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A Note on Generalized Mercer's Inequality

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Abstract We give an integral version and a refinement of M. Niezgoda's extension of the variant of Jensen's inequality given by A. McD. Mercer.

Keywords Convex functions · Jensen's inequality · Mercer's inequality

Mathematics Subject Classification Primary 26D15

1 Introduction and Preliminaries

Let us start with Jensen's inequality for convex functions, one of the most celebrated inequalities in mathematics and statistics (for detailed discussion and history, see [7] and [12]). Throughout the paper we assume that J and $[a, b]$ are intervals in $\mathbb{R}$.

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Proposition 1 *Let* $x_1, x_2, \ldots, x_n \in [a, b]$*, and let* $w_1, w_2, \ldots, w_n$ *be positive real numbers such that* $W_n = \sum_{i=1}^{n} w_i = 1$*. If* $\varphi : [a, b] \to \mathbb{R}$ *is a convex function, then the inequality*

$$\varphi\left(\sum_{i=1}^{n} w_i x_i\right) \leq \sum_{i=1}^{n} w_i \varphi(x_i)$$

holds.

In paper [8], A. McD. Mercer proved the following variant of Jensen's inequality, which we will refer to as Mercer's inequality.

Proposition 2 *Let* $x_1, x_2, \ldots, x_n \in [a, b]$*, and let* $w_1, w_2, \ldots, w_n$ *be positive real numbers such that* $W_n = \sum_{i=1}^{n} w_i = 1$*. If* $\varphi : [a, b] \to \mathbb{R}$ *is a convex function, then the inequality*

$$\varphi\left(m_1 + m_2 - \sum_{i=1}^{n} w_i x_i\right) \leq \varphi(m_1) + \varphi(m_2) - \sum_{i=1}^{n} w_i \varphi(x_i) \tag{1}$$

holds, where

$$m_1 = \min_{1 \leq i \leq n} \{x_i\} \quad \text{and} \quad m_2 = \max_{1 \leq i \leq n} \{x_i\}.$$

There are many versions, variants and generalizations of Proposition 1 and Proposition 2, see, e.g., [1], [3], [9] and [10]. Here we state few integral versions of Jensen's inequality from [12, pp. 58–59] which will be needed in the main theorems of our paper.

Proposition 3 *Let* $f : [a, b] \to J$ *be a continuous function. If the function* $H : [a, b] \to \mathbb{R}$ *is nondecreasing, bounded and* $H(a) \neq H(b)$*, then for every continuous convex function* $\varphi : J \to \mathbb{R}$ *the inequality*

$$\varphi\left(\frac{\int_a^b f(t)dH(t)}{\int_a^b dH(t)}\right) \leq \frac{\int_a^b \varphi(f(t))dH(t)}{\int_a^b dH(t)} \tag{2}$$

holds.

Inequality (2) can hold under different set of assumptions. For example, for a monotonic f, assumptions on H can be relaxed. The following proposition gives Jensen–Steffensen's inequality.

Proposition 4 *If* $f : [a, b] \to J$ *is continuous and monotonic (either nonincreasing or nondecreasing) and* $H : [a, b] \to \mathbb{R}$ *is either continuous or of bounded variation satisfying*

$$H(a) \leq H(t) \leq H(b) \quad \text{for all } t \in [a, b], \qquad H(b) > H(a),$$

then (2) holds.

If we replace the assumption of monotonicity of f over the whole interval $[a, b]$ in Proposition 4 with monotonicity over subintervals, we obtain the following, Jensen–Boas inequality.

Proposition 5 *If $H : [a, b] \to \mathbb{R}$ is continuous or of bounded variation satisfying*

$$H(a) \leq H(x_1) \leq H(y_1) \leq H(x_2) \leq \cdots \leq H(y_{k-1}) \leq H(x_k) \leq H(b)$$

for all $x_i \in (y_{i-1}, y_i)$ ($y_0 = a$, $y_k = b$), and $H(b) > H(a)$, and if f is continuous and monotonic (either nonincreasing or nondecreasing) in each of the k intervals (y_{i-1}, y_i), then inequality (2) holds.

In our construction for next proposition, we recall the definitions of majorization: For fixed $n \geq 2$,

$$\mathbf{x} = (x_1, \ldots, x_n), \quad \mathbf{y} = (y_1, \ldots, y_n)$$

denote two real n-tuples and

$$x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}, \quad y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$$

be their ordered components.

Definition 1 For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\mathbf{x} \prec \mathbf{y} \quad \text{if} \quad \begin{cases} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, & k \in \{1, \ldots, n-1\}, \\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}, \end{cases}$$

when $\mathbf{x} \prec \mathbf{y}$, $\mathbf{x}$ *is said to be majorized by* $\mathbf{y}$ *or* $\mathbf{y}$ *majorizes* $\mathbf{x}$.

This notion and notation of majorization was introduced by Hardy et al. in [4]. The following extension of (1) is given by M. Niezgoda in [9] which we will refer to as Niezgoda's inequality.

Proposition 6 *Let $\varphi : [a, b] \to \mathbb{R}$ be a continuous convex function. Suppose that $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\alpha_j \in J$, and $\boldsymbol{X} = (x_{ij})$ is a real $n \times m$ matrix such that $x_{ij} \in J$ for all $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$.*

If α majorizes each row of $\boldsymbol{X}$, i.e.,

$$\boldsymbol{x}_{i.} = (x_{i1}, \ldots, x_{im}) \prec (\alpha_1, \ldots, \alpha_m) = \alpha \text{ for each } i \in \{1, \ldots, n\},$$

then we have the inequality

$$\varphi\left(\sum_{j=1}^{m} \alpha_j - \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i x_{ij}\right) \leq \sum_{j=1}^{m} \varphi(\alpha_j) - \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i \varphi(x_{ij}), \tag{3}$$

where $\sum_{i=1}^{n} w_i = 1$ with $w_i \geq 0$.

The paper is organized as follows: In Sect. 2 we will give an integral generalization of Niezgoda's inequality. In the process we will use an integral majorization result of Pečarić [11] and prove a lemma which gives the Jensen–Boas inequality on disjoint set of subintervals. In Sect. 3 we will give a refinement of the inequality obtained in Sect. 2.

2 Generalized Mercer's Inequalities

Here we state some results needed in the main theorems of this section. The following proposition is a consequence of Theorem 1 in [11] (see also [12, p. 328]) and represents an integral majorization result.

Proposition 7 *Let $f,\ g : [a,b] \to J$ be two nonincreasing continuous functions, and let $H : [a,b] \to \mathbb{R}$ be a function of bounded variation. If*

$$\int_a^x f(t)\,dH(t) \le \int_a^x g(t)\,dH(t), \quad \textit{for each} \quad x \in (a,b),$$
$$\textit{and} \quad \int_a^b f(t)\,dH(t) = \int_a^b g(t)\,dH(t),$$

hold, then for every continuous convex function $\varphi : J \to R$ the following inequality holds

$$\int_a^b \varphi(f(t))\,dH(t) \le \int_a^b \varphi(g(t))\,dH(t). \tag{4}$$

Remark 1 If $f,\ g : [a,b] \to J$ are two nondecreasing continuous functions such that

$$\int_x^b f(t)\,dH(t) \le \int_x^b g(t)\,dH(t), \quad \text{for each} \quad x \in (a,b),$$
$$\text{and} \quad \int_a^b f(t)\,dH(t) = \int_a^b g(t)\,dH(t),$$

then again inequality (4) holds. In this paper we will state our results for nonincreasing f and g satisfying the assumption of Proposition 7, but they are still valid for nondecreasing f and g satisfying the above condition, see, for example, [6, p. 584].

The following lemma shows that the subintervals in the Jensen–Boas inequality (see Proposition 5) can be disjoint for the inequality of type (2) to hold.

Lemma 1 *Let $H : [a,b] \to \mathbb{R}$ be continuous or a function of bounded variation, and let $a \le a_1 \le b_1 \le a_2 \le \cdots \le a_k \le b_k \le b$ be a partition of the interval $[a,b]$, $I = \bigcup_{i=1}^k [a_i, b_i]$ and $L = \int_I dH(t)$. If*

$$H(a_i) \le H(t) \le H(b_i) \quad \textit{for all } t \in (a_i, b_i) \textit{ and } 1 \le i \le k \tag{5}$$

and $L > 0$, then for every function $f : [a,b] \to J$ which is continuous and monotonic (either nonincreasing or nondecreasing) in each of the k intervals (a_i, b_i) and every *convex and continuous function* $\varphi : J \to \mathbb{R}$*, the following inequality holds*

$$\varphi\left(\frac{1}{L}\int_I f(t)\,dH(t)\right) \le \frac{1}{L}\int_I \varphi(f(t))\,dH(t).$$

Proof Denote $w_i = \int_{a_i}^{b_i} dH(t)$. Due to (5), if $H(a_i) = H(b_i)$ then dH is a null-measure on $[a_i, b_i]$ and $w_i = 0$, while otherwise $w_i > 0$. Denote $S = \{i : w_i > 0\}$ and

$$x_i = \frac{1}{w_i}\int_{a_i}^{b_i} f(t)\,dH(t), \quad \text{for } i \in S.$$

Notice that

$$L = \int_I dH(t) = \sum_{i\in S} w_i > 0, \quad \int_I \varphi(f(t))\,dH(t) = \sum_{i\in S}\int_{a_i}^{b_i} \varphi(f(t))\,dH(t)$$

and, due to Proposition 4,

$$w_i\varphi(x_i) \le \int_{a_i}^{b_i} \varphi(f(t))\,dH(t), \quad \text{for } i \in S.$$

Therefore, taking into account the discrete Jensen's inequality,

$$\varphi\left(\frac{1}{L}\int_I f(t)\,dH(t)\right) = \varphi\left(\frac{1}{L}\sum_{i\in S} w_i x_i\right) \le \frac{1}{L}\sum_{i\in S} w_i\varphi(x_i)$$
$$\le \frac{1}{L}\sum_{i\in S}\int_{a_i}^{b_i} \varphi(f(t))\,dH(t) = \frac{1}{L}\int_I \varphi(f(t))\,dH(t).$$

The following theorem is our main result of this section, and it gives a generalization of Proposition 6.

Theorem 1 *Let* $a = b_0 \le a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \le a_{k+1} = b$, $I = \bigcup_{i=1}^{k}(a_i, b_i)$, $I^c = [a, b]\backslash I = \bigcup_{i=1}^{k+1}[b_{i-1}, a_i]$ *and* $H : [a, b] \to \mathbb{R}$ *be a function of bounded variation such that* $H(b_{i-1}) \le H(t) \le H(a_i)$ *for all* $t \in (b_{i-1}, a_i)$ *and* $1 \le i \le k+1$ *and* $L = \int_{I^c} dH(t) > 0$.

Furthermore, let (X, Σ, μ) *be a measure space with positive finite measure* μ*, let* $g : [a, b] \to J$ *be a nonincreasing continuous function, and let* $f : X \times [a, b] \to J$ *be a measurable function such that the mapping* $t \mapsto f(s, t)$ *is nonincreasing and continuous for each* $s \in X$*,*

$$\int_a^x f(s,t)\,dH(t) \le \int_a^x g(t)dH(t), \quad \textit{for each} \quad x \in (a, b),$$
$$\textit{and} \quad \int_a^b f(s,t)\,dH(t) = \int_a^b g(t)\,dH(t). \tag{6}$$

Then, for a continuous convex function $\varphi : J \to \mathbb{R}$ the following inequality holds

$$\varphi\left(\frac{1}{L}\left(\int_a^b g(t)dH(t) - \frac{1}{\mu(X)}\int_I\int_X f(s,t)d\mu(s)dH(t)\right)\right)$$
$$\leq \frac{1}{L}\left(\int_a^b \varphi(g(t))dH(t) - \frac{1}{\mu(X)}\int_I\int_X \varphi(f(s,t))d\mu(s)dH(t)\right). \quad (7)$$

Proof Using Fubini's theorem, equality (6) and the integral Jensen's inequality (2) we get

$$\varphi\left(\frac{1}{L}\left(\int_a^b g(t)dH(t) - \frac{1}{\mu(X)}\int_I\int_X f(s,t)d\mu(s)dH(t)\right)\right)$$
$$= \varphi\left(\frac{1}{\mu(X)}\int_X\left[\frac{1}{L}\int_{I^c} f(s,t)dH(t)\right]d\mu(s)\right)$$
$$\leq \frac{1}{\mu(X)}\int_X \varphi\left(\frac{1}{L}\int_{I^c} f(s,t)dH(t)\right)d\mu(s). \quad (8)$$

Applying Lemma 1 and Proposition 7, respectively, we have

$$\varphi\left(\frac{1}{L}\int_{I^c} f(s,t)dH(t)\right) \leq \frac{1}{L}\int_{I^c}\varphi(f(s,t))dH(t)$$
$$\leq \frac{1}{L}\left(\int_a^b \varphi(g(t))dH(t) - \int_I \varphi(f(s,t))dH(t)\right). \quad (9)$$

Finally, combining (8) and (9) we obtain inequality (7).

Corollary 1 *Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\alpha_j \in J$ and $\mathbf{X} = (x_{ij})$ be a real $n \times m$ matrix such that $x_{ij} \in J$ for all $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$, and let α majorize each row of $\mathbf{X}$, that is*

$$\mathbf{x}_{i.} = (x_{i1}, \ldots, x_{im}) \prec (\alpha_1, \ldots, \alpha_m) = \alpha \text{ for each } i \in \{1, \ldots, n\}.$$

Moreover, let $a_l, b_l \in \mathbb{N}$, $l \in 1, \ldots, k$, be such that $1 = b_0 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \leq a_{k+1} = m + 1$ and denote $L = \sum_{l=1}^{k+1}(a_l - b_{l-1})$. Then, for every continuous convex function $\varphi : J \to \mathbb{R}$ the inequality

$$\varphi\left(\frac{1}{L}\left(\sum_{j=1}^{m} a_j - \frac{1}{W_n}\sum_{l=1}^{k}\sum_{j=a_l}^{b_l-1}\sum_{i=1}^{n} w_i x_{ij}\right)\right)$$
$$\leq \frac{1}{L}\left(\sum_{j=1}^{m}\varphi(a_j) - \frac{1}{W_n}\sum_{l=1}^{k}\sum_{j=a_l}^{b_l-1}\sum_{i=1}^{n} w_i \varphi(x_{ij})\right)$$

holds, where $W_n = \sum_{i=1}^{n} w_i > 0$ with $w_i \geq 0$.

Proof The proof of this corollary follows from Theorem 1 by taking step functions. More concretely, for $a = b_0 = 1$, $b = a_{k+1} = m + 1$, $f(s,t) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij}\, \chi_{[i,i+1)}(s)\chi_{[j,j+1)}(t)$, $g(t) = \sum_{j=1}^{m} \alpha_j\, \chi_{[j,j+1)}(t)$, $X = [1, n+1)$, $d\mu(s) = \sum_{i=1}^{n} w_i\, \chi_{[i,i+1)}(s)\mathrm{d}s$ and $H(t) = t$.

Remark 2 If in Corollary 1 we take $k = 1$, $a_1 = 1$ and $b_1 = m$ and assume $W_n = \sum_{i=1}^{n} w_i = 1$, then we get Niezgoda's inequality (3).

Remark 3 For some similar results involving generalized convex functions, see [5].

3 Refinements

Throughout this section we assume that $E \subset X$ with $\mu(E), \mu(E^c) > 0$ and we use the following notations

$$W_E = \frac{\mu(E)}{\mu(X)}, \quad W_{E^c} = \frac{\mu(E^c)}{\mu(X)} = 1 - W_E.$$

The following refinement of (7) is valid.

Theorem 2 *Let $a = b_0 \le a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k \le a_{k+1} = b$, $I = \bigcup_{i=1}^{k}(a_i, b_i)$, $I^c = [a,b]\backslash I = \bigcup_{i=1}^{k+1}[b_{i-1}, a_i]$ and $H : [a,b] \to \mathbb{R}$ be a function of bounded variation such that $H(b_{i-1}) \le H(t) \le H(a_i)$ for all $t \in (b_{i-1}, a_i)$ and $1 \le i \le k+1$ and $L = \int_{I^c} dH(t) > 0$.*

Furthermore, let (X, Σ, μ) be a measure space with positive finite measure μ, let $g : [a,b] \to J$ be a nonincreasing continuous function, and let $f : X \times [a,b] \to J$ be a measurable function such that the mapping $t \mapsto f(s,t)$ is nonincreasing and continuous for each $s \in X$,

$$\int_a^x f(s,t)\, dH(t) \le \int_a^x g(t) dH(t), \quad \textit{for each} \quad x \in (a,b),$$
$$\textit{and} \quad \int_a^b f(s,t)\, dH(t) = \int_a^b g(t)\, dH(t).$$

Then, for a continuous convex function $\varphi : J \to \mathbb{R}$, the inequalities

$$\varphi\left(\frac{1}{L}\left(\int_a^b g(t)dH(t) - \frac{1}{\mu(X)}\int_I\int_X f(s,t)d\mu(s)dH(t)\right)\right) \le F(f,g,\varphi;E) \le \frac{1}{L}\left(\int_a^b \varphi(g(t))dH(t) - \frac{1}{\mu(X)}\int_I\int_X \varphi(f(s,t))d\mu(s)dH(t)\right) \tag{10}$$

hold, where

$$F(f,g,\varphi;E) = W_E\varphi\left(\frac{1}{L}\left(\int_a^b g(t)dH(t) - \frac{1}{\mu(E)}\int_I\int_E f(s,t)d\mu(s)dH(t)\right)\right) + W_{E^c}\varphi\left(\frac{1}{L}\left(\int_a^b g(t)dH(t) - \frac{1}{\mu(E^c)}\int_I\int_{E^c} f(s,t)d\mu(s)dH(t)\right)\right).$$

Proof Following the proof of Theorem 3 of [2], by using convexity of the function φ, we have

$$\begin{aligned}
&\varphi\left(\frac{1}{L}\left(\int_a^b g(t)dH(t)-\frac{1}{\mu(X)}\int_I\int_X f(s,t)d\mu(s)dH(t)\right)\right)\\
&=\varphi\left(W_E\left[\frac{1}{L}\left(\int_a^b g(t)dH(t)-\frac{1}{\mu(E)}\int_E\int_I f(s,t)dH(t)\right)d\mu(s)\right]\right.\\
&\quad\left.+\ W_{E^c}\left[\frac{1}{L}\left(\int_a^b g(t)dH(t)-\frac{1}{\mu(E^c)}\int_{E^c}\int_I f(s,t)dH(t)\right)d\mu(s)\right]\right)\\
&\le W_E\varphi\left(\frac{1}{L}\left(\int_a^b g(t)dH(t)-\frac{1}{\mu(E)}\int_E\int_I f(s,t)dH(t)\right)d\mu(s)\right)\\
&\quad+W_{E^c}\varphi\left(\frac{1}{L}\left(\int_a^b g(t)dH(t)-\frac{1}{\mu(E^c)}\int_{E^c}\int_I f(s,t)dH(t)\right)d\mu(s)\right)\\
&=F(f,g,\varphi;E)
\end{aligned}$$

for any E, which proves the first inequality in (10).

By inequality (7) we also have

$$\begin{aligned}
F(f,g,\varphi;E)&=W_E\varphi\left(\frac{1}{L}\left(\int_a^b g(t)dH(t)-\frac{1}{\mu(E)}\int_I\int_E f(s,t)d\mu(s)dH(t)\right)\right)\\
&\quad+W_{E^c}\varphi\left(\frac{1}{L}\left(\int_a^b g(t)dH(t)-\frac{1}{\mu(E^c)}\int_I\int_{E^c} f(s,t)d\mu(s)dH(t)\right)\right)\\
&\le W_E\left[\frac{1}{L}\left(\int_a^b \varphi(g(t))dH(t)-\frac{1}{\mu(E)}\int_I\int_E \varphi(f(s,t))d\mu(s)dH(t)\right)\right]\\
&\quad+W_{E^c}\left[\frac{1}{L}\left(\int_a^b \varphi(g(t))dH(t)-\frac{1}{\mu(E^c)}\int_I\int_{E^c} \varphi(f(s,t))d\mu(s)dH(t)\right)\right]\\
&=\frac{1}{L}\left(\int_a^b \varphi(g(t))dH(t)-\frac{1}{\mu(X)}\int_I\int_X \varphi(f(s,t))d\mu(s)dH(t)\right)
\end{aligned}$$

for any E, which proves the second inequality in (10).

Remark 4 Direct consequences of the previous theorem are the following two inequalities

$$\begin{aligned}
&\varphi\left(\frac{1}{L}\left(\int_a^b g(t)dH(t)-\frac{1}{\mu(X)}\int_I\int_X f(s,t)d\mu(s)dH(t)\right)\right)\\
&\le \inf_{\{E:0<\mu(E)<\mu(X)\}} F(f,g,\varphi;E)
\end{aligned}$$

and

$$\sup_{\{E:0<\mu(E)<\mu(X)\}} F(f, g, \varphi; E)$$
$$\leq \frac{1}{L}\left(\int_a^b \varphi(g(t))dH(t) - \frac{1}{\mu(X)}\int_I\int_X \varphi(f(s,t))d\mu(s)dH(t)\right).$$

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