leq i \leq m$  we consider the equalities

$$\begin{aligned} 0 = [y_{i+1}^2, f_2] &= [\beta_i^1 x_i^1 + \beta_i^2 x_i^2, f_2] = \beta_i^1 (b_1 x_i^1 + b_2 x_i^2) + \\ &+ \beta_i^2 (b_3 x_i^1 - b_1 x_i^2) = (\beta_i^1 b_1 + \beta_i^2 b_3) x_i^1 + (\beta_i^1 b_2 - \beta_i^2 b_1) x_i^2. \end{aligned}$$

Therefore,

$$\begin{cases} \beta_i^1 b_1 + \beta_i^2 b_3 = 0, \\ \beta_i^1 b_2 - \beta_i^2 b_1 = 0, \end{cases} \tag{4.5}$$

with  $0 \leq i \leq m$ .

If  $b_1^2 + b_2 b_3 \neq 0$ , then the system of equations (4.5) has only trivial solution, which is a contradiction. Hence,  $b_1^2 + b_2 b_3 = 0$ .

Similarly, from

$$0 = [y_{i+1}^1, e_2] = (\alpha_i^1 a_1 + \alpha_i^2 a_3) x_i^1 + (\alpha_i^1 a_2 - \alpha_i^2 a_1) x_i^2$$

we derive  $a_1^2 + a_2 a_3 = 0$ .

Thus, we have  $a_1 = \pm i \sqrt{a_2 a_3}$  and  $b_1 = \pm i \sqrt{b_2 b_3}$ .

Let us summarize the obtained products:

$$\begin{aligned} [x_i^1, e_2] &= a_1 x_i^1 + a_2 x_i^2, & [x_i^1, f_2] &= b_1 x_i^1 + b_2 x_i^2, \\ [x_i^2, e_2] &= a_3 x_i^1 - a_1 x_i^2, & [x_i^2, f_2] &= b_3 x_i^1 - b_1 x_i^2, \\ [x_i^1, h_2] &= (a_2 b_3 - a_3 b_2) x_i^1 + 2(a_1 b_2 - a_2 b_1) x_i^2, \\ [x_i^2, h_2] &= 2(a_3 b_1 - a_1 b_3) x_i^1 - (a_2 b_3 - a_3 b_2) x_i^2, \end{aligned} \tag{4.6}$$

with  $0 \leq i \leq m$  and the relations  $a_1^2 + a_2 a_3 = b_1^2 + b_2 b_3 = 0$ .

Taking the following basis transformation:

$$x_i^{1'} = Ax_i^1 + Bx_i^2, \quad x_i^{2'} = (Aa_1 + Ba_3)x_i^1 + (Aa_2 - Ba_1)x_i^2, \quad 0 \leq i \leq m$$

we can assume that the products (4.6) have the following form:

$$\begin{aligned} [x_i^1, e_2] &= x_i^2, & [x_i^2, e_2] &= 0, \\ [x_i^1, f_2] &= b_1 x_i^1 + b_1^2 x_i^2, & [x_i^2, f_2] &= -x_i^1 - b_1 x_i^2, \\ [x_i^1, h_2] &= -x_i^1 - 2b_1 x_i^2, & [x_i^2, h_2] &= x_i^2. \end{aligned}$$

Applying the change of basis as follows

$$x_i^{1'} = x_i^1 + b_1 x_i^2, \quad x_i^{2'} = x_i^2, \quad 0 \leq i \leq m,$$

we complete the proof of theorem. □

The following theorem establishes that condition (c) can be omitted because if conditions (a)–(b) are true, then condition (c) is always executable.

**Theorem 4.3** *Let  $L$  be a Leibniz algebra satisfying the conditions (a)–(b). Then either  $L$  satisfies the condition (c) or  $L \cong (sl_2^1 + I) \oplus sl_2^2$ .*

*Proof* Let  $L$  be a Leibniz algebra satisfying conditions (a) and (b), but not (c). In order to prove the assertion of theorem we have to establish that all modules  $I_{2,i}$ ,  $1 \leq i \leq m + 1$  are reducible over  $sl_2^2$ . Indeed, according to Theorem 2.5 we conclude that  $I_{2,i}$  are completely reducible modules over  $sl_2^2$ . In denotation of (4.4) we have  $I_{2,i} = \langle y_i^1 \rangle \oplus \langle y_i^2 \rangle$ , where  $\langle y_i^1 \rangle$ ,  $\langle y_i^2 \rangle$  are one-dimensional trivial  $sl_2^2$ -modules, that is,

$$[y_i^1, e_2] = [y_i^2, e_2] = [y_i^1, f_2] = [y_i^2, f_2] = [y_i^1, h_2] = [y_i^2, h_2] = 0.$$

We shall prove by contrary method, that is, we shall assume that not all modules  $I_{2,i}$  are reducible over  $sl_2^2$ . Then we can assume that there exist some  $s$ ,  $1 \leq s \leq m + 1$  and  $t$ ,  $1 \leq t \leq m + 1$ ,  $t \neq s$  such that  $I_{2,s}$  is irreducible, but  $I_{2,t}$  is reducible  $sl_2^2$ -modules. By renumerating of indexes, without loss of generality, we can suppose  $s = 2$  and  $t = 1$ .

From the products in the proof of Theorem 4.2 we have

$$[x_1^1, e_2] = [x_1^2, h_2] = x_1^2, \quad [x_1^1, h_2] = [x_1^2, f_2] = -x_1^1, \quad [x_1^2, e_2] = [x_1^1, f_2] = 0.$$

Consider the chain of equalities

$$\begin{aligned} 0 &= [y_1^1, [e_2, f_1]] = [[y_1^1, e_2], f_1] - [[y_1^1, f_1], e_2] = -[[y_1^1, f_1], e_2] = \\ &\quad - [[\alpha_0^1 x_0^1 + \alpha_0^2 x_0^2, f_1], e_2] = -[\alpha_0^1 x_1^1 + \alpha_0^2 x_1^2, e_2] = -\alpha_0^1 x_1^2, \\ 0 &= [y_1^1, [f_2, f_1]] = [[y_1^1, f_2], f_1] - [[y_1^1, f_1], f_2] = -[[y_1^1, f_1], f_2] = \\ &\quad - [[\alpha_0^1 x_0^1 + \alpha_0^2 x_0^2, f_1], e_2] = -[\alpha_0^1 x_1^1 + \alpha_0^2 x_1^2, f_2] = \alpha_0^2 x_1^1. \end{aligned}$$

Therefore,  $\alpha_0^1 = \alpha_0^2 = 0$ , which means  $y_1^1 = 0$ . Thus, we obtain a contradiction. □

### 5 Some Remarks on Leibniz Algebras with Semisimple Part $sl_n$

In this section we present the structure of Leibniz algebras with associated Lie algebra  $sl_n$  and with dimension of ideal  $I$  equal to 1 and 2.

**Proposition 5.1** *Let  $L/I$  be isomorphic to  $G$  where  $G = \langle e_1, e_2, \dots, e_m \rangle$  is a semisimple Lie algebra and  $\dim I = 1$ . Then  $L = G \oplus I$ .*

*Proof* We put  $I = \langle x \rangle$  and  $[x, e_i] = \alpha_i x$ . For the semisimple Lie algebra  $G$  we have  $[G, G] = G$ , that is, for any  $e_i$ ,  $1 \leq i \leq m$  there exist  $\beta_{p,q}$  such that  $e_i = \sum_{p,q} \beta_{p,q} [e_p, e_q]$ .

The following chain of equalities completes the proof of proposition.

$$\begin{aligned} [x, e_i] &= \left[ x, \sum_{p,q} \beta_{p,q} [e_p, e_q] \right] = \sum_{p,q} \beta_{p,q} [x, [e_p, e_q]] \\ &= \sum_{p,q} \beta_{p,q} ([ [x, e_p], e_q ] - [ [x, e_q], e_p ]) \\ &= \sum_{p,q} \beta_{p,q} (\alpha_p [x, e_q] - \alpha_q [x, e_p]) \\ &= \sum_{p,q} \beta_{p,q} (\alpha_p \alpha_q x - \alpha_q \alpha_p x) = 0. \end{aligned}$$

□

Let  $G$  be a simple Lie algebra with a basis  $\{e_1, e_2, \dots, e_m\}$  which satisfies the condition that for any  $e_i$  there exist  $p, q$  such that  $e_i = [e_p, e_q]$ .

We consider the case of  $L/I \cong G$  and  $\dim I = 2$ . We set  $\{x, y\}$  the basis elements of  $I$  and

$$[x, e_i] = \alpha_i x + \beta_i y, \quad [y, e_i] = \gamma_i x + \delta_i y, \quad 1 \leq i \leq n.$$

Consider

$$\begin{aligned} [x, e_i] &= [x, [e_p, e_q]] = [[x, e_p], e_q] - [[x, e_q], e_p] \\ &= (\beta_p \gamma_q - \beta_q \gamma_p)x + (\alpha_p \beta_q + \beta_p \delta_q - \alpha_q \beta_p - \beta_q \delta_p)y. \end{aligned}$$

On the other hand,  $[x, e_i] = \alpha_i x + \beta_i y$ .

Comparing the coefficients we derive

$$\begin{cases} \alpha_i = \beta_p \gamma_q - \beta_q \gamma_p, \\ \beta_i = \alpha_p \beta_q + \beta_p \delta_q - \alpha_q \beta_p - \beta_q \delta_p, \end{cases} \tag{5.1}$$

where  $1 \leq i \leq n$ .

Similarly, if we consider  $[y, e_i]$ , then

$$\begin{cases} \gamma_i = \gamma_p \alpha_q + \delta_p \gamma_q - \gamma_q \alpha_p - \delta_q \gamma_p, \\ \delta_i = \gamma_p \beta_q - \gamma_q \beta_p, \end{cases} \tag{5.2}$$

where  $1 \leq i \leq n$ .

From the systems (5.1) and (5.2) we obtain

$$[x, e_i] = \alpha_i x + \beta_i y, \quad [y, e_i] = \gamma_i x - \alpha_i y, \quad 1 \leq i \leq n. \tag{5.3}$$

Let  $L/I$  be isomorphic to  $sl_n$ . From [6] we have the standard basis  $\{h_k, e_{i,j}, 1 \leq k \leq n-1, 1 \leq i, j \leq n, i \neq j\}$  of  $sl_n$ . We recall the table of multiplication of  $sl_n$

$$\begin{aligned} [e_{ij}, e_{jk}] &= e_{ik}, & 1 \leq i, j, k \leq n, i \neq j, j \neq k, k \neq i, \\ [h_k, e_{in}] &= e_{in}, [h_k, e_{ni}] = -e_{ni}, & 1 \leq i, k \leq n-1, k \neq i, \\ [h_i, e_{ij}] &= e_{ij}, [h_j, e_{ij}] = -e_{ij}, [e_{ij}, e_{jj}] = h_i - h_j, & 1 \leq i, j \leq n-1, \\ [h_i, e_{in}] &= 2e_{in}, [h_i, e_{ni}] = -2e_{ni}, [e_{in}, e_{ni}] = h_i, & 1 \leq i \leq n-1. \end{aligned}$$

**Theorem 5.2** *Let  $L/I$  be isomorphic to  $sl_n$  ( $n \geq 3$ ), where  $\dim I = 2$ . Then  $L = sl_n \oplus I$ .*

*Proof* From (5.3) we have

$$\begin{aligned} [x, h_i] &= \alpha_i x + \beta_i y, \quad 1 \leq i \leq n-1, \quad [y, h_i] = \gamma_i x - \alpha_i y, \quad 1 \leq i \leq n-1, \\ [x, e_{ij}] &= \alpha_{ij} x + \beta_{ij} y, \quad [y, e_{ij}] = \gamma_{ij} x - \alpha_{ij} y, \quad 1 \leq i, j \leq n, \quad i \neq j. \end{aligned}$$

Applying the Leibniz identity for the following triples of elements

$$(x, h_i, e_{ij}), \quad (y, h_i, e_{ij}), \quad (x, h_i, e_{in}), \quad (y, h_i, e_{in}), \quad (x, h_i, e_{ni}), \quad (y, h_i, e_{ni}).$$

we deduce

$$\begin{cases} \alpha_{ij} + \gamma_i \beta_{ij} - \beta_i \gamma_{ij} = 0, \\ 2\beta_i \alpha_{ij} + (1 - 2\alpha_i) \beta_{ij} = 0, \\ -2\gamma_i \alpha_{ij} + (1 + 2\alpha_i) \gamma_{ij} = 0, \end{cases} \tag{5.4}$$

$$\begin{cases} 2\alpha_{in} + \gamma_i \beta_{in} - \beta_i \gamma_{in} = 0, \\ \beta_i \alpha_{in} + (1 - \alpha_i) \beta_{in} = 0, \\ -\gamma_i \alpha_{in} + (1 + \alpha_i) \gamma_{in} = 0, \end{cases} \tag{5.5}$$

$$\begin{cases} -2\alpha_{ni} + \gamma_i \beta_{ni} - \beta_i \gamma_{ni} = 0, \\ \beta_i \alpha_{ni} - (\alpha_i + 1) \beta_{ni} = 0, \\ -\gamma_i \alpha_{ni} + (\alpha_i - 1) \gamma_{ni} = 0. \end{cases} \tag{5.6}$$

Determinants of the systems (5.4)–(5.6) have the following values:

$$\begin{aligned} \text{Det} \begin{pmatrix} 1 & \gamma_i & -\beta_i \\ 2\beta_i & 1 - 2\alpha_i & 0 \\ -2\gamma_i & 0 & 1 + 2\alpha_i \end{pmatrix} &= 1 - 4(\alpha_i^2 + \beta_i \gamma_i), \\ \text{Det} \begin{pmatrix} 2 & \gamma_i & -\beta_i \\ \beta_i & 1 - \alpha_i & 0 \\ -\gamma_i & 0 & 1 + \alpha_i \end{pmatrix} &= 2 - 2(\alpha_i^2 + \beta_i \gamma_i), \\ \text{Det} \begin{pmatrix} -2 & \gamma_i & -\beta_i \\ \beta_i & -1 - \alpha_i & 0 \\ -\gamma_i & 0 & \alpha_i - 1 \end{pmatrix} &= -2 + 2(\alpha_i^2 + \beta_i \gamma_i). \end{aligned}$$

It is easy to see that these determinants do not equal to zero simultaneously. Therefore, we conclude that either

$$\alpha_{in} = \beta_{in} = \gamma_{in} = \alpha_{ni} = \beta_{ni} = \gamma_{ni} = 0, \quad 1 \leq i \leq n-1$$

or

$$\alpha_{ij} = \beta_{ij} = \gamma_{ij} = 0, \quad 1 \leq i, j \leq n-1.$$

*Case 1* Let  $\alpha_{in} = \beta_{in} = \gamma_{in} = \alpha_{ni} = \beta_{ni} = \gamma_{ni} = 0$  where  $1 \leq i \leq n-1$ . Then the nonzero products  $[I, sl_n]$  are the following

$$\begin{aligned} [x, h_i] &= \alpha_i x + \beta_i y, & [y, h_i] &= \gamma_i x - \alpha_i y, & 1 \leq i \leq n-1, \\ [x, e_{ij}] &= \alpha_{ij} x + \beta_{ij} y, & [y, e_{ij}] &= \gamma_{ij} x - \alpha_{ij} y, & 1 \leq i, j \leq n-1, \quad i \neq j. \end{aligned}$$

From the equalities

$$\begin{aligned} [x, h_i] &= [x, [e_{in}, e_{ni}]] = [[x, e_{in}], e_{ni}] - [[x, e_{ni}], e_{in}] = 0, \\ [y, h_i] &= [y, [e_{in}, e_{ni}]] = [[y, e_{in}], e_{ni}] - [[y, e_{ni}], e_{in}] = 0, \\ [x, e_{ij}] &= [x, [e_{in}, e_{nj}]] = [[x, e_{in}], e_{nj}] - [[x, e_{nj}], e_{ni}] = 0, \\ [y, e_{ij}] &= [y, [e_{in}, e_{nj}]] = [[y, e_{in}], e_{nj}] - [[y, e_{nj}], e_{ni}] = 0, \end{aligned}$$

we obtain  $[I, sl_n] = 0$ .

*Case 2* Let  $\alpha_{ij} = \beta_{ij} = \gamma_{ij} = 0$  where  $1 \leq i, j \leq n-1$ . Then we have

$$\begin{aligned} [x, h_i] &= \alpha_i x + \beta_i y, & [y, h_i] &= \gamma_i x - \alpha_i y, & 1 \leq i \leq n-1, \\ [x, e_{in}] &= \alpha_{in} x + \beta_{in} y, & [y, e_{in}] &= \gamma_{in} x - \alpha_{in} y, & 1 \leq i \leq n-1, \\ [x, e_{ni}] &= \alpha_{ni} x + \beta_{ni} y, & [y, e_{ni}] &= \gamma_{ni} x - \alpha_{ni} y, & 1 \leq i \leq n-1, \\ [x, e_{ij}] &= [y, e_{ij}] = 0, & & & 1 \leq i, j \leq n-1, \quad i \neq j. \end{aligned}$$

Using Leibniz identity for the following triples

$$\begin{aligned} (x, e_{ij}, e_{ji}), & (y, e_{ij}, e_{ji}), (x, h_j, e_{in}), (y, h_j, e_{in}), \\ (x, h_j, e_{ni}), & (y, h_j, e_{ni}), (x, e_{in}, e_{ni}), (y, e_{in}, e_{ni}), \end{aligned}$$

we obtain

$$[x, h_i] = [y, h_i] = [x, e_{in}] = [y, e_{in}] = [x, e_{ni}] = [y, e_{ni}] = 0, \quad 1 \leq i \leq n-1.$$

□

**Acknowledgements** The authors are sincerely grateful to anonymous referees for critical remarks.



## References

1. Albeverio, S., Omirov, B.A., Rakhimov, I.S.: Varieties of nilpotent complex Leibniz algebras of dimension less than five. *Commun. Algebra* **33**, 1575–1585 (2005)
2. Ayupov, S.A., Omirov, B.A.: On Leibniz algebras. *Algebra and operators theory. Proceedings of the Colloquium in Tashkent 1997*, pp. 1–13. Kluwer Academic Publishers, Dordrecht (1998)
3. Barnes, D.W.: On Levi's theorem for Leibniz algebras. *Bull. Aust. Math. Soc.* **86**(2), 184–185 (2012)
4. Camacho, L.M., Gómez-Vidal, S., Omirov, B.A.: Leibniz algebras with associated Lie algebras  $sl_2 + R$ , ( $\dim R = 2$ ). [arXiv:1111.4631v1](https://arxiv.org/abs/1111.4631v1), pp. 11. *Commun. Algebra* **43**(10), 4403–4414 (2015)
5. Gómez-Vidal, S., Omirov, B.A., Khudoyberdiyev, A.K.: Some remarks on semisimple Leibniz algebras. *J. Algebra* **410**, 526–540 (2014)
6. Jacobson, N.: *Lie algebras*. Wiley, New York (1962)
7. Loday, J.-L.: Une version non commutative des algèbres de Lie: les algèbres de Leibniz. *Ens. Math.* **39**, 269–293 (1993)
8. Loday, J.-L., Pirashvili, T.: Universal enveloping algebras of Leibniz algebras and (co)homology. *Math. Ann.* **296**, 139–158 (1993)
9. Omirov, B.A., Rakhimov, I.S., Turdibaev, R.M.: On description of Leibniz algebras corresponding to  $sl_2$ . *Algebra Represent. Theory* **16**(5), 1507–1519 (2013)
10. Rakhimov, I.S., Rikhsiboev, I.M., Khudoyberdiyev, A.K., Karimjanov, I.A.: Description of some classes of Leibniz algebras. *Linear Algebra Appl.* **437**(9), 2209–2227 (2012)
11. Rakhimov, I.S., Atan, K.A.M.: On contractions and invariants of Leibniz algebras. *Bull. Malays. Math. Sci. Soc.* **35**(2), 557–565 (2012)