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Biderivations of Triangular Rings Revisited

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Abstract We consider the problem of describing the form of biderivations of a triangular ring. Our approach is based on the notion of the maximal left ring of quotients, which enables us to generalize Benkovič's result on biderivations (Benkovič in Linear Algebra Appl 431:1587–1602, 2009). Our result is applied to block upper triangular matrix rings and nest algebras.

Keywords Biderivation · Derivation · Triangular ring · Upper triangular matrix ring · Nest algebra · Maximal left ring of quotients · Utumi left quotient ring · Extended centroid

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1 Introduction

Let R be an associative ring. For $x, y \in R$ we denote $xy - yx$ by $[x, y]$. An additive map $d : R \to R$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. For example, if we take an element $a \in R$, then the map $x \mapsto [a, x]$ is a derivation. These kinds of derivations are called *inner derivations*. Next, a biadditive map $B : R \times R \to R$ is said to be a *biderivation* if it is a derivation in each argument, i.e., for each $y \in R$ the maps

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$$x \mapsto B(x, y) \quad \text{and} \quad x \mapsto B(y, x)$$

are derivations. For example, if we take a central element $\lambda \in Z(R)$, then the map $(x, y) \mapsto \lambda[x, y]$ is a biderivation. Such kind of biderivations is called *inner biderivations*.

Biderivations appear in many areas. Brešar et al. [5] have shown that each biderivation B of a noncommutative prime ring R is of the form $B(x, y) = \lambda[x, y]$, for some element λ in the extended centroid of R. It has turned out that this result can be applied to the problem of describing the form of commuting maps. The reader is referred to a survey paper [4] where applications of biderivations to some other areas are described. Biderivations were also studied on nest algebras by Zhang et al. [12] and on upper triangular matrix algebras by Zhao et al. [13]. In 2009, Benkovič [2] considered biderivations on a certain class of triangular algebras. He proved that a bilinear biderivation B of a triangular algebra R satisfying certain conditions (see conditions (a)–(e) on page 4) is of the form

$$B(x, y) = \lambda[x, y] + [x, [y, r]]$$

for some elements $\lambda \in Z(R)$ and $r \in R$. On the other hand, Ghosseiri [9] considered (biadditive) biderivations of an arbitrary triangular ring R (not assuming that eRf is a faithful bimodule). He proved that each biderivation $B : R \times R \to R$ can be written as

$$B(x, y) = B_1(x, y) + [x, [y, r]] + \Delta(x, y),$$

for some element $r \in R$ and some biderivations B_1 and Δ satisfying certain conditions. However, the explicit form of biderivations B_1 and Δ was not described. The goal of this paper is to generalize Benkovič's result [2, Theorem 4.11]. Namely, using the notion of the maximal left ring of quotients, we shall describe the form of (biadditive) biderivations for a much larger class of triangular rings than the one considered in [2] (see our main result, Theorem 3.3). In this context, we shall obtain a refinement of the result of Ghosseiri [9, Theorem 2.4] giving explicit form of biderivations B_1, Δ and consequently the one of B. We shall also apply Theorem 3.3 to (block) upper triangular matrix rings and nest algebras, obtaining a generalization of [2, Corollary 4.13] and an extension of [2, Corollary 4.14].

2 Preliminaries

The maximal left ring of quotients (or Utumi left quotient ring) of an associative ring R was introduced in 1956 by Utumi [14]. It turns out that each associative ring R, whose right annihilator $\text{rann}_R(R) := \{x \in R \mid Rx = 0\}$ is zero (in particular, if R is unital), has its maximal left ring of quotients, denoted by $Q(R)$. A left ideal I of R is called *dense* if for every $0 \neq r_1 \in R$, $r_2 \in R$ there exists an element $r \in R$ such that $rr_1 \neq 0$ and $rr_2 \in I$. Let us denote the set of all dense left ideals of R by $\mathcal{D}_l(R)$. The maximal left ring of quotients of a unital ring R can be characterized in the following way.

Proposition 2.1 (Theorem 24.8 in [11]) *Let R be a unital ring. The maximal left ring of quotients $Q(R)$ satisfies the following properties:*

(i) *R is a subring of $Q(R)$ with the same 1,*
(ii) *for any $q \in Q(R)$, there exists a dense left ideal I of R such that $Iq \subseteq R$,*
(iii) *if $0 \neq q \in Q(R)$ and I is a dense left ideal of R, then $Iq \neq 0$,*
(iv) *for any dense left ideal I of R and a left R-module homomorphism $f : I \to R$ there exists $q \in Q(R)$ such that f is a right multiplication by q.*

Moreover, the properties (i)–(iv) characterize $Q(R)$ up to an isomorphism, which is the identity on R.

By $C(R)$, we denote the center $Z(Q(R))$ of $Q(R)$, and we call it the *extended centroid* of R. Note that Proposition 2.1 implies that

$$C(R) = \{q \in Q(R) |\ qx = xq \text{ for all } x \in R\} \supseteq Z(R).$$

A unital ring R with a nontrivial idempotent e is said to be a *triangular ring* if eRf is a faithful (eRe, fRf)-bimodule and $fRe = 0$, where f denotes the idempotent $1 - e$. Each triangular ring R has the following Peirce decomposition:

$$R = eRe \oplus eRf \oplus fRf.$$

Obviously, a unital ring R is triangular if and only if there exist unital rings A, B and a unital faithful (A, B)-bimodule M such that R is isomorphic to the ring

$$\mathrm{Tri}(A, M, B) := \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix};\quad a \in A,\ m \in M,\ b \in B \right\}$$

with the usual matrix addition and multiplication. The most important examples of triangular rings are upper triangular matrix rings, block upper triangular matrix rings, and nest algebras. We shall need the following three results on triangular rings, which were obtained in our previous paper [8].

Proposition 2.2 (Proposition 2.6 in [8]) *Let R be a triangular ring. Then eR is a dense left ideal of R and for each $q \in Q(R)$ the following holds:*

(i) *$eRfq = 0$ implies $fq = 0$,*
(ii) *$qeRf = 0$ implies $qe = 0$.*

Proposition 2.3 (Proposition 2.7 in [8]) *Let R be a triangular ring. Then the following holds:*

(i) *$Z(R) = \{c \in eRe \oplus fRf \mid c \cdot exf = exf \cdot c$ for all $x \in R\}$,*
(ii) *$C(R) = \{q \in eQe \oplus fQf \mid q \cdot exf = exf \cdot q$ for all $x \in R\}$,*
(iii) *$Z(eRe) \subseteq C(R)e$,*

(iv) *There exists a unique ring isomorphism* $\tau : C(R)e \to C(R)f$ *such that* $\lambda e \cdot exf = exf \cdot \tau(\lambda e)$ *for all* $x \in R$, $\lambda \in C(R)$. *Moreover,* $\tau\big(Z(R)e\big) = Z(R)f$.

Lemma 2.4 (Lemma 3.1 in [8]) *Let* R *be a triangular ring. Suppose that* $F, G : eRf \to C(R)e$ *are arbitrary maps such that*

$$F(exf)eyf + G(eyf)exf = 0 \tag{2.1}$$

for all $x, y \in R$. *Then there exist* $s, t \in Q(R)$ *such that*

$$F(exf) = exfse \quad \text{and} \quad G(exf) = exfte,$$

and $exfse - ftexf, exfte - fsexf \in C(R)$ *for all* $x \in R$.

Remark 2.5 If both F and G from Lemma 2.4 map eRf to $Z(R)e$, then $exfse - ftexf \in Z(R)$ and $exfte - fsexf \in Z(R)$ for all $x \in R$.

Let R be a triangular ring. We know that

$$Z(R)e \subseteq Z(eRe) \subseteq C(R)e \quad \text{and} \quad Z(R)f \subseteq Z(fRf).$$

In general, however, $Z(fRf) \nsubseteq C(R)f$ (see [8, Section 6]). We shall see that the set

$$\mathcal{S}(R) := \{q \in fQ(R)f \mid [q, fRf] = 0 \quad \text{and} \quad eRq \subseteq R\}$$

plays an important role in our treatise. Obviously,

$$Z(R)f \subseteq Z(fRf) \subseteq \mathcal{S}(R).$$

In the following proposition, we give a sufficient condition for a triangular ring R to satisfy $\mathcal{S}(R) \subseteq C(R)f$. As in [2], we say that an (R, R)-bimodule endomorphism ψ of eRf is of the *standard form* if there exist $a \in Z(eRe)$ and $b \in Z(fRf)$ such that $\psi(exf) = aexf + exfb$ for all $x \in R$.

Proposition 2.6 *If* R *is a triangular ring such that:*

(i) $Z(fRf) \subseteq C(R)f$,
(ii) *each* (R, R)*-bimodule endomorphism of* eRf *is of the standard form,*

then $\mathcal{S}(R) \subseteq C(R)f$. *In particular, if* $Z(eRe) = Z(R)e$ *and* $Z(fRf) = Z(R)f$, *then* $\mathcal{S}(R) = Z(R)f$.

Proof Let q be an arbitrary element in $\mathcal{S}(R)$. Then $[q, fRf] = 0$ and $eRq \subseteq R$. We claim that $q \in C(R)f$. Let us define a map $\psi : eRf \to eRf$ by $\psi(exf) = exfq$. Obviously, ψ is an (R, R)-bimodule endomorphism. Thus, there exist $a \in Z(eRe) \subseteq C(R)e$ and $b \in Z(fRf) \subseteq C(R)f$ such that

$$\psi(exf) = aexf + exfb$$

for all $x \in R$. Consequently, we get $\psi(exf) = exf\big(\tau(a) + b\big)$ and so

$$eRf(q - \tau(a) - b) = 0.$$

Hence, $q = \tau(a) + b \in C(R)f$. In particular, if $Z(eRe) = Z(R)e$ and $Z(fRf) = Z(R)f$, then $q = \tau(a) + b \in Z(R)f$. □

Let R be an arbitrary ring. For any subsets $X, Y \subseteq R$, we define the sets: $\operatorname{ann}_X(Y) := \{x \in X \mid xY \cup Yx = 0\}$ and $\operatorname{rann}_X(Y) := \{x \in X \mid Yx = 0\}$.

3 The Main Result

In [2], Benkovič described the form of bilinear biderivations of a triangular algebra R satisfying the following assumptions:

(a) $Z(R)e = Z(eRe)$,
(b) $Z(R)f = Z(fRf)$,
(c) either eRe or fRf is noncommutative,
(d) if $0 \neq \alpha \in Z(R)$ and $0 \neq a \in R$, then $\alpha a \neq 0$,
(e) each linear derivation of R is inner.

Namely, he proved that in this case, each biderivation $B : R \times R \to R$ is of the form

$$B(x, y) = \lambda[x, y] + [x, [y, r]] \tag{3.1}$$

for some $\lambda \in Z(R)$ and $r \in R$. The goal of this paper is to show that using the notion of the maximal left ring of quotients, we are able to describe the form of (biadditive) biderivations of a triangular ring R assuming only that $\mathcal{S}(R) \subseteq C(R)f$. Note that this condition is considerably weaker than assumptions (a)–(e) together. Namely, assuming only (b) and (e), it follows already that $\mathcal{S}(R) \subseteq C(R)f$ (see [2, Corollary 3.4] and Proposition 2.6). However, when dealing with this more general class of triangular rings, we may also encounter biderivations different from those described in (3.1). For example, suppose that $R = T_2(K)$ is the ring of all upper triangular 2×2 matrices over a commutative ring K with unity. Let $e := e_{11}$ and $f := e_{22}$. Pick any $0 \neq q \in fM_2(K)e$. Then it is easy to see that a map $B : R \times R \to R$ defined by

$$B(x, y) = exfqeyf$$

is a biderivation, which is not of the form (3.1) (see also [2, Remark 4.15]). The main result of this paper, Theorem 3.3, states that each biderivation B of a triangular ring R such that $\mathcal{S}(R) \subseteq C(R)f$ is of the form

$$B(x, y) = \lambda[x, y] + [x, [y, \mu e + r]] + exfqeyf,$$

for some elements $\lambda, \mu \in C(R)$, $r \in R$, and $q \in Q(R)$.

Let R be a unital ring, and let $B : R \times R \to R$ be a biderivation. Then

$$B(x, 1) = B(1, x) = 0 \text{ and } B(x, 0) = B(0, x) = 0,$$

and according to [3, Corollary 2.4], we have

$$B(x, y)[u, v] = [x, y]B(u, v) \tag{3.2}$$

for all $x, y, u, v \in R$.

Lemma 3.1 *Let R be a triangular ring. If $B : R \times R \to R$ is a biderivation, then*

(i) $B(e, e) = -B(e, f) = -B(f, e) = B(f, f)$,
(ii) $B(e, e)[R, R] = 0 = [R, R]B(e, e)$,
(iii) $B(x, e), B(e, x), B(x, f), B(f, x) \in eRf$,
(iv) $B(exf, eyf) \in eRf$,
(v) $B(exe, e) = B(e, exe) = exeB(e, e)f$,
(vi) $B(fxf, f) = B(f, fxf) = eB(f, f)fxf$
(vii) $B(exe, eyf), B(eyf, exe), B(fxf, eyf), B(eyf, fxf) \in eRf$
(viii) $B(exe, fyf) = exeB(e, f)fyf = B(fyf, exe)$

for all $x, y \in R$.

Proof Obviously, the proof of (i) is straightforward (see [2, Lemma 4.2(iii)]) and (ii) follows immediately from (3.2). Note that [2, Lemma 4.3] implies (iv). Since

$$B(x, e) = B(x, e)e + eB(x, e)$$

we get $eB(x, e)e = 0$ and $fB(x, e)f = 0$. Hence, $B(x, e) = eB(x, e)f \in eRf$ for all $x \in R$. Similarly, $B(e, x), B(x, f), B(f, x) \in eRf$ for all $x \in R$ and so (iii) holds true. Using (iii), we see that

$$B(exe, e) = eB(exe, e)f = eB(exe, e)ef + exeB(e, e)f = exeB(e, e)f$$

and similarly $B(e, exe) = eB(e, exe)f = exeB(e, e)f$ for all $x \in R$, which proves (v). Analogously, (vi) holds true. Using (iii), we get

$$B(exe, eyf) = B(exe, eyf)f = eB(exe, eyf)f \in eRf$$

for all $x, y \in R$. Analogously, $B(eyf, exe), B(fxf, eyf), B(eyf, fxf) \in eRf$ for all $x, y \in R$ and so (vii) holds true. For all $x, y \in R$, we have

$$\begin{aligned} B(exe, fyf) &= B(exe, fyf)e + exeB(e, fyf) \\ &= fB(exe, fyf)e + B(exe, f)fyfe + exeB(e, f)fyf \\ &= exeB(e, f)fyf \end{aligned}$$

and similarly $B(fyf, exe) = exeB(e, f)fyf$, which proves (viii). □

Lemma 3.2 *Let R be a triangular ring. If $B : R \times R \to R$ is a biderivation such that $B(e, e) = 0$, then there exist $s, s_1 \in \mathcal{S}(R)$ and additive maps $p, p' : eRf \to \mathcal{S}(R)$ such that*

(i) $B(x, y) - B(exe, eye) = [x, y]s_1 + [y, exf]s + exfp(eyf)$,
(ii) $B(exe, eye)ezf = [exe, eye]ezfs_1$,

where $[R, R]eRs = 0 = f[R, R]s$ and $exfp(eyf) = eyfp'(exf)$ for all $x, y \in R$.

Proof Since $B(e, e) = 0$, Lemma 3.1 yields

$$\begin{aligned} &B(exe, e) = B(e, exe) = 0,\\ &B(fxf, f) = B(f, fxf) = 0,\\ &B(exe, fyf) = B(fyf, exe) = 0, \end{aligned} \tag{3.3}$$

and hence $B(exe, eye) \in eRe$ and $B(fxf, fyf) \in fRf$ for all $x, y \in R$. Let us define a map $\varphi_1 : eR \to eRf$ by

$$\varphi_1(ex) = B(e, exf).$$

Then

$$\begin{aligned} \varphi_1(yex) = \varphi_1(eyex) = B(e, eye \cdot exf) &= eyeB(e, exf) + B(e, eye)exf\\ &= eyeB(e, exf) = y\varphi_1(ex) \end{aligned}$$

for all $x, y \in R$. Thus, $\varphi_1 : eR \to R$ is a left R-module homomorphism. Moreover, since eR is a dense left ideal of R, Proposition 2.1 implies that there exists $s_1 \in Q(R)$ such that $\varphi_1(ex) = exs_1$ for all $x \in R$. Since $es_1 = \varphi_1(e) = B(e, 0) = 0$ and $\varphi_1(eR) \subseteq eRf$, we see that $s_1 = fs_1f \in fQ(R)f$. Thus,

$$B(e, exf) = exfs_1$$

for all $x \in R$. Analogously, defining a map $\varphi_2 : eR \to eRf$ by $\varphi_2(ex) = B(exf, e)$, we obtain an element $s_2 \in fQ(R)f$ such that

$$B(exf, e) = exfs_2$$

for all $x \in R$. Consequently,

$$\begin{aligned} B(exe, eyf) = B(exe \cdot e, eyf) &= B(exe, eyf)e + exeB(e, eyf)\\ &= eB(exe, eyf)e + exeyfs_1f \end{aligned}$$

and since $eB(exe, eyf)e = 0$ (see Lemma 3.1(vii)), we get

$$B(exe, eyf) = exeyfs_1f \tag{3.4}$$

for all $x, y \in R$. Analogously,

$$B(eyf, exe) = exeyfs_2f \tag{3.5}$$

for all $x, y \in R$. Note that

$$\begin{aligned} B(exf, f) &= eB(exf, f) = B(exf, ef) - B(exf, e)f \\ &= -exfs_2f \end{aligned}$$

and hence

$$B(exf, fyf) = fB(exf, fyf) + B(exf, f)fyf = -exfs_2fyf.$$

Thus,

$$B(exf, fyf) = -exfs_2fyf \tag{3.6}$$

and analogously

$$B(fyf, exf) = -exfs_1fyf \tag{3.7}$$

for all $x, y \in R$. Next, we claim that $s_1, s_2 \in \mathcal{S}(R)$. Namely, using Lemma 3.1(iii) and (3.4), we see that

$$\begin{aligned} 0 = exfB(e, fyf) &= B(e, exfyf) - B(e, exf)fyf \\ &= exfyfs_1f - exfs_1fyf = exf[fyf, s_1] \end{aligned}$$

for all $x, y \in R$. Hence, Proposition 2.2 implies that $[s_1, fRf] = 0$ and so $s_1 \in \mathcal{S}(R)$. Similarly, $s_2 \in \mathcal{S}(R)$. Setting $x = exe$, $y = eye$, $u = e$, $v = ezf$ in (3.2) and using (3.4), we obtain

$$B(exe, eye)ezf = [exe, eye]ezfs_1f \tag{3.8}$$

for all $x, y, z \in R$. Similarly, setting $x = exe$, $y = eye$, $u = ezf$, $v = f$ in (3.2) and using (3.6), we get

$$B(exe, eye)ezf = -[exe, eye]ezfs_2f \tag{3.9}$$

for all $x, y, z \in R$. Consequently, $[R, R]eRf(s_1 + s_2) = 0$. On the other hand, setting $x = e$, $y = ezf$, $u = fxf$, $v = fyf$ in (3.2) and using (3.4), we obtain

$$ezfs_1f[fxf, fyf] = ezfB(fxf, fyf)$$

and so $eRf\big(s_1[fxf, fyf] - B(fxf, fyf)\big) = 0$ for all $x, y \in R$. Hence, Proposition 2.2 yields

$$B(fxf, fyf) = s_1[fxf, fyf] \tag{3.10}$$

for all $x, y \in R$. Similarly, we see that

$$B(fxf, fyf) = -s_2[fxf, fyf] \tag{3.11}$$

for all $x, y \in R$. Thus, $(s_1 + s_2)[fRf, fRf] = 0$. Next, for each $m \in eRf$, we define a map $\psi_m : eR \to eRf$ by

$$\psi_m(ex) = B(exf, m).$$

Since ψ_m is additive and

$$\begin{aligned}\psi_m(yex) = \psi_m(eyex) &= B(eyexf, m)\\ &= eyeB(exf, m) + B(eye, m)exf\\ &= y\psi_m(ex)\end{aligned}$$

for all $x, y \in R$, we see that $\psi_m : eR \to R$ is a left R-module homomorphism. Hence, there exists $p_m \in Q(R)$ such that

$$\psi_m(ex) = exp_m = exfp_mf$$

for all $x \in R$. Moreover, using Proposition 2.2, we see that fp_mf is uniquely determined by m. Consequently, a map $p : eRf \to fQ(R)f$ given by $p : exf \mapsto fp_{exf}f$ is well defined. Thus,

$$B(exf, eyf) = exfp(eyf) \tag{3.12}$$

for all $x, y \in R$. Analogously, defining a map $\psi'_m : eR \to eRf$ by $\psi'_m(ey) = B(m, eyf)$, we see that there exists a map $p' : eRf \to fQ(R)f$ such that

$$B(exf, eyf) = eyfp'(exf) \tag{3.13}$$

for all $x, y \in R$. Comparing (3.12) and (3.13), we get

$$exfp(eyf) - eyfp'(exf) = 0$$

for all $x, y \in R$. Moreover, since B is a biderivation, we have

$$exf[fzf, p(eyf)] = B(exfzf, eyf) - B(exf, eyf)fzf = 0$$

for all $x, y, z \in R$ and so Proposition 2.2 yields that $p(eRf) \subseteq \mathcal{S}(R)$. Analogously, $p'(eRf) \subseteq \mathcal{S}(R)$. Now, using (3.3), (3.11), (3.4), (3.7), (3.5), (3.6), and (3.12), we obtain

$$\begin{aligned} B(x, y) - B(exe, eye) &= B(fxf, fyf) + B(exe, eyf) + B(fxf, eyf) \\ &\quad + B(exf, eye) + B(exf, fyf) + B(exf, eyf) \\ &= [fxf, fyf]s_1 + exeyfs_1 - eyfs_1fxf + eyexfs_2 \\ &\quad - exfs_2fyf + exfp(eyf) \\ &= [x, y]s_1 + eyexfs_1 - exfyfs_1 + eyexfs_2 \\ &\quad - exfyfs_2 + exfp(eyf) \\ &= [x, y]s_1 + eyexfs - exfyfs + exfp(eyf) \end{aligned}$$

for all $x, y \in R$, where $s := s_1 + s_2$. Thus,

$$B(x, y) - B(exe, eye) = [x, y]s_1 + [y, exf]s + exfp(eyf)$$

for all $x, y \in R$, where $f[R, R]s = 0$ and $[R, R]eRs = 0$. □

We are now ready to prove our main theorem.

Theorem 3.3 *Let R be a triangular ring such that $\mathcal{S}(R) \subseteq C(R)f$. If $B : R \times R \to R$ is a biderivation, then there exist $\lambda \in C(R)$, $\mu \in \operatorname{ann}_{C(R)}(f[R, R] \cup [R, R]e)$, $r \in \operatorname{ann}_R([R, R])$, and $q, q' \in fQ(R)e$ such that*

$$B(x, y) = \lambda[x, y] + [x, [y, \mu e + r]] + exfqeyf \tag{3.14}$$

and $exfqeyf = eyfq'exf \in eRf$ for all $x, y \in R$.

In particular, if $\mathcal{S}(R) = Z(R)f$, then $\lambda, \mu, exfq'e + fqexf, exfqe + fq'exf \in Z(R)$ for each $x \in R$.

Proof According to [2, Proposition 4.10]

$$B(x, y) = B_1(x, y) + [x, [y, r]] \tag{3.15}$$

for all $x, y \in R$, where $r = B(e, e) \in eRf$, $r[R, R] = 0 = [R, R]r$, and $B_1 : R \times R \to R$ is a biderivation such that $B_1(e, e) = 0$. Hence, Lemma 3.2 implies that there exist elements $s, s_1 \in \mathcal{S}(R)$ and additive maps $p, p' : eRf \to \mathcal{S}(R)$ such that

$$B_1(x, y) = B_1(exe, eye) + [x, y]s_1 + [y, exf]s + exfp(eyf), \tag{3.16}$$

$$B_1(exe, eye)ezf = [exe, eye]ezfs_1, \tag{3.17}$$

and

$$exfp(eyf) - eyfp'(exf) = 0 \tag{3.18}$$

for all $x, y, z \in R$, where $f[R, R]s = 0$ and $[R, R]eRs = 0$. Since $s_1 \in \mathcal{S}(R) \subseteq C(R)f$, (3.17) yields

$$\Big(B_1(exe, eye) - \tau^{-1}(s_1)[exe, eye]\Big)eRf = 0$$

and hence

$$B_1(exe, eye) = \tau^{-1}(s_1)[exe, eye]$$

for all $x, y \in R$. Thus, (3.16) can be rewritten as

$$B_1(x, y) = \tau^{-1}(s_1)[exe, eye] + [x, y]s_1 + [y, exf]s + exfp(eyf) \tag{3.19}$$

for all $x, y \in R$. Note that

$$\begin{aligned}\tau^{-1}(s_1)[exe, eye] + [x, y]s_1 &= \tau^{-1}(s_1)e[x, y]e + e[x, y]fs_1 + f[x, y]fs_1\\ &= (\tau^{-1}(s_1) + s_1)\big(e[x, y]e + e[x, y]f + f[x, y]f\big)\\ &= (\tau^{-1}(s_1) + s_1)[x, y]\end{aligned}$$

and so

$$\tau^{-1}(s_1)[exe, eye] + [x, y]s_1 = \lambda[x, y] \tag{3.20}$$

for all $x, y \in R$, where $\lambda' := \tau^{-1}(s_1) + s_1 \in C(R)$. Let $\mu := -\tau^{-1}(s) - s \in C(R)$. Note that in case $Z(R)f = \mathcal{S}(R)$, it follows that $\lambda', \mu \in Z(R)$. Obviously, $\mu f[R, R] = 0$ and $[R, R]eRf\mu = 0$. Thus, $\mu[R, R]e = 0$ and

$$[y, exf]s = \mu[exf, y] \tag{3.21}$$

for all $x, y \in R$. Since (3.18) can be rewritten as

$$\tau^{-1}(p(eyf))exf - \tau^{-1}(p'(exf))eyf = 0$$

for all $x, y \in R$, Lemma 2.4 implies that there exist $q, q' \in fQ(R)e$ such that

$$\tau^{-1}(p(exf)) = exfq'e \quad \text{and} \quad \tau^{-1}(p'(exf)) = exfqe,$$

where

$$exfq'e + fqexf \in C(R) \text{ and } exfqe + fq'exf \in C(R)$$

for all $x \in R$. Note that if $Z(R)f = \mathcal{S}(R)$, then $\tau^{-1}(p(eRf)), \tau^{-1}(p'(eRf)) \subseteq Z(R)e$ and so $exfq'e + fqexf, exfqe + fq'exf \in Z(R)$ for each $x \in R$ (see Remark 2.5). Hence,

$$p(exf) = fqexf \quad \text{and} \quad p'(exf) = fq'exf \tag{3.22}$$

for all $x \in R$. Now, using (3.19), (3.20), (3.21), and (3.22), we obtain

$$B_1(x, y) = \lambda'[x, y] + \mu[exf, y] + exfqeyf, \tag{3.23}$$

where $eyfq'exf = exfqeyf \in eRf$ for all $x, y \in R$. Since $\mu[eRe, eRe] = 0$ and $\mu[fRf, fRf] = 0$, it follows that

$$\mu[exf, y] = \mu[x, y] + [x, [y, \mu e]] \tag{3.24}$$

for all $x, y \in R$. Thus, using (3.15), (3.23), and (3.24), we may conclude that

$$B(x, y) = \lambda[x, y] + [x, [y, \mu e + r]] + exfqeyf$$

for all $x, y \in R$, where $\lambda := \lambda' + \mu \in C(R)$ (in case $Z(R)f = \mathcal{S}(R)$, we have $\lambda \in Z(R)$). □

Corollary 3.4 *Let R be a triangular ring such that $Z(R)f = \mathcal{S}(R)$. If either eRe or fRf does not contain nonzero central ideals, then each biderivation B of R is of the form*

$$B(x, y) = \lambda[x, y] + [x, [y, \mu e + r]] \qquad (x, y \in R)$$

for some $\lambda \in Z(R)$, $\mu \in \mathrm{ann}_{Z(R)}(f[R, R] \cup [R, R]e)$, and $r \in \mathrm{ann}_R([R, R])$.

Proof Since $\mathcal{S}(R) = Z(R)f \subseteq C(R)f$ Theorem 3.3 implies that there exist $\lambda \in Z(R)$, $\mu \in \mathrm{ann}_{Z(R)}(f[R, R] \cup [R, R]e)$, $r \in \mathrm{ann}_R([R, R])$, and $q, q' \in fQ(R)e$ such that

$$B(x, y) = \lambda[x, y] + [x, [y, \mu e + r]] + exfqeyf$$

and

$$exfq'e + fqexf,\ exfqe + fq'exf \in Z(R)$$

for all $x, y \in R$. Consequently, $eRfqe \subseteq Z(R)e$ and $fq'eRf \subseteq Z(R)f$. Moreover, $eRfqe$ is a central ideal of eRe and $fq'eRf$ is a central ideal of fRf. Thus, the assumption implies that either $eRfqe = 0$ or $fq'eRf = 0$. Now, using Proposition 2.2, it follows that $q = q' = 0$. □

Corollary 3.5 *Let R be a triangular ring such that $\mathcal{S}(R) \subseteq C(R)f$. If either $eR[R, R]Re = eRe$ or $fR[R, R]Rf = fRf$, then each biderivation B of R is of the form*

$$B(x, y) = \lambda[x, y] \qquad (x, y \in R)$$

for some $\lambda \in C(R)$. In particular, if $\mathcal{S}(R) = Z(R)f$, then $\lambda \in Z(R)$.

Proof According to Theorem 3.3, there exist $\lambda \in C(R)$, $\mu \in \operatorname{ann}_{C(R)}(f[R,R] \cup [R,R]e)$, $r \in \operatorname{ann}_R([R,R])$, and $q, q' \in fQ(R)e$ such that

$$B(x, y) = \lambda[x, y] + [x, [y, \mu e + r]] + exfqeyf$$

and

$$exfq'e + fqexf,\ exfqe + fq'exf \in C(R)$$

for all $x, y \in R$. Hence, $eRfqe \subseteq C(R)e$ and $fq'eRf \subseteq C(R)f$ and so

$$exe(eye \cdot exfqe) = (eye \cdot exfqe)exe = (exfqe \cdot eye)exe = eye(exe \cdot exfqe)$$

for all $x, y, z \in R$. Thus, $[exe, eye]eRfqe = 0$ and analogously $fq'eRf[fxf, fyf] = 0$ for all $x, y \in R$. Then $eR[R, R]ReRfqe = 0$ and $fq'eRfR[R, R]Rf = 0$. Now, the assumption yields that $q = q' = 0$. Next, since $r \in \operatorname{ann}_R([R, R]) \subseteq eRf$, it follows that $[eRe, eRe]r = 0$ and $r[fRf, fRf] = 0$. Consequently, $eR[R, R]Rer = 0$ and $rfR[R, R]Rf = 0$. Now, the assumption implies that $r = erf = 0$. Similarly, since $\mu = \mu e + \mu f \in C(R)$ and $\mu eR[R, R]Re = 0$, $\mu fR[R, R]Rf = 0$ our assumption yields that $\mu = 0$. Note that according to Theorem 3.3 $\lambda \in Z(R)$ if $\mathcal{S}(R) = Z(R)f$. ◻

Let us give an example of a triangular ring R such that $\mathcal{S}(R) \nsubseteq C(R)f$ and an example of a biderivation of R, which is not of the form (4.3).

Example 3.6 Let $F[X, Y]$ be the unital ring of all polynomials in commuting indeterminates X and Y with coefficients in a field F with $\operatorname{char}(F) = 0$. By M, we denote the quotient ring $F[X, Y]/(X^2, Y^2, XY)$. Let $A := F[X]/(X^2)$ and $B := F[Y]/(Y^2)$. Obviously, A and B are unital subrings of M. Moreover, M is an (A, B)-bimodule, which is faithful as a left A-module and also as a right B-module. Let R be the triangular ring

$$\operatorname{Tri}(A, M, B) = \left\{ \begin{bmatrix} \alpha_0 + \alpha_1 X & \mu_0 + \mu_1 X + \mu_2 Y \\ 0 & \beta_0 + \beta_1 Y \end{bmatrix};\ \alpha_i,\ \beta_i,\ \mu_i \in F \right\}.$$

Then $Z(R) = FI$, $Z(eRe) = eRe$, and $Z(fRf) = fRf$, where

$$e := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } f := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

According to [8, Section 4], it turns out that

$$C(R) = \left\{ \begin{bmatrix} \alpha_0 + \alpha_1 X & 0 \\ 0 & \alpha_0 + \alpha_1 X \end{bmatrix}; \alpha_0, \alpha_1 \in F \right\}.$$

Since $fRf = Z(fRf) \subseteq \mathcal{S}(R)$, it follows that $\mathcal{S}(R) \nsubseteq C(R)f$. Thus, R does not satisfy the assumption from Theorem 3.3. Let us define a map $\phi : R \to R$ by

$$\phi\left(\begin{bmatrix} \alpha_0 + \alpha_1 X & \mu_0 + \mu_1 X + \mu_2 Y \\ 0 & \beta_0 + \beta_1 Y \end{bmatrix}\right) = \begin{bmatrix} 0 & \mu_0 Y \\ 0 & (\beta_0 - \alpha_0) Y \end{bmatrix}.$$

We leave it to the reader to verify that a map $B : R \times R \to R$ defined by

$$B(x, y) = [\phi(x), y]$$

is a biderivation, which is not of the form (4.3).

4 Applications

(Block) upper triangular matrix rings Let S be a unital ring and let $n \geq 2$. Suppose that $\bar{k} = (k_1, k_2, \ldots, k_m) \in \mathbb{N}^m$ is an ordered m-tuple of positive integers such that $k_1 + k_2 + \cdots + k_m = n$. The block upper triangular matrix ring $B_n^{\bar{k}}(S)$ is a subring of $M_n(S)$ of the form

$$B_n^{\bar{k}}(S) = \begin{pmatrix} M_{k_1}(S) & M_{k_1 \times k_2}(S) & \cdots & M_{k_1 \times k_m}(S) \\ 0 & M_{k_2}(S) & \cdots & M_{k_2 \times k_m}(S) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{k_m}(S) \end{pmatrix}.$$

Obviously, the full matrix ring $M_n(S)$ and the upper triangular matrix ring $T_n(S)$ are just special examples of block upper triangular matrix rings. It turns out that $Q(M_n(S)) = M_n(Q(S))$ (see, e.g., [1, Proposition 2.4]) and hence $C(M_n(S)) = C(S)I$. By e_{ij}, we denote the (i, j)th matrix unit. Since $e_{11}M_n(S) \subseteq T_n(S) \subseteq B_n^{\bar{k}}(S) \subseteq M_n(S)$ and since $e_{11}M_n(S)$ is a right ideal of $M_n(S)$ such that

$$\mathrm{rann}_{M_n(S)}(e_{11}M_n(S)) = 0.$$

it follows that

$$Q(T_n(S)) = Q(B_n^{\bar{k}}(S)) = M_n(Q(S))$$

(see Exercise 9 on page 380 in [10]). Hence

$$C(T_n(S)) = C(B_n^{\bar{k}}(S)) = C(S)I.$$

Suppose that $B_n^{\bar{k}}(S) \neq M_n(S)$. Then $B_n^{\bar{k}}(S)$ can be represented as a triangular ring. Namely, pick any $l \in \{1, 2, \ldots, m-1\}$. Setting $n' := k_1 + \cdots + k_l$, $e := e_{1,1} + \cdots + e_{n',n'}$, and $f := I - e$, we see that $fB_n^{\bar{k}}(S)e = 0$, $eB_n^{\bar{k}}(S)e \cong B_{n'}^{(k_1,\ldots,k_l)}(S)$, $fB_n^{\bar{k}}(S)f \cong B_{n-n'}^{(k_{l+1},\ldots,k_m)}(S)$, and $eB_n^{\bar{k}}(S)f \cong M_{n' \times (n-n')}(S)$ is a faithful $(eB_n^{\bar{k}}(S)e, fB_n^{\bar{k}}(S)f)$-bimodule. Accordingly, we may consider $B_n^{\bar{k}}(S)$ as a triangular ring of the form

$$B_n^{\bar{k}}(S) = \begin{pmatrix} B_{n'}^{(k_1,\ldots,k_l)}(S) & M_{n'\times(n-n')}(S) \\ & B_{n-n'}^{(k_{l+1},\ldots,k_m)}(S) \end{pmatrix}. \tag{4.1}$$

Moreover, we claim that $Z(B_n^{\bar{k}}(S))f = \mathcal{S}(B_n^{\bar{k}}(S))$. Namely, let $q \in \mathcal{S}(B_n^{\bar{k}}(S))$. Then $q \in fM_n(Q(S))f$, $[q, fB_n^{\bar{k}}(S)f] = 0$, and $eB_n^{\bar{k}}(S)fqf \subseteq eB_n^{\bar{k}}(S)f$. Hence, $e_{1i} \in eB_n^{\bar{k}}(S)f$ and so

$$\sum_{j=n'+1}^{n} [q]_{ij} \cdot e_{1j} = e_{1i}q \in eB_n^{\bar{k}}(S)f$$

for each $i \in \{n'+1, \ldots, n\}$, where $[q]_{ij}$ denotes the (i, j)th term of q. Consequently, $[q]_{ij} \in S$ for all $i, j \in \{n'+1, \ldots, n\}$. Thus, $q \in fM_n(S)f$ and $[q, fB_n^{\bar{k}}(S)f] = 0$, which yields that $q \in Z(fB_n^{\bar{k}}(S)f) = Z(S)f = Z(B_n^{\bar{k}}(S))f$.

Applying our results from the previous section to $B_n^{\bar{k}}(S)$, we obtain the following corollary, which is a generalization of [2, Corollary 4.13].

Corollary 4.1 *Let S be a unital ring and let $n \geq 3$. Suppose that B is a biderivation of $B_n^{\bar{k}}(S)$, where $B_n^{\bar{k}}(S) \neq M_n(S)$. Then there exist $\lambda \in Z(S)I$, $r \in \operatorname{ann}_{B_n^{\bar{k}}(S)}([B_n^{\bar{k}}(S), B_n^{\bar{k}}(S)])$ such that*

$$B(x, y) = \lambda[x, y] + [x, [y, r]], \tag{4.2}$$

for all $x, y \in B_n^{\bar{k}}(S)$. In particular, if $k_1 > 1$ or $k_m > 1$, then $r = 0$.

Proof Suppose that $k_1 > 1$. Let $n' := k_1$, $e := e_{11} + \cdots + e_{n'n'}$, and $f := I - e$. Then $eB_n^{\bar{k}}(S)e \cong M_{n'}(S)$. Since $n' > 1$, it follows that $M_{n'}(S)$ coincides with its ideal generated by the set $[M_{n'}(S), M_{n'}(S)]$. Thus, $Z(B_n^{\bar{k}}(S))f = \mathcal{S}(B_n^{\bar{k}}(S))$ and $eB_n^{\bar{k}}(S)[B_n^{\bar{k}}(S), B_n^{\bar{k}}(S)]B_n^{\bar{k}}(S)e = eB_n^{\bar{k}}(S)e$ and so Corollary 3.5 implies that B is an inner biderivation.

Next, suppose that $k_m > 1$. Let $n' := k_1 + \cdots + k_{m-1}$, $e := e_{11} + \cdots + e_{n'n'}$, and $f := I - e$. Then $fB_n^{\bar{k}}(S)f \cong M_{k_m}(S)$. Hence, $fB_n^{\bar{k}}(S)[B_n^{\bar{k}}(S), B_n^{\bar{k}}(S)]B_n^{\bar{k}}(S)f = fB_n^{\bar{k}}(S)f$ and so Corollary 3.5 implies that B is an inner biderivation.

It remains to consider the case when $k_1 = k_m = 1$. Let $e := e_{11}$ and $f := I - e$. Then $fB_n^{\bar{k}}(S)f \cong B_{n-1}^{\bar{k}}(S) \neq M_{n-1}(S)$. Hence, $fB_n^{\bar{k}}(S)f$ is a triangular ring and so it does not contain nonzero central ideals. Thus, according to Corollary 3.4, there exist $\lambda \in Z(S)I$, $\mu \in \operatorname{ann}_{Z(S)I}(f[B_n^{\bar{k}}(S), B_n^{\bar{k}}(S)] \cup [B_n^{\bar{k}}(S), B_n^{\bar{k}}(S)]e)$, and $r \in \operatorname{ann}_{B_n^{\bar{k}}(S)}([B_n^{\bar{k}}(S), B_n^{\bar{k}}(S)])$ such that

$$B(x, y) = \lambda[x, y] + [x, [y, \mu e + r]]$$

for all $x, y \in B_n^{\bar{k}}(S)$. Obviously, there exist i, j such that $i < j$ and $e_{ii}, e_{ij} \in fB_n^{\bar{k}}(S)f$. Since $e_{ij} = [e_{ii}, e_{ij}] \in f[B_n^{\bar{k}}(S), B_n^{\bar{k}}(S)]f$, it follows that $\mu e_{ij} = 0$ and so $\mu = 0$. □

Now, we are able to describe biderivations of $T_n(S)$, where $n \geq 3$.

Corollary 4.2 *Let S be a unital ring and let $n \geq 3$. Suppose that B is a biderivation of $T_n(S)$. Then there exist $\lambda \in Z(S)I$, $r \in \mathrm{ann}_{T_n(S)}([T_n(S), T_n(S)])$ such that*

$$B(x, y) = \lambda[x, y] + [x, [y, r]],$$

for all $x, y \in T_n(S)$.

Proof Obviously, $T_n(S) = B_n^{(1,\dots,1)}(S)$ and hence Corollary 4.1 yields the conclusion.

Next, applying Theorem 3.3, we also describe biderivations of $T_2(S)$.

Corollary 4.3 *Let S be a unital ring and let B be a biderivation of $T_2(S)$. Then there exist $\lambda \in Z(S)$, $\mu, \nu \in \mathrm{ann}_{Z(S)}([S, S])$, and $s \in \mathrm{ann}_S([S, S])$ such that*

$$B(x, y) = \lambda[x, y] + [x, [y, \mu e_{11} + s e_{12}]] + \nu e_{11} x e_{21} y e_{22} \tag{4.3}$$

for all $x, y \in T_2(S)$.

Proof Let $e = e_{11}$ and $f = e_{22}$. Then Theorem 3.3 implies that there exist $\lambda, \mu \in Z(S)$, $r \in T_2(S)$, and $q, q' \in fM_2(Q(S))e$ such that

$$B(x, y) = \lambda[x, y] + [x, [y, \mu e + r]] + exfqeyf \tag{4.4}$$

and $exfqe + fq'exf \in Z(S)I$ for all $x, y \in T_2(S)$, where $\mu f[T_2(S), T_2(S)]f = 0$, $\mu e[T_2(S), T_2(S)]e$, $r[T_2(S), T_2(S)] = 0$, and $[T_2(S), T_2(S)]r = 0$. Hence, $\mu \in \mathrm{ann}_{Z(S)}([S, S])$, and there exists $s \in S$ such that $r = se_{12}$ and $s[S, S] = 0 = [S, S]s$. Thus, $\mu e + r = \mu e_{11} + se_{12}$. Next, since $exfqe + fq'exf \in Z(S)I$ for all $x, y \in T_2(S)$, it follows that $q = q' = \nu e_{21}$ for some $\nu \in Z(S)$ such that $\nu S \subseteq Z(S)$. Therefore, $\nu \in \mathrm{ann}_{Z(S)}([S, S])$ and (4.4) can be rewritten as

$$B(x, y) = \lambda[x, y] + [x, [y, \mu e_{11} + s e_{12}]] + \nu e_{11} x e_{21} y e_{22}$$

for all $x, y \in T_2(S)$. □

Let us mention that Corollary 4.3 is a generalization of [2, Proposition 4.16].

Nest algebras Recall that a *nest* is a chain $\mathcal{N}$ of closed subspaces of a complex Hilbert space H containing $\{0\}$ and H, which is closed under arbitrary intersections and closed linear spans of its elements. The algebra

$$\mathcal{T}(\mathcal{N}) = \{T \in \mathcal{B}(H) \mid T(N) \subseteq N \text{ for all } N \in \mathcal{N}\}.$$

is called a *nest algebra* associated with $\mathcal{N}$. A nest $\mathcal{N}$ is called trivial if $\mathcal{N} = \{0, H\}$. We refer the reader to [7] for the general theory of nest algebras. According to [6, Proposition 5] and [7, Chapter 2], each nest algebra associated with a nontrivial nest can be considered as a triangular ring. Namely, let $\mathcal{N}$ be a nontrivial nest and pick any $N \in \mathcal{N} \setminus \{0, H\}$. Let e be the orthonormal projection onto N and $f := I - e$. Then $\mathcal{N}_1 := e(\mathcal{N})$ and $\mathcal{N}_2 := f(\mathcal{N})$ are nests of N and $N^{\perp}$, respectively. Moreover, $\mathcal{T}(\mathcal{N}_1) = e\mathcal{T}(\mathcal{N})e$ and $\mathcal{T}(\mathcal{N}_2) = f\mathcal{T}(\mathcal{N})f$ are nest algebras, $f\mathcal{T}(\mathcal{N})e = 0$, and $e\mathcal{T}(\mathcal{N})f$ is a faithful $(\mathcal{T}(\mathcal{N}_1), \mathcal{T}(\mathcal{N}_2))$-bimodule. Thus,

$$\mathcal{T}(\mathcal{N}) = \begin{pmatrix} \mathcal{T}(\mathcal{N}_1) & e\mathcal{T}(\mathcal{N})f \\ & \mathcal{T}(\mathcal{N}_2) \end{pmatrix} \tag{4.5}$$

is a triangular ring and $Z(\mathcal{T}(\mathcal{N})) = \mathbb{C}I$. It is easy to see that $Z(e\mathcal{T}(\mathcal{N})e) = \mathbb{C}e = Z(\mathcal{T}(\mathcal{N}))e \subseteq C(\mathcal{T}(\mathcal{N}))e$ and $Z(f\mathcal{T}(\mathcal{N})f) = \mathbb{C}f = Z(\mathcal{T}(\mathcal{N}))f \subseteq C(\mathcal{T}(\mathcal{N}))f$ and so (i) from Proposition 2.6 holds true. Next, let $\phi : e\mathcal{T}(\mathcal{N})f \to e\mathcal{T}(\mathcal{N})f$ be a $(\mathcal{T}(\mathcal{N}), \mathcal{T}(\mathcal{N}))$-bimodule endomorphism. We define a map $d : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N})$ by $d(x) = \phi(exf)$. Obviously, d is a $\mathbb{C}$-linear derivation of $\mathcal{T}(\mathcal{N})$. Since each $\mathbb{C}$-linear derivation of a nest algebra is inner (see, e.g., [7, Theorem 19.7]), there exists $a \in \mathcal{T}(\mathcal{N})$ such that $d(x) = ax - xa$ and so

$$\phi(exf) = aexf - exfa$$

for all $x \in \mathcal{T}(\mathcal{N})$. Thus, ϕ is of the standard form and hence (ii) from Proposition 2.6 holds true. Now, Proposition 2.6 implies that $\mathcal{S}(\mathcal{T}(\mathcal{N})) = Z(\mathcal{T}(\mathcal{N}))f$.

Using Corollaries 3.4 and 4.3, we obtain the following extension of Benkovič's result [2, Corollary 4.14].

Corollary 4.4 *Let $\mathcal{N}$ be a nontrivial nest of a complex Hilbert space H. Suppose that B is a biderivation of $\mathcal{T}(\mathcal{N})$. Then the following hold.*

(i) If $\dim H = 2$*, then* $\mathcal{T}(\mathcal{N}) \cong T_2(\mathbb{C})$ *and B is of the form* (4.3).
(ii) If $\dim H \geq 3$*, then there exist* $\lambda \in \mathbb{C}I$ *and* $r \in \operatorname{ann}_{\mathcal{T}(\mathcal{N})}([\mathcal{T}(\mathcal{N}), \mathcal{T}(\mathcal{N})])$ *such that*

$$B(x, y) = \lambda[x, y] + [x, [y, r]],$$

for all $x, y \in \mathcal{T}(\mathcal{N})$.

Proof Obviously, (i) follows from Corollary 4.3. Next, suppose that $\dim H \geq 3$ and pick $N \in \mathcal{N} \setminus \{0, H\}$. Let e be the orthonormal projection onto N and $f := I - e$. Since $\dim H \geq 3$, it follows that either $\dim N > 1$ or $\dim N^{\perp} > 1$. If $\dim N > 1$, then either $e\mathcal{T}(\mathcal{N})e = B(N)$ is a noncommutative prime ring or $e\mathcal{T}(\mathcal{N})e$ is a triangular ring. Similarly, if $\dim N^{\perp} > 1$, then either $f\mathcal{T}(\mathcal{N})f = B(N^{\perp})$ is a noncommutative prime ring or $f\mathcal{T}(\mathcal{N})f$ is a triangular ring. Consequently, either $e\mathcal{T}(\mathcal{N})e$ or $f\mathcal{T}(\mathcal{N})f$ does not contain nonzero central ideals. Moreover, since $\mathcal{S}(\mathcal{T}(\mathcal{N})) = Z(\mathcal{T}(\mathcal{N}))f$ Corollary 3.4 implies that there exist $\lambda \in \mathbb{C}I$,

$\mu \in \text{ann}_{\mathbb{C}I}(f[\mathcal{T}(\mathcal{N}), \mathcal{T}(\mathcal{N})] \cup [\mathcal{T}(\mathcal{N}), \mathcal{T}(\mathcal{N})]e)$, and $r \in \text{ann}_{\mathcal{T}(\mathcal{N})}([\mathcal{T}(\mathcal{N}), \mathcal{T}(\mathcal{N})])$ such that

$$B(x, y) = \lambda[x, y] + [x, [y, \mu e + r]]$$

for all $x, y \in \mathcal{T}(\mathcal{N})$. Since either $e\mathcal{T}(\mathcal{N})e$ or $f\mathcal{T}(\mathcal{N})f$ is noncommutative, it follows that $\mu = 0$. □

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