Bull. Malays. Math. Sci. Soc. (2018) 41:1859–1877
https://doi.org/10.1007/s40840-016-0431-2

BULLETIN of the
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840

CrossMark

The Cauchy Problem on a Generalized Novikov Equation

Kunquan Li[1] · Meijing Shan[2] · Chongbin Xu[1] · Zhengguang Guo[1]

Received: 10 August 2016 / Published online: 24 October 2016
© Malaysian Mathematical Sciences Society and Penerbit Universiti Sains Malaysia 2016

Abstract We give asymptotic description of strong solutions in its lifespan with compactly supported initial momentum and investigate the persistence property in weighted space and blow-up phenomena for a generalized Novikov equation.

Keywords Novikov equation · Compact support · Persistence property · Blow-up

Mathematics Subject Classification 37L05 · 35Q53 · 35L05

1 Introduction

We, in this paper, are interested in the following generalized Novikov equation:

$$u_t - u_{txx} + (a+b)u^2u_x = auu_xu_{xx} + bu^2u_{xxx}, \tag{1.1}$$

Communicated by Yong Zhou.

✉ Zhengguang Guo
gzgmath@wzu.edu.cn

Kunquan Li
kquanmath@163.com

Meijing Shan
shanmeijing@ecupl.edu.cn

Chongbin Xu
xuchongbin1977@126.com

1 College of Mathematics and Information Science, Wenzhou University, Wenzhou 325035, Zhejiang, China

2 Institute of Information Science and Technology, East China University of Political Science and Law, Shanghai 201620, China

with associated initial data $u(x, t = 0) = u_0(x)$, where $a > 0$ and $b > 0$ are the arbitrary constants. It is obvious when a and b are fixed by 3 and 1, respectively, (1.1) reduces to the celebrated Novikov equation

$$u_t - u_{txx} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx}, \tag{1.2}$$

which was obtained by using the perturbative symmetry approach in a symmetry classification of nonlocal partial differential equations with quadratic or cubic nonlinearity [24]; subsequently, Novikov found a scalar Lax pair for it, then proved that (1.2) is integrable. Taking convolution with Green's function $G(x) = e^{-|x|}/2,\ x \in \mathbb{R}$ for the operator $(1 - \partial_x^2)^{-1}$ gives the following equivalent nonlocal form:

$$u_t + u^2 u_x + G * (3uu_x u_{xx} + 2u_x^3 + 3u^2 u_x) = 0. \tag{1.3}$$

By comparison with the celebrated Camassa–Holm equation [3]

$$u_t + uu_x + \partial_x G * \left(u^2 + \frac{1}{2}u_x^2\right) = 0, \tag{1.4}$$

and also the Degasperis–Procesi equation [5]

$$u_t + uu_x + \partial_x G * \left(\frac{3}{2}u^2\right) = 0, \tag{1.5}$$

it is easy to observe that Novikov equation has nonlinear terms that are cubic rather than quadratic. It has drawn much attention since it appears. Well-posedness of Cauchy problem for (1.2) in Sobolev space on the torus was first established by Tiglay [27], the cases of the line and the circle were done by Himonas et al. [13]. Grayshan [6] studied the data-to-solution map in the Sobolev space. Hone et al. [15] calculated the explicit formulas for multipeakon solutions of (1.2) using the matrix Lax pair found by Hone and Wang. This multipeakon property is common with the Camassa–Holm equation and Degasperis–Procesi equations [4]. Hone and Wang [16] presented a matrix Lax pair for (1.2), and showed how it was related by a reciprocal transformation to a negative flow in the Swasa–Kotera hierarchy. Infinitely, many conserved quantities as well as bi-Hamiltonian structures were found. Sufficient conditions on the initial data to guarantee finite time blow-up were established by [17,30], and the global in time solution was obtained by assuming that $(1 - \partial_x^2)u_0$ does not change sign. Ni and Zhou [25] proved that the Cauchy problem for (1.2) is locally well-posed in Besov space $B^s_{2,r}$ with the critical index $s = 3/2$ and in Sobolev space H^s with $s > 3/2$ with the aid of Kato's semigroup theory [19]. Also Yan et al. [29] proved local well-posedness in Besov space under certain assumptions. Global weak solution was also shown by the authors [20,22,28]. Asymptotic profile and measure of the momentum support are the recent works [8,12]. For some related issues, the readers are referred to [9–11,18,21,23,32] and references therein.

It is not difficult to find that the coefficients of the Camassa–Holm equation from the terms uu_x, $u_x u_{xx}$, and uu_{xxx} are 3, 2, and 1, which satisfy in order that $3 = 2 + 1$, similar fact happens for the Degasperis–Procesi equation, i.e., $4 = 3 + 1$, then it does from the corresponding terms of the Novikov equation. Note that it is formally simple mathematical relationship among the coefficients, but it actually plays important role in the analysis of dynamical properties of both equations. As we know, the Camassa–Holm and Degasperis–Procesi equations possess different conservation laws, for example,

$$\int_{\mathbb{R}}(u^2+u_x^2)\mathrm{d}x,\ \int_{\mathbb{R}}(u^3+uu_x^2)\mathrm{d}x,\ \int_{\mathbb{R}}\sqrt{y}\mathrm{d}x$$

are conserved for the Camassa–Holm equation (1.4), while

$$\int_{\mathbb{R}}u^3\mathrm{d}x,\ \int_{\mathbb{R}}yv\mathrm{d}x,\ \int_{\mathbb{R}}y^{1/3}\mathrm{d}x$$

are conserved for the Degasperis–Procesi (1.5), where $v=(4-\partial_x^2)u$ and $y=(1-\partial_x^2)u$. Lots of research also indicated the differences between (1.4) and (1.5). One purpose of this paper is to show how these coefficients affect the properties of solutions. On the other hand, we try to discover something new even for (1.2) to extend some previous results. Precisely, in this paper, we find that the H^1 -norm of strong solutions to (1.1) is an invariant if $a=3b$, which usually plays an important role in the study of shallow water type equations (the Camassa–Holm equation, Degasperis–Procesi equation, and Novikov equation). Note that even for Novikov equation (1.2), the asymptotic description and persistence properties in weighted space investigated here are completely new. Based on Kato's semigroup theory [19], one can prove local well-posedness for (1.1) as what was done by Zhou [25] for the Novikov equation. We are not going to repeat it but focus on the following issues. In Sect. 2, the detailed asymptotic profiles of strong solution are shown with compactly supported initial momentum $y_0(x)$ rather than $u_0(x)$, which can be comparable with the work [14]. We, in Sect. 3, determine the persistence property of strong solutions in weighted space in the sense that the solutions to (1.1) will retain this property as their initial values do. In the final section, we discuss finite time blow-up phenomenon and global in time solutions.

2 Preliminaries

Set $y=(1-\partial_x^2)u$, which is usually called momentum. Then $u(x,t)$ and $u_x(x,t)$ can be expressed by

$$u(x,t)=\frac{1}{2}e^{-x}\int_{-\infty}^{x}e^{\xi}y(\xi,t)\mathrm{d}\xi+\frac{1}{2}e^{x}\int_{x}^{\infty}e^{-\xi}y(\xi,t)\mathrm{d}\xi, \tag{2.1}$$

$$u_x(x,t)=-\frac{1}{2}e^{-x}\int_{-\infty}^{x}e^{\xi}y(\xi,t)\mathrm{d}\xi+\frac{1}{2}e^{x}\int_{x}^{\infty}e^{-\xi}y(\xi,t)\mathrm{d}\xi. \tag{2.2}$$

For convenience of later use, we have an equivalent form of (1.1)

$$y_t+by_xu^2+ayuu_x=0. \tag{2.3}$$

Let $q(x,t)$ be the particle line evolved by the solution $u(x,t)$:

$$\frac{\mathrm{d}q(x,t)}{\mathrm{d}t} = bu^2(q(x,t),t), \quad q(x,0) = x. \tag{2.4}$$

Differentiating it with respect to x

$$\frac{\mathrm{d}q_t(x,t)}{\mathrm{d}x} = 2buu_x(q,t)q_x, \quad q_x(x,0) = 1.$$

Then, we have

$$q_x(x,t) = \exp\left(2b\int_0^t uu_x(q,s)\mathrm{d}s\right)$$

which is always positive before blow-up time. Therefore, the function $q(x,t)$ is an increasing diffeomorphism of the line. From (2.3), we can prove that

$$y(q(x,t),t)q_x^{\frac{a}{2b}} = y_0(x). \tag{2.5}$$

In fact, by straightforward computation, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[y(q(x,t),t)q_x^{\frac{a}{2b}}(x,t)\right] &= (y_t + y_x q_t)q_x^{\frac{a}{2b}} + \frac{a}{2b}yq_x^{\frac{a}{2b}-1}q_{xt} \\ &= (y_t + y_x bu^2)q_x^{\frac{a}{2b}} + \frac{a}{2b}yq_x^{\frac{a}{2b}-1}2buu_x q_x \\ &= (y_t + by_x u^2 + ayuu_x)q_x^{\frac{a}{2b}} = 0.\end{aligned}$$

In order to investigate the blow-up phenomenon, we shall borrow a lemma from Zhou's work [31], which can be proved by making full use of comparison theorem of ordinary differential equations.

Lemma 2.1 *Suppose that* $\Phi(t) \in C^2(0,\infty)$ *satisfying*

$$\begin{cases}\Phi''(t) \geqslant C_0\Phi'(t)\Phi(t), & C_0 > 0, \\ \Phi(0) > 0, \Phi'(0) > 0. & \end{cases}$$

Then $\Phi(t)$ *blows up in finite time and the blow-up time can be estimated as*

$$T \leqslant \max\left\{\frac{2}{C_0\Phi(0)}, \quad \frac{\Phi(0)}{\Phi'(0)}\right\}.$$

3 Asymptotic Description

The purpose of this section is to give a more detailed description on the corresponding strong solution $u(x,t)$ to (1.1) in its lifespan with initial momentum being compactly supported and nonnegative. As a byproduct, an explicit formula for $u(x,t)$ in space direction is shown in the following theorem.

Theorem 3.1 *Assume that for some* $T > 0$ *and* $s > 5/2$, $u(x,t) \in C([0,T]; H^s(\mathbb{R}))$ *is a nontrivial strong solution of* (1.1) *with associated initial value* $u_0(x) \in H^s(\mathbb{R})$. *If* $0 < a \le 6b$ *and initial momentum* $y_0(x) = (1-\partial_x^2)u_0(x)$ *are nonnegative with compact support in an interval* $[\alpha, \beta]$. *Then we have*

$$u(x,t) = \begin{cases} \frac{1}{2}E_1(t)e^{-x}, & for\ x > q(\beta,t), \\ \frac{1}{2}E_2(t)e^{x}, & for\ x < q(\alpha,t), \end{cases} \tag{3.1}$$

where $E_1(t)$ *and* $E_2(t)$ *are continuous positive functions. Furthermore,* $E_1(t)$ *is strictly increasing, while* $E_2(t)$ *is strictly decreasing for all* $t \in [0,T]$.

Remark 3.1 We remark that (3.1) gives not only exponential decay behavior of strong solutions in space variable, but also the information that $u(x,t)$ can be expressed by separation of variables into a product of two single variable functions. In our case the solution $u(x,t)$ lies above x-axis, we conjecture it is the term u^2u_x whose index of u is 2, an even number that results in it in some sense.

Proof Since $y_0(x)$ has compact support in the interval $[\alpha,\beta]$, it follows by (2.5) that $y(\cdot,t)$ has its compact support in the interval $[q(\alpha,t), q(\beta,t)]$ for any $t \in [0,T]$. Hence the following functions are well defined

$$E_1(t) = \int_{\mathbb{R}} e^{\xi} y(\xi,t)\mathrm{d}\xi \text{ and } E_2(t) = \int_{\mathbb{R}} e^{-\xi} y(\xi,t)\mathrm{d}\xi \tag{3.2}$$

with

$$E_1(0) = \int_{\mathbb{R}} e^{\xi} y_0(\xi)\mathrm{d}\xi > 0 \text{ and } E_2(0) = \int_{\mathbb{R}} e^{-\xi} y_0(\xi)\mathrm{d}\xi > 0$$

by the positivity of initial momentum $y_0(x)$. The relations between $u(x,t)$ and $y(x,t)$ yield

$$u(x,t) = \frac{1}{2}e^{-|x|} * y(x,t) = \frac{1}{2}e^{-x}E_1(t), \quad x > q(\beta,t)$$

and

$$u(x,t) = \frac{1}{2}e^{-|x|} * y(x,t) = \frac{1}{2}e^{x}E_2(t), \quad x < q(\alpha,t).$$

Now it remains to show the monotonicity of $E_1(t)$ and $E_2(t)$. It follows from (3.2) that

$$\frac{\mathrm{d}E_1(t)}{\mathrm{d}t} = \int_{\mathbb{R}} e^{\xi} y_t(\xi,t)\mathrm{d}\xi.$$

Thus, in view of (2.3) and after integration by parts, we have

$$\begin{aligned}\frac{dE_1(t)}{\mathrm{d}t} &= \int_{\mathbb{R}} e^{\xi} y_t(\xi,t)\mathrm{d}\xi = -\int_{\mathbb{R}} e^{\xi}(bu^2y_x + ayuu_x)\mathrm{d}\xi \\ &= -b\int_{\mathbb{R}} e^{\xi}u^2\mathrm{d}y - a\int_{\mathbb{R}} e^{\xi}yuu_x\mathrm{d}\xi \\ &= b\int_{\mathbb{R}} e^{\xi}u^2y\mathrm{d}\xi + 2b\int_{\mathbb{R}} e^{\xi}uu_xy\mathrm{d}\xi - a\int_{\mathbb{R}} e^{\xi}uu_xy\mathrm{d}\xi\end{aligned}$$

$$
\begin{aligned}
&= b\int_{\mathbb{R}} e^{\xi}u^2(u-u_{xx})\mathrm{d}\xi + (2b-a)\int_{\mathbb{R}} e^{\xi}uu_x y\mathrm{d}\xi \\
&= b\int_{\mathbb{R}} e^{\xi}u^3\mathrm{d}\xi - b\int_{\mathbb{R}} e^{\xi}u^2u_{xx}\mathrm{d}\xi + (2b-a)\int_{\mathbb{R}} e^{\xi}uu_x y\mathrm{d}\xi \\
&= \frac{a}{3}\int_{\mathbb{R}} e^{\xi}u^3\mathrm{d}\xi + \frac{6b-a}{2}\int_{\mathbb{R}} e^{\xi}uu_x^2\mathrm{d}\xi + \frac{2b-a}{2}\int_{\mathbb{R}} e^{\xi}u_x^3\mathrm{d}\xi. \qquad (3.3)
\end{aligned}
$$

Since $y(\cdot,t)$ is compactly supported in $[q(\alpha,t), q(\beta,t)]$, a direct consequence of (2.1) and (2.2) implies that $u(x,t)\geq 0$ and $u(x,t)+u_x(x,t)>0$ for all $t\in[0,T]$ and $x\in\mathbb{R}$. Therefore, when $0<a\leq 2b$,

$$
\begin{aligned}
\frac{dE_1(t)}{\mathrm{d}t} &= \frac{a}{3}\int_{\mathbb{R}} e^{\xi}u^3\mathrm{d}\xi + \frac{6b-a}{2}\int_{\mathbb{R}} e^{\xi}uu_x^2\mathrm{d}\xi + \frac{2b-a}{2}\int_{\mathbb{R}} e^{\xi}u_x^3\mathrm{d}\xi \\
&= \frac{a}{3}\int_{\mathbb{R}} e^{\xi}u^3\mathrm{d}\xi + 2b\int_{\mathbb{R}} e^{\xi}uu_x^2\mathrm{d}\xi + \frac{2b-a}{2}\int_{\mathbb{R}} e^{\xi}u_x^2(u+u_x)\mathrm{d}\xi \\
&> 0.
\end{aligned}
$$

Moreover, $dE_1(t)/\mathrm{d}t$ can also be expressed by

$$
\begin{aligned}
\frac{dE_1(t)}{\mathrm{d}t} &= \frac{a}{3}\int_{\mathbb{R}} e^{\xi}u^3\mathrm{d}\xi + \frac{6b-a}{2}\int_{\mathbb{R}} e^{\xi}uu_x^2\mathrm{d}\xi + \frac{2b-a}{2}\int_{\mathbb{R}} e^{\xi}u_x^3\mathrm{d}\xi \\
&= \frac{6b-a}{6}\int_{\mathbb{R}} e^{\xi}uu^2\mathrm{d}\xi + \frac{6b-a}{2}\int_{\mathbb{R}} e^{\xi}uu_x^2\mathrm{d}\xi \\
&\quad - \frac{2b-a}{2}\int_{\mathbb{R}} e^{\xi}(u^3-u_x^3)\mathrm{d}\xi.
\end{aligned}
$$

Note that

$$
u(x,t)-u_x(x,t) = e^{-x}\int_{-\infty}^{x} e^{\xi}y(\xi,t)\mathrm{d}\xi \geq 0,
$$

then it follows that

$$
\begin{aligned}
u^3-u_x^3 &= (u-u_x)\,(u^2+uu_x+u_x^2) \\
&= (u-u_x)\left[\left(u+\frac{1}{2}u_x\right)^2 + \frac{3}{4}u_x^2\right] \geq 0.
\end{aligned}
$$

Then

$$
\frac{\mathrm{d}E_1(t)}{\mathrm{d}t} > 0, \quad \text{for } t\in[0,T]
$$

holds when $2b<a\leq 6b$.

Similarly, for $E_2(t)$, we have after integration by parts that

$$
\begin{aligned}
\frac{\mathrm{d}E_2(t)}{\mathrm{d}t} &= \int_{\mathbb{R}} e^{-\xi}y_t(\xi,t)\mathrm{d}\xi = -b\int_{\mathbb{R}} e^{-\xi}u^2y_x\mathrm{d}\xi - a\int_{\mathbb{R}} e^{-\xi}uu_x y\mathrm{d}\xi \\
&= -b\int_{\mathbb{R}} e^{-\xi}u^2y\mathrm{d}\xi + (2b-a)\int_{\mathbb{R}} e^{-\xi}uu_x y\mathrm{d}\xi
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{a}{3}\int_{\mathbb{R}} e^{-\xi}u^3 \mathrm{d}\xi - \frac{6b-a}{2}\int_{\mathbb{R}} e^{-\xi}uu_x^2 \mathrm{d}\xi + \frac{2b-a}{2}\int_{\mathbb{R}} e^{-\xi}u_x^3 \mathrm{d}\xi \\
&= -\frac{a}{3}\int_{\mathbb{R}} e^{-\xi}u^3 \mathrm{d}\xi - 2b\int_{\mathbb{R}} e^{-\xi}uu_x^2 \mathrm{d}\xi \\
&\quad -\frac{2b-a}{2}\int_{\mathbb{R}} e^{-\xi}u_x^2(u-u_x) \mathrm{d}\xi \qquad (3.4) \\
&= -\frac{6b-a}{6}\int_{\mathbb{R}} e^{-\xi}uu^2 \mathrm{d}\xi - \frac{6b-a}{2}\int_{\mathbb{R}} e^{-\xi}uu_x^2 \mathrm{d}\xi \\
&\quad +\frac{2b-a}{2}\int_{\mathbb{R}} e^{-\xi}(u^3+u_x^3) \mathrm{d}\xi. \qquad (3.5)
\end{aligned}
$$

If $0 < a \leq 6b$, as above for $E_1(t)$, then (3.4) and (3.5) can lead to

$$
\frac{\mathrm{d}E_2(t)}{\mathrm{d}t} < 0, \quad \text{for } t \in [0, T].
$$

This completes the proof of Theorem 3.1. □

4 Persistence Property in Weighted Space

Generally, we say persistence property of solutions to (1.1): the solution $u(x,t)$ and its derivatives retain this property as their initial values do. In this part, we intend to find a class of weight functions φ such that

$$
||\varphi u(t)||_p + ||\varphi u_x(t)||_p + ||\varphi u_{xx}(t)||_p < \infty,
$$

which can generalize some previous results, where $||\cdot||_p$ is the usual L^p norm. We can obtain a persistence result of solutions in the weighted space $L^p(\mathbb{R}, \varphi^p(x)\mathrm{d}x)$. We will work with moderate weight functions which have been systematically used to lead to optimal results for the Camassa–Holm equation in [2]. To this end, let us first recall some standard definitions and basic results in time-frequency analysis [1,7].

Definition 4.1 A nonnegative function $v\colon \mathbb{R}^n \to \mathbb{R}$ is called sub-multiplicative if $v(x+y) \leq v(x)v(y)$ holds for all $x, y \in \mathbb{R}^n$. Given a sub-multiplicative function v, a positive function φ is called v-moderate if there exists a constant $C > 0$ such that $\varphi(x+y) \leq Cv(x)\varphi(y)$ holds for all $x, y \in \mathbb{R}^n$.

It is proved in [2] that φ is v-moderate if and only if the weighted Young inequality

$$
\|(f_1 * f_2)\varphi\|_p \leq C\,\|f_1 v\|_1\,\|f_2\varphi\|_p \tag{4.1}
$$

holds for any two measurable functions f_1, f_2 and $1 \leq p \leq \infty$.

Definition 4.2 We say that $\varphi\colon \mathbb{R} \to (0, +\infty)$ is an admissible weight for (1.1) if it is locally absolutely continuous such that $|\varphi'(x)| \leq A|\varphi(x)|$ for some $A > 0$ and a.e. $x \in \mathbb{R}$, and that is v-moderate with a sub-multiplicative function v satisfying $\inf_{\mathbb{R}} v > 0$ and

$$\int_{\mathbb{R}} v(x)e^{-|x|}\mathrm{d}x < \infty. \tag{4.2}$$

Now we show the main result of this section.

Theorem 4.3 *Let $u_0 \in H^s(\mathbb{R})$ with $s > 5/2$, $2 \le p \le \infty$, and $u \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$ be the strong solution to (1.1) starting from u_0 such that $\varphi u_0, \varphi u_{0x}, \varphi u_{0xx} \in L^p(\mathbb{R})$ for an admissible weight function φ of (1.1). Then for all $t \in [0,T)$, we have the estimate*

$$\begin{aligned}&\|\varphi u(\cdot,t)\|_p + \|\varphi u_x(\cdot,t)\|_p + \|\varphi u_{xx}(\cdot,t)\|_p\\ &\quad\le \exp(CM^2 t)\left(\|\varphi u_0(x)\|_p + \|\varphi u_{0x}(x)\|_p + \|\varphi u_{0xx}(x)\|_p\right),\end{aligned}$$

where the constant C depending only on the weights v and φ and

$$M := \sup_{t\in[0,T)} \left(\|u(\cdot,t)\|_\infty + \|u_x(\cdot,t)\|_\infty + \|u_{xx}(\cdot,t)\|_\infty\right).$$

Remark 4.1 The standard examples for admissible weight functions are given by the family of functions [7]

$$\varphi(x) = \varphi_{a,b,c,d}(x) = e^{a|x|^b}(1+|x|)^c \log(e+|x|^{\mathrm{d}}),$$

where we require that $a \ge 0$, $0 \le b \le 1$, and $ab < 1$.

- If we take $\varphi(x) = \varphi_{0,0,c,0}(x)$ with $c > 0$ and choose $p = \infty$ in Theorem 4.3, then the initial datum decays algebraically like $(1+|x|)^{-c}$, which is preserved by the solution in its lifespan, i.e., the solution $u(x,t)$ and its derivative also decay like $(1+|x|)^{-c}$ asymptotically. This is exactly the result of Ni and Zhou [26] for the Camassa–Holm equation.
- If we take $\varphi(x) = \varphi_{a,1,0,0}(x)$ if $x \ge 0$ and $\varphi(x) = 1$ if $x \le 0$ with $0 \le a < 1$. Such weight clearly satisfies the admissibility conditions of Definition 4.2. The result from Theorem 4.3 with $p = \infty$ implies that the pointwise decay of initial values can be preserved during the evolution. Similar decay properties were also determined by [14,25] for the Camassa–Holm and Novikov equations, respectively. So, Theorem 4.3 is actually a generalized version of them, and can also be viewed as an intermediate state of decay behavior.

Proof For convenience, we rewrite (1.1) as the following transport equation:

$$u_t + bu^2u_x + G * F(u) = 0, \tag{4.3}$$

where $F(u) = au^2u_x - (a-6b)uu_xu_{xx} + 2bu_x^3$ and $G(x) = e^{-|x|}/2$ is the Green's function of the operator $\left(1-\partial_x^2\right)^{-1}$. For any $N \in \mathbb{N}\backslash\{0\}$, we define the N-truncations: $\varphi_N(x) = \min\{\varphi(x), N\}$. Then it is easy to check that $\varphi_N(x) : \mathbb{R} \to \mathbb{R}$ is locally absolutely continuous function satisfying $\|\varphi_N(x)\|_\infty \leq N$ and $|\varphi_N'(x)| \leq A\varphi_N(x)$ a.e. on $\mathbb{R}$.

For $p \in [2, +\infty)$, multiplying (4.3) by $\varphi_N |\varphi_N u|^{p-2}\varphi_N u$ and integrating both sides on the line, one gets

$$\|\varphi_N u\|_p^{p-1} \frac{\mathrm{d}}{\mathrm{d}t} \|\varphi_N u\|_p + b\int_{\mathbb{R}} u|\varphi_N u|^p u_x \mathrm{d}x + \int_{\mathbb{R}} \varphi_N G * F(u)|\varphi_N u|^{p-2}\varphi_N u \mathrm{d}x = 0.$$

We observe that

$$\left|\int_{\mathbb{R}} u|\varphi_N u|^p u_x \mathrm{d}x\right| \leq M^2 \|\varphi_N u\|_p^p,$$

and by Hölder's inequality that

$$\left|\int_{\mathbb{R}} \varphi_N G * F(u)|\varphi_N u|^{p-2}\varphi_N u \mathrm{d}x\right| \leq \|\varphi_N G * F(u)\|_p \|\varphi_N u\|_p^{p-1}.$$

Moreover, we have by using (4.1) and (4.2) that

$$\|\varphi_N G * F(u)\|_p \lesssim \|Gv\|_1 \|\varphi_N F(u)\|_p \lesssim \|\varphi_N F(u)\|_p .$$

Hence, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\varphi_N u\|_p \lesssim M^2 \|\varphi_N u\|_p + \|\varphi_N F(u)\|_p . \tag{4.4}$$

Differentiating (4.3) with respect to variable x produces the following equation:

$$u_{tx} + bu^2 u_{xx} + 2buu_x^2 + \partial_x(G * F(u)) = 0.$$

Multiplying the above equation by $\varphi_N|\varphi_N u_x|^{p-2}\varphi_N u_x$ and integrating over the line, one has

$$\begin{aligned} &||\varphi_N u_x||_p^{p-1} \frac{\mathrm{d}}{\mathrm{d}t}||\varphi_N u_x||_p + \int_{\mathbb{R}} \varphi_N \partial_x(G * F(u))|\varphi_N u_x|^{p-2}\varphi_N u_x \mathrm{d}x \\ &+ \ b\int_{\mathbb{R}} u^2 u_{xx}\varphi_N|\varphi_N u_x|^{p-2}\varphi_N u_x \mathrm{d}x + 2b\int_{\mathbb{R}} uu_x^2\varphi_N|\varphi_N u_x|^{p-2}\varphi_N u_x \mathrm{d}x = 0. \end{aligned}$$

We estimate that

$$\begin{aligned} &\int_{\mathbb{R}} u^2 u_{xx}\varphi_N|\varphi_N u_x|^{p-2}\varphi_N u_x \mathrm{d}x \\ &\quad = \int_{\mathbb{R}} u^2|\varphi_N u_x|^{p-2}\varphi_N u_x \left[(u_x\varphi_N)_x - u_x\partial_x\varphi_N\right] \mathrm{d}x \\ &\quad = \int_{\mathbb{R}} u^2 \partial_x\left(\frac{|\varphi_N u_x|^p}{p}\right) \mathrm{d}x - \int_{\mathbb{R}} u^2|\varphi_N u_x|^{p-2}\varphi_N u_x \left(u_x\varphi_N'\right) \mathrm{d}x. \end{aligned}$$

Note that $|\varphi_N'(x)| \leq A\varphi_N(x)$ a.e. on $\mathbb{R}$, then it follows that

$$\left|\int_{\mathbb{R}} u^2 u_{xx}\varphi_N|\varphi_N u_x|^{p-2}\varphi_N u_x \mathrm{d}x\right|$$
$$\leq 2p^{-1}M^2 \|\varphi_N u_x\|_p^p + A||u||_{L^\infty}^2 \|\varphi_N u_x\|_p^p$$
$$\leq M^2(1+A)\|\varphi_N u_x\|_p^p$$

and

$$\left|\int_{\mathbb{R}} uu_x^2\varphi_N|\varphi_N u_x|^{p-2}\varphi_N u_x \mathrm{d}x\right| \leq M^2 \|\varphi_N u_x\|_p^p$$
$$\left|\int_{\mathbb{R}} \varphi_N \partial_x(G * F(u))|\varphi_N u_x|^{p-2}\varphi_N u_x \mathrm{d}x\right| \leq \|\varphi_N \partial_x(G * F(u))\|_p \|\varphi_N u_x\|_p^{p-1}.$$

By using the fact $\partial_x G = -\frac{1}{2}\mathrm{sgn}(x)e^{-|x|}$ in the weak sense and using (4.1) and (4.2) again, there holds

$$\|\varphi_N \partial_x(G * F(u))\|_p \lesssim \|(\partial_x G)v\|_1 \|\varphi_N F(u)\|_p \lesssim \|\varphi_N F(u)\|_p.$$

Thus, we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_N u_x\|_p \lesssim C_1 M^2 \|\varphi_N u_x\|_p + \|\varphi_N F(u)\|_p. \tag{4.5}$$

Differentiating (4.3) twice with respect to spatial variable x gives the following equation:

$$u_{txx} + bu^2 u_{xxx} + 6buu_x u_{xx} + 2bu_x^3 + \partial_x^2(G * F(u)) = 0.$$

Multiplying this equation by $\varphi_N|\varphi_N u_{xx}|^{p-2}\varphi_N u_{xx}$ and integrating over the line, we have

$$\|\varphi_N u_{xx}\|_p^{p-1}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_N u_{xx}\|_p + b\int_{\mathbb{R}} u^2 u_{xxx}\varphi_N|\varphi_N u_{xx}|^{p-2}\varphi_N u_{xx}\mathrm{d}x$$
$$+\int_{\mathbb{R}}\varphi_N\partial_x^2(G * F(u))|\varphi_N u_{xx}|^{p-2}\varphi_N u_{xx}\mathrm{d}x$$
$$+ \ 6b\int_{\mathbb{R}} uu_x|\varphi_N u_{xx}|^p \mathrm{d}x + 2b\int_{\mathbb{R}} u_x^3\varphi_N|\varphi_N u_{xx}|^{p-2}\varphi_N u_{xx}\mathrm{d}x = 0.$$

We first estimate the second term of the above equation. For this purpose, we rewrite it as follows:

$$b\int_{\mathbb{R}}[\partial_x(\varphi_N u_{xx}) - \partial_x(\varphi_N)u_{xx}]\, u^2|\varphi_N u_{xx}|^{p-2}\varphi_N u_{xx}\mathrm{d}x,$$

then we have, in view of $|\partial_x(\varphi_N)| \leq A|\varphi_N|$, that

$$\left|\int_{\mathbb{R}} \partial_x(\varphi_N) u_{xx} u^2 |\varphi_N u_{xx}|^{p-2} \varphi_N u_{xx} \mathrm{d}x\right| \leq AM^2 \left\|\varphi_N u_{xx}\right\|_p^p,$$

and

$$\begin{aligned}&\left|\int_{\mathbb{R}} \partial_x(\varphi_N u_{xx}) u^2 |\varphi_N u_{xx}|^{p-2} \varphi_N u_{xx} \mathrm{d}x\right| \\ &\quad \leq \left|\int_{\mathbb{R}} u^2 \partial_x \left(\frac{|\varphi_N u_{xx}|^p}{p}\right) \mathrm{d}x\right| = \frac{2}{p}\left|\int_{\mathbb{R}} u u_x |\varphi_N u_{xx}|^p \mathrm{d}x\right| \\ &\quad \lesssim M^2 \left\|\varphi_N u_{xx}\right\|_p^p,\end{aligned}$$

by using integration by parts in view of Sobolev's theorem. Hence, we obtain

$$\left|\int_{\mathbb{R}} u^2 u_{xxx} \varphi_N |\varphi_N u_{xx}|^{p-2} \varphi_N u_{xx} \mathrm{d}x\right| \leq (A+1) M^2 \left\|\varphi_N u_{xx}\right\|_p^p.$$

Moreover, for the other terms, we have

$$\begin{aligned}\left|\int_{\mathbb{R}} u u_x |\varphi_N u_{xx}|^p \mathrm{d}x\right| &\leq M^2 \left\|\varphi_N u_{xx}\right\|_p^p, \\ \left|\int_{\mathbb{R}} u_x^3 \varphi_N |\varphi_N u_{xx}|^{p-2} \varphi_N u_{xx} \mathrm{d}x\right| &\leq M^2 \left\|\varphi_N u_x\right\|_p \left\|\varphi_N u_{xx}\right\|_p^{p-1}, \\ \left|\int_{\mathbb{R}} \varphi_N \partial_x^2 (G * F(u)) |\varphi_N u_{xx}|^{p-2} \varphi_N u_{xx} \mathrm{d}x\right| &\leq \left\|\varphi_N \partial_x^2 (G * F(u))\right\|_p \left\|\varphi_N u_{xx}\right\|_p^{p-1}.\end{aligned}$$

Thus, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\|\varphi_N u_{xx}\right\|_p \lesssim C_2 M^2 \left(\left\|\varphi_N u_x\right\|_p + \left\|\varphi_N u_{xx}\right\|_p + \left\|\varphi_N F(u)\right\|_p\right), \tag{4.6}$$

where we have used the fact

$$\left\|\varphi_N \partial_x^2 (G * F(u))\right\|_p \lesssim \| \varphi_N F(u) \|_p.$$

Furthermore, we conclude easily by the definition of $F(u)$ that

$$\left\|\varphi_N F(u)\right\|_p \lesssim C_3 M^2 \left(\left\|\varphi_N u\right\|_p + \left\|\varphi_N u_x\right\|_p + \left\|\varphi_N u_{xx}\right\|_p\right). \tag{4.7}$$

Combining (4.4), (4.5), and (4.6), there exists a constant C such that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} \left(\left\|\varphi_N u\right\|_p + \left\|\varphi_N u_x\right\|_p + \left\|\varphi_N u_{xx}\right\|_p\right) \\ &\quad \leq C M^2 \left(\left\|\varphi_N u\right\|_p + \left\|\varphi_N u_x\right\|_p + \left\|\varphi_N u_{xx}\right\|_p\right).\end{aligned}$$

Now by the Gronwall's lemma, we have

$$\begin{aligned}&\|\varphi_N u\|_p + \|\varphi_N u_x\|_p + \|\varphi_N u_{xx}\|_p\\&\quad \le \exp(CM^2 t)\left(\|\varphi_N u_0\|_p + \|\varphi_N u_{0x}\|_p + \|\varphi_N u_{0xx}\|_p\right).\end{aligned}$$

Since for a.e. $x \in \mathbb{R}$, φ_N goes to φ as N tends to ∞, by the assumptions we arrive at

$$\begin{aligned}&\|\varphi u\|_p + \|\varphi u_x\|_p + \|\varphi u_{xx}\|_p\\&\quad \le \exp(CM^2 t)\left(\|\varphi u_0\|_p + \|\varphi u_{0x}\|_p + \|\varphi u_{0xx}\|_p\right).\end{aligned} \tag{4.8}$$

The remaining is to treat the case of $p = \infty$. We have $u_0, u_{0x}, u_{0xx} \in L^2 \cap L^\infty$, and $\varphi_N \in L^\infty$. Therefore, for all $q \ge 2$, we have as before

$$\begin{aligned}&\|\varphi_N u\|_q + \|\varphi_N u_x\|_q + \|\varphi_N u_{xx}\|_q\\&\quad \le \exp(CM^2 t)\left(\|\varphi_N u_0\|_q + \|\varphi_N u_{0x}\|_q + \|\varphi_N u_{0xx}\|_q\right)\end{aligned}$$

with the factor $\exp(CM^2 t)$ being independent on q. Let $q \to \infty$ and using the fact that the L^∞-norm is the limit of L^q norm as $q \to \infty$, it implies that

$$\begin{aligned}&\|\varphi_N u\|_\infty + \|\varphi_N u_x\|_\infty + \|\varphi_N u_{xx}\|_\infty\\&\quad \le \exp(CM^2 t)\left(\|\varphi_N u_0\|_\infty + \|\varphi_N u_{0x}\|_\infty + \|\varphi_N u_{0xx}\|_\infty\right).\end{aligned}$$

Note that $\exp(CM^2 t)$ is independent on N, the above inequality implies that (4.8) still holds true for $p = \infty$ by taking the limit as $N \to \infty$. This completes the whole proof. □

5 Blow-Up and Global Solutions

There exist strong solutions to (1.1) that exist globally and strong solutions that blow up in finite time. This section is devoted to the study of blow-up condition which is given to guarantee that an initially smooth solution develops singularity in finite time and to looking for sufficient condition for the global existence of the solution $u(x, t)$. We first show a scenario to understand what really happens as the solution blows up.

Theorem 5.1 *Let $u_0(x) \in H^s(\mathbb{R})$, $s \ge 2$, $a > b > 0$, and T be the maximal existence time of the solution $u(x, t)$ to (1.1) arising from $u_0(x)$. Then the corresponding solution blows up in finite time if and only if*

$$\lim_{t \uparrow T} \liminf_{x \in \mathbb{R}} (u u_x)(x, t) = -\infty.$$

Proof Multiplying both sides of (2.3) by y and integrating on the line, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} y^2 \mathrm{d}x = -2b\int_{\mathbb{R}} u^2 y y_x \mathrm{d}x - 2a\int_{\mathbb{R}} u u_x y^2 \mathrm{d}x$$

$$= 2(b-a)\int_{\mathbb{R}} uu_x y^2 \mathrm{d}x.$$

If uu_x is bounded from blow on $[0, T)$, then there exist two constants $C_1, C_2 > 0$ such that

$$\int_{\mathbb{R}} y^2 \mathrm{d}x \le C_1 \int_0^t \int_{\mathbb{R}} y^2 \mathrm{d}x\mathrm{d}s + C_2.$$

Due to Gronwall's inequality, one has

$$\int_{\mathbb{R}} y^2 \mathrm{d}x = ||y||_{L^2}^2 \le C_2 \left(1 + C_1 t e^{C_1 t}\right),$$

which implies that the H^2-norm of solution is also bounded on $[0, T)$ by the fact

$$||u||_{H^2}^2 \le ||y||_{L^2}^2 \le 2||u||_{H^2}^2. \tag{5.1}$$

On the other hand, since $u = G * y$ and $u_x = G_x * y$, one has

$$\|uu_x\|_{L^\infty} \le ||G||_{L^2}||G_x||_{L^2}||y||_{L^2}^2 \le 2||G||_{L^2}||G_x||_{L^2}||u||_{H^2}^2, \tag{5.2}$$

where we have used (5.1). Hence, (5.2) indicates that $\|uu_x\|_{L^\infty}$ can be bounded by $||u||_{H^2}$. Note that for $a = b$, it is clear that the L^2-norm of y is conserved with respect to time, which means the solution $u(x, t)$ does not blow up, and for $a < b$, similar arguments as above show that blow-up occurs if and only if uu_x tends to $+\infty$. We complete the proof. □

We now present the following condition to show the existence of blow-up solutions.

Theorem 5.2 *Suppose that $u_0 \in H^s(\mathbb{R})$, $s \ge 2$, $a = 3b > 0$ and there exists an $x_0 \in \mathbb{R}$ such that $u_0(x_0) \ge 0$, $y_0(x_0) = 0$. Furthermore, the nontrivial $y_0(x)$ satisfies that*

$$y_0(x) \ge 0 \quad \textit{for } x \in (-\infty, x_0) \quad \textit{and} \quad y_0(x) \le 0 \quad \textit{for } x \in (x_0, +\infty).$$

Then the corresponding solution $u(x, t)$ blows up in finite time.

Proof In view of the initial condition and (2.5), we have $y(q(x_0, t), t) = 0$ and

$$\begin{cases} y(q(x,t),t) \ge 0, & x \in (-\infty, x_0) \\ y(q(x,t),t) \le 0, & x \in (x_0, +\infty) \end{cases}.$$

It is straightforward to obtain from (2.1) and (2.2) that

$$u + u_x = e^x \int_x^\infty e^{-\xi} y(\xi)\mathrm{d}\xi, \quad u - u_x = e^{-x} \int_{-\infty}^x e^{\xi} y(\xi)\mathrm{d}\xi. \tag{5.3}$$

In the following, we derive an equation for uu_x, then show it is possible to go to $-\infty$ in finite time. Differentiating $uu_x(q(x_0, t), t)$ with respect to t, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(2uu_x(q,t) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(e^{q}\int_q^{\infty} e^{-\xi}y(\xi,t)\mathrm{d}\xi\right)^2 - \left(e^{-q}\int_{-\infty}^{q} e^{\xi}y(\xi,t)\mathrm{d}\xi\right)^2\right]\\
&= bu^2\left(e^{q}\int_q^{\infty} e^{-\xi}y(\xi,t)\mathrm{d}\xi\right)^2 + e^{2q}\left(\int_q^{\infty} e^{-\xi}y(\xi,t)\mathrm{d}\xi\right)\left(\int_q^{\infty} e^{-\xi}y_t(\xi,t)\mathrm{d}\xi\right)\\
&\quad + bu^2\left(e^{-q}\int_{-\infty}^{q} e^{\xi}y(\xi,t)\mathrm{d}\xi\right)^2 - e^{-2q}\left(\int_{-\infty}^{q} e^{\xi}y(\xi,t)\mathrm{d}\xi\right)\left(\int_{-\infty}^{q} e^{\xi}y_t(\xi,t)\mathrm{d}\xi\right)\\
&= bu^2(u-u_x)^2(q,t) - (u-u_x)(q(x_0,t),t)e^{-q}\int_{-\infty}^{q} e^{\xi}y_t(\xi,t)\mathrm{d}\xi\\
&\quad + bu^2(u+u_x)^2(q,t) + (u+u_x)(q(x_0,t),t)e^{q}\int_q^{\infty} e^{-\xi}y_t(\xi,t)\mathrm{d}\xi, \qquad (5.4)
\end{aligned}
$$

where we used $y(q(x_0,t),t)=0$. Rewriting (2.3) as $y_t = -b(yu^2)_x - (a-2b)yuu_x$, we estimate the following terms in the above equation as follows:

$$
\begin{aligned}
&e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}y_t(\xi,t)\mathrm{d}\xi\\
&\quad = -e^{-q(x_0,t)}\left(b\int_{-\infty}^{q(x_0,t)} e^{\xi}(yu^2)_\xi\mathrm{d}\xi + (a-2b)\int_{-\infty}^{q(x_0,t)} e^{\xi}yuu_\xi\mathrm{d}\xi\right)\\
&\quad = e^{-q(x_0,t)}\left(\int_{-\infty}^{q(x_0,t)} e^{\xi}(bu^3 - bu^2u_{\xi\xi} - (a-2b)u^2u_\xi + (a-2b)uu_\xi u_{\xi\xi})\mathrm{d}\xi\right)\\
&\quad = -bu^2(q(x_0,t),t)u_x(q(x_0,t),t) + \frac{a-2b}{2}u(q(x_0,t),t)u_x^2(q(x_0,t),t)\\
&\qquad + e^{-q(x_0,t)}\left(\int_{-\infty}^{q(x_0,t)} e^{\xi}(bu^3 + (3b-a)u^2u_\xi + \frac{6b-a}{2}uu_\xi^2 - \frac{a-2b}{2}u_\xi^3)\mathrm{d}\xi\right).
\end{aligned}
$$

The initial condition and (5.3) imply that

$$
(u-u_x)(x,t) \geq 0, \quad x \in (-\infty, q(x_0,t)),
$$

which gives

$$
u^3 + 3uu_x^2 - u_x^3 \geq 3u^2u_x.
$$

Therefore, for $a = 3b > 0$, we have

$$
\begin{aligned}
&\int_{-\infty}^{q(x_0,t)} e^{\xi}(bu^3 + (3b-a)u^2u_\xi + \frac{6b-a}{2}uu_\xi^2 - \frac{a-2b}{2}u_\xi^3)\mathrm{d}\xi\\
&\quad = \frac{b}{2}\int_{-\infty}^{q(x_0,t)} e^{\xi}(2u^3 + 3uu_\xi^2 - u_\xi^3)\mathrm{d}\xi \geq \frac{b}{2}e^{q(x_0,t)}u^3(q(x_0,t),t).
\end{aligned}
$$

Thus, it follows that

$$e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}y_t(\xi,t)\mathrm{d}\xi \\ \geq \left(-bu^2u_x+\frac{b}{2}uu_x^2+\frac{b}{2}u^3\right).(q(x_0,t),t) \tag{5.5}$$

Similarly, one can get

$$e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}y_t(\xi,t)\mathrm{d}\xi \\ \geqslant \left(-bu^2u_x-\frac{b}{2}uu_x^2-\frac{b}{2}u^3\right)(q(x_0,t),t). \tag{5.6}$$

Combining (5.5), (5.6), and (5.4), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(2uu_x(q(x_0,t),t)\leq bu^2(u^2-u_x^2)(q(x_0,t),t). \tag{5.7}$$

Claim $uu_x(q(x_0,t)$ is negative and decreasing, and $u^2(q(x_0,t),t)<u_x^2(q(x_0,t),t)$ for all $t\geq 0$. We prove it by contradiction. Suppose that there exists a t_0 such that $u^2(q(x_0,t),t)<u_x^2(q(x_0,t),t)$ on $[0,t_0)$ but $u^2(q(x_0,t_0),t_0)\geq u_x^2(q(x_0,t_0),t_0)$, then

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(u(u-u_x)(q(x_0,t),t))\\&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-q}\int_{-\infty}^{q}e^{\xi}y(\xi,t)\mathrm{d}\xi\right)\left(e^{-q}\int_{-\infty}^{q}e^{\xi}y(\xi,t)\mathrm{d}\xi\right)\\&\quad+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{q}\int_{q}^{\infty}e^{-\xi}y(\xi,t)\mathrm{d}\xi\right)\left(e^{-q}\int_{-\infty}^{q}e^{\xi}y(\xi,t)\mathrm{d}\xi\right)\\&=-\left(bu^2(u-u_x)^2\right)(q(x_0,t),t)+(u-u_x)\left(e^{-q}\int_{-\infty}^{q}e^{\xi}y_t(\xi,t)\mathrm{d}\xi\right)\\&\quad+\frac{1}{2}\left(e^{q}\int_{q}^{\infty}e^{-\xi}y_t(\xi,t)\mathrm{d}\xi\right)(u-u_x)+\frac{1}{2}\left(e^{-q}\int_{-\infty}^{q}e^{\xi}y_t(\xi,t)\mathrm{d}\xi\right)(u+u_x)\\&\geq\frac{b}{2}u^2(u_x^2-u^2)(q(x_0,t),t)>0,\ \text{on } [0,t_0),\end{aligned}$$

where we have used (5.5) and (5.6). Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(u+u_x)(q(x_0,t),t))\leq-\frac{b}{2}u^2(u_x^2-u^2)(q(x_0,t_0),t_0)<0$$

holds on $[0,t_0)$. It follows in view of the continuity property of ODEs that

$$u^2(u_x^2-u^2)(q(x_0,t_0),t_0)$$

$$
\begin{aligned}
&= [u(u-u_x)][-u(u+u_x)]\,(q(x_0,t_0),t_0)\\
&\geq [u_0(u_0-u_{0x})(x_0)]\,[-u_0(u_0+u_{0x})(x_0)]\\
&= -u_0^2\left(\int_{-\infty}^{x_0} e^{\xi}y_0(\xi)\mathrm{d}\xi\right)\left(\int_{x_0}^{\infty} e^{-\xi}y_0(\xi)\mathrm{d}\xi\right) > 0,
\end{aligned}
$$

where the initial conditions on y_0 were used again. It is an obvious contradiction with assumption. Hence the claim is true. Moreover, based on the above analysis, we have the following inequality:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}u^2(u_x^2-u^2)(q(x_0,t),t)\\
&\quad= -u(u+u_x)(q(x_0,t),t)\frac{\mathrm{d}}{\mathrm{d}t}(u(u-u_x)(q(x_0,t),t))\\
&\qquad -(u(u-u_x)(q(x_0,t),t))\frac{\mathrm{d}}{\mathrm{d}t}u(u+u_x)(q(x_0,t),t)\\
&\quad\geq -b\left(uu_xu^2(u_x^2-u^2)\right)(q(x_0,t),t).
\end{aligned}
$$

It follows by integrating both sides of (5.7) from $[0,t]$ that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}u^2(u_x^2-u^2)(q(x_0,t),t)\\
&\quad\geqslant \frac{1}{2}bu^2(u_x^2-u^2)(q,t)\left(\int_0^t u^2(u_x^2-u^2)(q,s)\mathrm{d}s-2u_0u_{0x}\right).
\end{aligned}
$$

The proof can be completed by letting

$$
\Phi(t)=\int_0^t u^2(u_x^2-u^2)(q(x_0,s),s)\mathrm{d}s-2u_0u_{0x}
$$

and $C_0=b/2$ in Lemma 2.1. □

Not all strong solutions develop singularities in finite time. The following result shows that (1.1) also admits global solutions.

Theorem 5.3 *If $u_0\in H^s$ with $s\geq 3/2$, $a=3b$, and $(1-\partial_x^2)u_0\geq 0$, then the corresponding solution $u(x,t)$ to* (1.1) *satisfies*

$$
||u_x(\cdot,t)||_{L^\infty}\leq \|u\|_{L^\infty}\leq Const,
$$

i.e., $u(x,t)$ can exist globally in time.

Remark 5.1 This result implies that it is the sign not the size of initial momentum that can determine global solutions. Moreover, condition $(1-\partial_x^2)u_0\geq 0$ can also be substituted by $(1-\partial_x^2)u_0\leq 0$.

Proof It suffices to consider $s \geq 3$ to prove this result by the standard density argument. Since $y_0(x) = u_0(x) - u_{0xx}(x) \geq 0$, it follows from (2.5) that $y(x,t) \geq 0$ for all $x \in \mathbb{R}$. Then, one has

$$\begin{aligned} u_x(x,t) &= -u(x,t) + e^x \int_x^\infty e^{-\xi} y(\xi,t)\mathrm{d}\xi \\ &\geq -u(x,t). \end{aligned} \tag{5.8}$$

On the other hand, it follows from (2.2) that

$$\begin{aligned} u_x(x,t) &= u(x,t) - e^{-x} \int_{-\infty}^x e^{\xi} y(\xi,t)\mathrm{d}\xi \\ &\leq u(x,t). \end{aligned} \tag{5.9}$$

Therefore,

$$||u_x(\cdot,t)||_{L^\infty} \leq \|u(\cdot,t)\|_{L^\infty}.$$

It remains to show that $\|u\|_{L^\infty}$ is bounded. Multiplying both sides of (2.3) by $u(x,t)$ and then integrating by parts on $\mathbb{R}$, we obtain

$$\int_{\mathbb{R}} u y_t \mathrm{d}x + \int_{\mathbb{R}} b u^3 y_x \mathrm{d}x + \int_{\mathbb{R}} a u^2 u_x y \mathrm{d}x = 0.$$

Hence,

$$\int_{\mathbb{R}} u y_t \mathrm{d}x - 3b \int_{\mathbb{R}} u^2 u_x y \mathrm{d}x + a \int_{\mathbb{R}} u^2 u_x y \mathrm{d}x = 0,$$

which gives

$$\int_{\mathbb{R}} u y_t \mathrm{d}x + (a - 3b) \int_{\mathbb{R}} u^2 u_x (u - u_{xx})\mathrm{d}x = 0.$$

If $a = 3b$, then

$$\int_{\mathbb{R}} u y_t \mathrm{d}x = 0,$$

while

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|u(x,t)\|_{H^1}^2 = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} (u^2 + u_x^2)\mathrm{d}x = \int_{\mathbb{R}} u y_t \mathrm{d}x,$$

which implies that the H^1-norm of $u(x,t)$ is an invariant and also gives the bound of $u(x,t)$ by the following fact:

$$\|u(\cdot,t)\|_{L^\infty}^2 \leq \frac{1}{2} \|u(x,t)\|_{H^1}^2 = \frac{1}{2} \|u_0(x)\|_{H^1}^2.$$

We complete the proof by Theorem 5.1. □

Acknowledgements The authors would like to thank the referees for their careful reading and helpful comments. This work was partially supported by the National Natural Science Foundation of China, under Grant No. 11301394.

References

1. Aldroubi, A., Gröchenig, K.: Nonuniform sampling and reconstruction in shift-invariant spaces. SIAM Rev. **43**(4), 585–620 (2001)
2. Brandolese, L.: Breakdown for the Camassa–Holm equation using decay criteria and persistence in weighted spaces. Int. Math. Res. Not. **22**, 5161–5181 (2012)
3. Camassa, R., Holm, D.: An integrable shallow water equation with peaked solitons. Phys. Rev. Lett. **71**, 1661–1664 (1993)
4. Constantin, A., Lannes, D.: The hydro-dynamical relevance of the Camassa–Holm and Degasperis–Procesi equations. Arch. Ration. Mech. Anal. **193**, 165–186 (2009)
5. Degasperis, A., Procesi, M.: In: Degasperis, A., Gaeta, G. (eds.) Symmetry and Perturbation Theory, SPT 98, Rome, 16–22, December 1998, p. 23. World Scientific, River Edge (1999)
6. Grayshan, K.: Peakon solutions of the Novikov equation and the properties of the date-to-solution map. J. Math. Anal. Appl. **397**, 515–521 (2013)
7. Gröchenig, K.: Weight functions in time-frequency analysis. In: Pseudo-differential Operators: Partial Differential Equations and Time–Frequency Analysis. Fields Institute Communications, vol. 52, pp. 343–366. American Mathematical Society, Providence, RI (2007)
8. Guo, Z.: On an integrable Camassa–Holm type equation with cubic nonlinearity. Nonlinear Anal. Real World Appl. **34**, 225–232 (2017)
9. Guo, Z., Zhou, Y.: On solutions to a two-component generalized Camassa–Holm system. Stud. Appl. Math. **124**, 307–322 (2010)
10. Guo, Z.: Blow-up and global solutions to a new integrable model with two components. J. Math. Anal. Appl. **372**, 316–327 (2010)
11. Guo, Z., Zhu, M.: Wave breaking for a modified two-component Camassa–Holm system. J. Differ Equ. **252**, 2759–2770 (2012)
12. Guo, Z., Zhu, M.: Wave breaking and measure of momentum support for an integrable Camassa–Holm system with two components. Stud. Appl. Math. **130**, 417–430 (2013)
13. Himonas, A., Holliman, C.: The Cauchy problem for the Novikov equation. Nonlinearity **25**, 449–479 (2012)
14. Himonas, A., Misiolek, G., Ponce, G., Zhou, Y.: Persistence properties and unique continuation of solutions of Camassa–Holm equation. Commun. Math. Phys. **271**, 511–522 (2007)
15. Hone, A.N.W., Lundmark, H., Szmigielski, J.: Explicit multipeakon solutions of Novikov's cubically nonlinear integrable Camassa–Holm equation. Dyn. Partial Diff. Equ. **6**, 253–289 (2009)
16. Hone, A.N.W., Wang, J.P.: Integrable peakon equations with cubic nonlinearity. J. Phys. A **41**, 372002 (2008)
17. Jiang, Z., Ni, L.: Blow-up phenomena for the integrable Novikov equation. J. Math. Anal. Appl. **385**, 551–558 (2012)
18. Jiang, Z., Ni, L., Zhou, Y.: Wave breaking for the Camassa–Holm equation. J. Nonlinear Sci. **22**, 235–245 (2012)
19. Kato, T.: Quasi-linear equations of evolution with applications to partial differential equations. In: Spectral Theory and Differential Equations. Lecture Notes in Mathematics, vol. 448, pp. 25–70. Springer, Berlin (1975)
20. Lai, S.: Global weak solutions to the Novikov equation. J. Funct. Anal. **265**, 520–544 (2013)
21. Lai, S., Wu, M.: The local strong and weak solutions to a generalized Novikov equation. J. Math. Anal. Appl. **385**, 551–558 (2012)
22. Lai, S., Li, N., Wu, Y.: The existence of global strong and weak solutions for the Novikov equation. J. Math. Anal. Appl. **399**, 682–691 (2013)
23. Mi, Y., Mu, C.: On the Cauchy problem for the modified Novikov equation with peakon solutions. J. Differ. Equ. **254**, 961–982 (2013)
24. Novikov, V.S.: Generalizations of the Camassa–Holm equation. J. Phys. A **42**, 342002 (2009)
25. Ni, L., Zhou, Y.: Well-posedeness and persistence properties for the Novikov equation. J. Differ. Equ. **250**, 3002–3021 (2011)
26. Ni, L., Zhou, Y.: A new asymptotic behavior for solutions of the Camassa–Holm equation. Proc. Am. Math. Soc. **140**, 607–614 (2012)
27. Tiglay, F.: The periodic Cauchy problem for Novikov's equation. Int. Math. Res. Not. **20**, 4633–4648 (2011)
28. Wu, X., Yin, Z.: Global weak solutions for the Novikov equation. J. Phys. A **44**, 055202 (2011)

29. Yan, W., Li, Y., Zhang, Y.: The Cauchy problem for the integrable Novikov equation. J. Differ. Equ. **253**, 298–318 (2012)
30. Yan, W., Li, Y., Zhang, Y.: The Cauchy problem for the Novikov equation. Nonlinear Differ. Equ. Appl. **20**, 1157–1169 (2013)
31. Zhou, Y.: On solutions to the Holm-Staley b-family of equations. Nonlinearity **23**(2), 369–381 (2010)
32. Zhou, S., Chen, R.: A few remarks on the generalized Novikov equation. J. Inequalities Appl. **2013**, 560 (2013)