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An Algorithm for Vector Optimization Problems

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Abstract In this paper, we consider of finding efficient solution and weakly efficient solution for nonconvex vector optimization problems. When X and Y are normed spaces, F is an anti-Lipschitz mapping from X to Y, and the ordering cone is regular, we present an algorithm to guarantee that the generated sequence converges to an efficient solution with respect to normed topology. If the domain of the mapping is compact, we prove that the generated sequence converges to an efficient solution with respect to normed topology without requiring that mapping is anti-Lipschitz. We also give an algorithm to guarantee that the generated sequence converges to a weakly efficient solution with respect to normed topology.

Keywords Vector optimization problems · Efficient solution · Weakly efficient solution · Algorithm

Mathematics Subject Classification 90C26 · 90C29

1 Introduction

Recently, some numerical methods for solving convex multiobjective optimization problems have been proposed in the following papers: The steepest descent method

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for multiobjective optimization was dealt with in [1], and an extension of the projective gradient method to the case of convex constrained vector optimization can be found in [2]. Bonnel et al. [3] constructed a vector-valued proximal point algorithm to investigate convex vector optimization problem in Hilbert space, and they generalized the famous Rockafellar's results [4] from scalar case to vector case. Ceng and Yao generalized Bonnel's results to approximate case in [5]. Chen and Zhao [6] proposed a generalized proximal point algorithm for convex vector optimization problems in uniformly convex and uniformly smooth Banach spaces. Chen et al. [7] introduced a vector-valued Tikhonov-type regularization algorithm for an extended-valued multiobjective optimization problem, and under some mild conditions, they proved that any sequence generated by the algorithm converges to a weak Pareto optimal solution of the multiobjective optimization problem. Similar study can also be found in [8].

In this paper, we consider of finding efficient solution and weakly efficient solution for nonconvex vector optimization problems. By the method of scalarization, under the condition that mapping is anti-Lipschitz, and the ordering cone is regular, we present an algorithm to guarantee that the generated sequence converges to an efficient solution with respect to normed topology. If the domain of the mapping is compact, we prove that the generated sequence converges to an efficient solution with respect to normed topology without requiring that the mapping is anti-Lipschitz. We also give an algorithm to guarantee that the generated sequence converges to a weakly efficient solution with respect to normed topology.

2 Preliminaries and Definitions

Throughout this paper, let X and Y be real normed linear spaces. Let Y^* be the topological dual space of Y. Let C be a closed convex pointed cone in Y. The cone C induces a partial ordering in Y defined by

$$x \leq y \quad \text{if and only if} \quad y - x \in C.$$

Let

$$C^* = \{f \in Y^* : f(y) \geq 0, \quad \text{for all} y \in C\}$$

be the dual cone of C. Denote the quasi-interior of C^* by $C^\sharp$, i.e.,

$$C^\sharp := \{f \in Y^* : f(y) > 0 \ \text{ for all } y \in C \backslash \{0\}\}.$$

Let D be a nonempty subset of Y. The cone hull of D is defined as

$$\text{cone}(D) = \{td : t \geq 0, d \in D\}.$$

Denote the closure of D by $\text{cl}(D)$ and the interior of D by $\text{int}D$.

A nonempty convex subset B of the convex cone C is called a base of C if $C=\text{cone}(B)$ and $0 \notin \text{cl}(B)$.

By the separation theorem of convex sets (see [9], Theorem 3.20, Theorem 3.16), we can get the following two lemmas.

Lemma 2.1 *If C is a closed convex pointed cone in Y, then $C^\sharp \neq \emptyset$ if and only if C has a base.*

Lemma 2.2 *If C is a closed convex pointed cone in Y with $\text{int}C \neq \emptyset$, then $C^* \backslash \{0\} \neq \emptyset$.*

Let A be a nonempty subset of X, and let $F: A \to Y$ be a mapping. We consider the following vector optimization problem (in short (VOP)):

$$\min_{x \in A} F(x).$$

Definition 2.1 A vector $x_0 \in A$ is called an efficient solution to the (VOP) if

$$\{F(x_0)\} = (F(x_0) - C) \cap F(A).$$

The set of efficient solutions to the (VOP) is denoted by $E(A, F)$.

Definition 2.2 Let $\text{int}C \neq \emptyset$. A vector $x_0 \in A$ is called a weakly efficient solution to the (VOP) if

$$(F(x_0) - \text{int}C) \cap F(A) = \emptyset.$$

The set of weakly efficient solutions to the (VOP) is denoted by $E_W(A, F)$.

A set $D \subset Y$ is said to be bounded from below if there exists some $y_0 \in Y$, such that

$$y_0 \leq y \quad \text{for all} \quad y \in D.$$

Definition 2.3 (see [10]) The closed convex pointed cone C is said to be regular if every decreasing sequence which is bounded from below is convergent in norm topology.

Remark 2.1 The spaces R^n, c_0, l, l^2, $L[a, b]$, and $L^2[a, b]$ are Banach spaces, whose positive cones are regular.

Definition 2.4 Let A be a nonempty subset of X. A mapping $F: A \to Y$ is said to be anti-Lipschitz, if there exists a real number $\alpha > 0$ such that

$$\|x - y\| \leq \alpha \|F(x) - F(y)\| \quad \text{for all} \quad x, y \in A.$$

Remark 2.2 Let X be a real Hilbert with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let A be a nonempty subset of X. Let $F: A \to X$ be a mapping. If F is strong monotone on A, that is , there exists some $\beta > 0$ such that

$$\langle F(x) - F(y), x - y \rangle \geq \beta \|x - y\|^2 \quad \text{for all } x, \ y \ \in A,$$

then, by Schwarz inequality, we have

$$\|F(x) - F(y)\|\|x - y\| \geq \langle F(x) - F(y), x - y\rangle \geq \beta\|x - y\|^2 \text{ for all } x,\ y\ \in A.$$

Thus, we have

$$\|F(x) - F(y)\| \geq \beta\|x - y\| \text{ for all } x,\ y\ \in A.$$

From this, we can see that if X is a real Hilbert space, A is a nonempty subset of X, and $F\colon A \to X$ is strong monotone on A, then F is anti-Lipschitz.

3 Algorithm

In this section, let C be a closed convex pointed cone in Y, A be a nonempty subset of X, and $F\colon A \to Y$ be a mapping. Assume that $F(A)$ is bounded from below.

The first algorithm (to be called A1) is given as follows.

Let C have a base, and let $f \in C^\sharp$ (see Lemma 2.1). The method generates a sequence $\{x_n\} \subset X$ in the following way:

Initialization: Choose $x_1 \in A$.
Stopping rule: Given $x_n \in A$, if $x_n \in E(A, F)$, then $x_{n+p} = x_n$ for all $p \geq 1$.
Iterative step: Given $x_n \in A$, if $x_n \notin E(A, F)$, then take as the next iterate any $x_{n+1} \in A$ such that

$$F(x_{n+1}) \in (F(x_n) - C) \cap F(A), \tag{3.1}$$

and

$$f(F(x_{n+1})) < \inf\{f(y):\ y \in (F(x_n) - C) \cap F(A)\} + 1/2^n. \tag{3.2}$$

The second algorithm (to be called A2) is given as follows.

Let $\text{int}C \neq \emptyset$, and let $f \in C^*\backslash\{0\}$ (see Lemma 2.2). The method generates a sequence $\{x_n\} \subset X$ in the following way:

Initialization: Choose $x_1 \in A$.
Stopping rule: Given $x_n \in A$, if $x_n \in E_W(A, F)$, then $x_{n+p} = x_n$ for all $p \geq 1$.
Iterative step: Given $x_n \in A$, if $x_n \notin E_W(A, F)$, then take as the next iterate any $x_{n+1} \in A$ such that

$$F(x_{n+1}) \in (F(x_n) - \text{int}C) \cap F(A), \tag{3.3}$$

and

$$f(F(x_{n+1})) < \inf\{f(y):\ y \in (F(x_n) - \text{int}C) \cap F(A)\} + 1/2^n. \tag{3.4}$$

Under some conditions, we will prove that the generated sequence converges to an efficient solution.

Theorem 3.1 *Let X and Y be real normed linear spaces, and C be a closed convex pointed cone in Y. Let A be a nonempty subset of X, and let $F\colon A \to Y$ be a mapping. Assume that the following conditions are satisfied:*

(i) *C has a base, and C is regular;*
(ii) *$F(A)$ is closed and bounded from below;*
(iii) *F is an anti-Lipschitz mapping.*

Then, any sequence $\{x_n\}$ generated by algorithm A1 converges to an efficient solution of the (VOP) with respect to norm topology.

Proof By assumption, C has a base, in view of Lemma 2.1, $C^\sharp \neq \emptyset$. Let $f \in C^\sharp$. Given $x_1 \in A$, if $x_1 \in E(A, F)$, then $x_{1+p} = x_1$ for all $p \geq 1$.

If $x_1 \notin E(A, F)$. Since $F(A)$ is bounded from below, there exists some $y_0 \in Y$ such that

$$y_0 \leq F(x) \quad \text{for all } x \in A. \tag{3.5}$$

By $f \in C^\sharp$, we have

$$f(y_0) \leq f(F(x)) \quad \text{for all } x \in A.$$

Thus, $\inf\{f(y){:}y \in (F(x_1) - C) \cap F(A)\}$ is a real number. By the definition of infimum, there exists $x_2 \in A$ such that

$$F(x_2) \in (F(x_1) - C) \cap F(A),$$

and

$$f(F(x_2)) < \inf\{f(y) : \ y \in (F(x_1) - C) \cap F(A)\} + 1/2.$$

So, we can obtain the conclusion through finite iterations to get some x_n such that $x_n \in E(A, F)$, by picking $x_{n+p} = x_n$ for all $p \geq 1$, or else we can get a sequence $\{F(x_n)\}$ with $x_n \in A$ such that

$$F(x_{n+1}) \in (F(x_n) - C) \cap F(A), \tag{3.6}$$

and

$$f(F(x_{n+1})) < \inf\{f(y) : \ y \in (F(x_n) - C) \cap F(A)\} + 1/2^n. \tag{3.7}$$

By (3.5) and (3.6), we have

$$y_0 \leq \cdots \leq F(x_{n+1}) \leq F(x_n) \leq \cdots \leq F(x_1).$$

Since C is regular, $\{F(x_n)\}$ converges to some $\bar{y} \in Y$ in norm. As $F(A)$ is closed, we have $\bar{y} \in F(A)$. Thus, there exists some $\bar{x} \in A$ such that $\bar{y} = F(\bar{x})$. So

$$\lim_{n\to\infty} F(x_n) = F(\bar{x}). \tag{3.8}$$

Now we claim that $\bar{x} \in E(A, F)$. If not, then there exists some $\acute{x} \in A$ such that

$$F(\bar{x}) - F(\acute{x}) \in C\backslash\{0\}. \tag{3.9}$$

Since $f \in C^\sharp$, we have

$$f(F(\bar{x})) > f(F(\acute{x})). \tag{3.10}$$

Noting that $\{F(x_n)\}$ is a decreasing sequence, for any fixed n, when $m \geq n$, we have that

$$F(x_m) \leq F(x_n). \tag{3.11}$$

Taking the limit on the both sides of (3.11), let $m \to \infty$, by (3.8) and by the closedness of C, we get

$$F(\bar{x}) \leq F(x_n).$$

Thus,

$$F(\bar{x}) \leq F(x_n) \quad \text{for all} \quad n. \tag{3.12}$$

By (3.12), for each n, there exists $c_n \in C$ such that

$$F(\bar{x}) = F(x_n) - c_n.$$

This together with (3.9), we have

$$F(\acute{x}) \in (F(x_n) - C) \cap F(A) \quad \text{for all } n.$$

From (3.7), we have

$$f(F(x_{n+1})) \leq f(F(\acute{x})) + 1/2^n \quad \text{for all } n. \tag{3.13}$$

Taking limit on the both sides of (3.13), by the continuity of f, we get

$$f(F(\bar{x})) \leq f(F(\acute{x})).$$

It contradicts (3.10). Thus, $\bar{x} \in E(A, F)$.

Now, we show that $\{x_n\}$ converges to $\bar{x}$ with respect to norm topology. If not, then there exist some $\varepsilon_0 > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\|x_{n_k} - \bar{x}\| \geq \varepsilon_0. \tag{3.14}$$

By assumption, there exists some $\alpha > 0$ such that

$$\|x - y\| \leq \alpha \|F(x) - F(y)\| \quad \text{for any } x, y \in A.$$

This together with (3.14), for all k, we have

$$\varepsilon_0 \leq \|x_{n_k} - \bar{x}\| \leq \alpha \|F(x_{n_k}) - F(\bar{x})\|.$$

This contradicts (3.8). The proof is completed. □

Theorem 3.2 *Let X and Y be real normed linear spaces, and C be a closed convex pointed cone in Y with $\text{int}C \neq \emptyset$. Let A be a nonempty subset of X, and let $F\colon A \to Y$ be a mapping. Assume that the following conditions are satisfied:*

(i) *C is regular;*
(ii) *$F(A)$ is closed and bounded from below;*
(iii) *F is an anti-Lipschitz mapping.*

Then, any sequence $\{x_n\}$ generated by algorithm A2 converges to a weakly efficient solution of the (VOP) with respect to norm topology.

Proof Since $\text{int}C \neq \emptyset$, by Lemma 2.2, $C^*\backslash\{0\} \neq \emptyset$. Let $f \in C^*\backslash\{0\}$. Given $x_1 \in A$, if $x_1 \in E_W(A, F)$, then $x_{1+p} = x_1$ for all $p \geq 1$. Let $x_1 \notin E_W(A, F)$. Since $F(A)$ is bounded from below, there exists some $y_0 \in Y$ such that

$$y_0 \leq F(x) \quad \text{for all } x \in A.$$

By $f \in C^*\backslash\{0\}$, we have that

$$f(y_0) \leq f(F(x)) \text{ for all } x \in A. \tag{3.15}$$

Since $x_1 \notin E_W(A, F)$,

$$(F(x_1) - \text{int}C) \cap F(A) \neq \emptyset.$$

This together with (3.15), we know that

$$\inf\{f(y) : y \in (F(x_1) - \text{int}C) \cap F(A)\}$$

is a real number. By the definition of infimum, there exists $x_2 \in A$ such that

$$F(x_2) \in (F(x_1) - \text{int}C) \cap F(A),$$

and

$$f(F(x_2)) < \inf\{f(y): \ y \in (F(x_1) - \text{int}C) \cap F(A)\} + 1/2.$$

So, we can obtain the conclusion through finite iterations to get some x_n such that $x_n \in E_W(A, F)$, and by picking $x_{n+p} = x_n$ for all $p \geq 1$, or else we can get a sequence $\{F(x_n)\}$ by induction with $x_{n+1} \in A$ such that

$$F(x_{n+1}) \in (F(x_n) - \text{int}C) \cap F(A), \tag{3.16}$$

and

$$f(F(x_{n+1})) < \inf\{f(y): \ y \in (F(x_n) - \text{int}C) \cap F(A)\} + 1/2^n. \tag{3.17}$$

By (3.15) and (3.16), we have

$$y_0 \leq \cdots \leq F(x_{n+1}) \leq F(x_n) \leq \cdots \leq F(x_1).$$

Since C is regular, $\{F(x_n)\}$ converges to some $\bar{y} \in Y$ in norm. As $F(A)$ is closed, we have $\bar{y} \in F(A)$. Thus, there exists some $\bar{x} \in A$ such that $\bar{y} = F(\bar{x})$. So

$$\lim_{n \to \infty} F(x_n) = F(\bar{x}). \tag{3.18}$$

Now we claim that $\bar{x} \in E_W(A, F)$.

If not, then there exists some $\acute{x} \in A$ such that

$$F(\bar{x}) - F(\acute{x}) \in \text{int}C. \tag{3.19}$$

This together with $f \in C^* \backslash \{0\}$, we have

$$f(F(\bar{x})) > f(F(\acute{x})). \tag{3.20}$$

Noting that $\{F(x_n)\}$ is a decreasing sequence, for any fixed n, when $m \geq n$, we have that

$$F(x_m) \leq F(x_n). \tag{3.21}$$

Taking the limit on the both sides of (3.21), let $m \to \infty$, by (3.18) and by the closedness of C, we get

$$F(\bar{x}) \leq F(x_n).$$

Thus,

$$F(\bar{x}) \leq F(x_n) \quad \text{for all} \ \ n. \tag{3.22}$$

By (3.22), for each n, there exists $c_n \in C$ such that

$$F(\bar{x}) = F(x_n) - c_n.$$

This together with (3.19), we have

$$F(\acute{x}) \in (F(x_n) - \text{int}C) \cap F(A) \quad \text{for all } n.$$

From (3.17), we have

$$f(F(x_{n+1})) \leq f(F(\acute{x})) + 1/2^n \quad \text{for all } n. \tag{3.23}$$

Taking limit on the both sides of (3.23), by the continuity of f, we get

$$f(F(\bar{x})) \leq f(F(\acute{x})).$$

It contradicts (3.20). Thus $\bar{x} \in E_W(A, F)$.

In a fashion similar to Theorem 3.1, we can see that $\lim_{n\to\infty} x_n = \bar{x}$. The proof is completed. □

Theorem 3.3 *Let X and Y be real normed linear spaces, and C be a closed convex pointed cone in Y. Let A be a nonempty subset of X, and let $F: A \to Y$ be a mapping. Assume that the following conditions are satisfied:*

(i) *A is a nonempty compact subset of X;*
(ii) *C has a base, and C is regular;*
(iii) *F is continuous on A, and $F(A)$ is bounded from below;*
(iv) *F is injective.*

Then, any sequence $\{x_n\}$ generated by algorithm A1 converges to an efficient solution of the (VOP) with respect to norm topology.

Proof Let $f \in C^{\sharp}$. By assumption, we can see that $F(A)$ is closed. In a fashion similar to Theorem 3.1, we can obtain the conclusion through finite iterations to get some x_n such that $x_n \in E(A, F)$, and by picking $x_{n+p} = x_n$ for all $p \geq 1$, or else we can get a sequence $\{F(x_n)\}$ by induction with $x_{n+1} \in A$ such that

$$F(x_{n+1}) \in (F(x_n) - C) \cap F(A),$$

and

$$f(F(x_{n+1})) < \inf\{f(y) : \ y \in (F(x_n) - C) \cap F(A)\} + 1/2^n,$$

and there exists some $\bar{x} \in E(A, F)$ such that $\lim_{n\to\infty} F(x_n) = F(\bar{x})$. Now, we show that $x_n \to \bar{x}$. We pick any subsequence $\{x_{n_k}\}$ of $\{x_n\}$. By the compactness of A, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$, $\{x_{n_{k_j}}\}$ converges to some $\grave{x} \in A$. By the continuity of F, we have $\lim_{j\to\infty} F(x_{n_{k_j}}) = F(\grave{x})$. Since $\lim_{j\to\infty} F(x_{n_{k_j}}) = F(\bar{x})$, we have $F(\grave{x}) = F(\bar{x})$. As F is injective, we have $\grave{x} = \bar{x}$. Thus, for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$, $\{x_{n_{k_j}}\}$ converges to same $\bar{x} \in A$. We can see that $\{x_n\}$ converges to $\bar{x}$. The proof is completed. □

Similar to the proof of Theorem 3.2 and Theorem 3.3, we can get the following theorem.

Theorem 3.4 *Let X and Y be real normed linear spaces, C be a closed convex pointed cone in Y with $\mathrm{int}C \neq \emptyset$. Let A be a nonempty subset of X, and let $F: A \to Y$ be a mapping. Assume that the following conditions are satisfied:*

(i) *A is a nonempty compact subset of X;*
(ii) *C is regular;*
(iii) *F is continuous on A, and $F(A)$ is bounded from below;*
(iv) *F is injective.*

Then, any sequence $\{x_n\}$ generated by algorithm A2 converges to a weakly efficient solution of the (VOP) with respect to norm topology.

Now, we give two examples illustrating Theorem 3.1 and Theorem 3.2.

Example 3.1 Let

$$X = R, Y = R^2, C = R^2_+ = \{u = (x, y) : x \geq 0, y \geq 0\}, A = [1, +\infty).$$

Let $F: A \to R^2$ be defined by

$$F(x) = (x^2, x^{\frac{1}{2}}), x \in A.$$

It is clear that F is not convex.

By the mean value theorem, for any $x, y \in A$ with $x < y$, there exist $\xi \in (x, y)$ and $\eta \in (x, y)$ such that

$$\begin{aligned} \|F(x) - F(y)\| &= \|(x^2 - y^2, x^{\frac{1}{2}} - y^{\frac{1}{2}})\| = \|(2\xi(x - y), \frac{1}{2}\eta^{-\frac{1}{2}}(x - y))\| \\ &= \sqrt{(2\xi(x - y))^2 + (\frac{1}{2}\eta^{-\frac{1}{2}}(x - y))^2} \geq 2|x - y|. \end{aligned}$$

It is clear the conditions of Theorem 3.1 and Theorem 3.2 are satisfied.

Example 3.2 Let

$$X = Y = R^n, C = R^n_+ = \{x = (x_1, x_2, \ldots, x_n) : x_i \geq 0, i = 1, 2, \ldots, n\},$$
$$A_1 = \{x = (x_1, \ldots, x_n) : x_1^2 + x_2^2 + \cdots + x_n^2 = 1\}, A = A_1 + R^n_+,$$

and let $a_1, a_2, \ldots, a_n$ be positive real numbers. Let $F: R^n \to R^n$ be defined by

$$F(x) = (a_1x_1, a_2x_2, \ldots, a_nx_n), \quad x = (x_1, x_2, \ldots, x_n) \in R^n.$$

We have

$$\langle F(x) - F(y), x - y\rangle = \langle (a_1(x_1 - y_1), \dots, a_n(x_n - y_n)), (x_1 - y_1, \dots, x_n - y_n)\rangle$$
$$= \sum_{i=1}^{n} a_i(x_i - y_i)^2 \geq \beta \sum_{i=1}^{n} |x_i - y_i|^2 = \beta\|x - y\|^2,$$

where $\beta = \min\{a_1, a_2, \dots, a_n\}$. This means that $F\colon A \to Y$ is anti-Lipschitz. It is easy to see that $F(A_1)$ is closed and bounded, and so is compact. We can also see that $F(A) = F(A_1) + R^n_+$. Since $F(A_1)$ is compact and R^n_+ is closed, $F(A)$ is closed. It follows from int$C \neq \emptyset$ and $F(A_1)$ is bounded that $F(A)$ is bounded from below. Thus, the conditions of Theorem 3.1 and Theorem 3.2 are satisfied.

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