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Strong Commutativity Preserving Skew Derivations in Semiprime Rings

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Abstract Let R be a semiprime ring of characteristic different from 2, C its extended centroid, $Z(R)$ its center, F and G non-zero skew derivations of R with associated automorphism α and m, n positive integers such that

$$[F(x), G(y)]_m = [x, y]^n \text{ for all } x, y \in R.$$

Then R is commutative.

Keywords Skew derivations · Automorphism · Generalized polynomial identity (GPI) · Prime and semiprime ring · Strong commutativity preserving

Mathematics Subject Classification 16N60 · 16U80 · 16W25

1 Introduction

Let R be a prime ring of characteristic different from 2 with center $Z(R)$, extended centroid C, right Martindale quotient ring Q_r, and symmetric Martindale quotient ring Q.

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An additive map $d : R \to R$ is a *derivation* on R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Let $a \in R$ be a fixed element. A map $d : R \to R$ defined by $d(x) = [a, x] = ax - xa$, $x \in R$, is a derivation on R, which is called *the inner derivation* defined by a. Many results in the literature indicate how the global structure of a ring R is often tightly connected to the behavior of additive maps defined on R. A well-known result of Posner [18], states that if d is a derivation of R such that $[d(x), x] \in Z(R)$ for any $x \in R$, then either $d = 0$ or R is commutative.

In this paper, we study the structure of prime and semiprime rings having skew derivations satisfying strong commutativity preserving conditions. Specifically, let α be an automorphism of a ring R. An additive map $D : R \to R$ is called an α-*derivation* (or a *skew derivation*) on R if $D(xy) = D(x)y + \alpha(x)D(y)$ for all $x, y \in R$. In this case, α is called an *associated automorphism* of D. Basic examples of α-derivations are the usual derivations and the map $\alpha - id$, where id denotes the identity map. Let $b \in Q$ be a fixed element. Then a map $D : R \to R$ defined by $D(x) = bx - \alpha(x)b$, $x \in R$, is an α-derivation on R and it is called an *inner α-derivation* (an *inner skew derivation*) defined by b. If a skew derivation D is not inner, then it is called *outer*.

If $\mathcal{S} \subseteq \mathcal{R}$, the map $F : \mathcal{R} \to \mathcal{R}$ is called commutativity preserving on $\mathcal{S}$ if $[x, y] = 0$ implies $[F(x), F(y)] = 0$; it is called strong commutativity preserving (for brevity we will always say SCP) on $\mathcal{S}$ if $[F(x), F(y)] = [x, y]$, for all $x, y \in \mathcal{S}$.

In [1], Bell and Daif proved that if R is a semiprime ring admitting a derivation d which is SCP on the right ideal I of R, then $I \subseteq Z$. The natural possibility when an additive map preserves commutativity appears in a paper by Bresar and Miers [2]. They showed that any additive map F which is SCP on a semiprime ring R is of the form $F(x) = \lambda x + \mu(x)$, where $\lambda \in C$, $\lambda^2 = 1$, and $\mu : R \to C$ is an additive map of R into C.

Later in [15], Lin and Liu extended this result to Lie ideals, in case the ring R is prime. More precisely they proved that if L is a non-central Lie ideal of R and F is an additive map satisfying $[F(x), F(y)] - [x, y] \in C$ for all $x, y \in L$, then $F(x) = \lambda x + \mu(x)$, where $\lambda \in C$, $\lambda^2 = 1$, and $\mu : R \to C$, unless when $char(R) = 2$ and R satisfies the standard identity s_4 of degree 4.

More recently, in [16] Liu showed that if R is a semiprime ring, I a non-zero right ideal of R, F, and G non-zero skew derivations of R, with associated automorphism α such that $[F(x), G(y)] = [x, y]$ for all $x, y \in I$, then $[\alpha(x), x] = 0$ for any $x \in I$. Moreover, if $\alpha(I) \subseteq I$ then $\sigma(I) \subseteq Z(R)$. Finally, it is proved that if R is prime then R is commutative.

Here we continue this line of investigation and we examine what happens in case F and G are skew derivations of R such that $[F(x), G(y)]_m = [x, y]^n$ for all $x, y \in I$, where I is a non-zero ideal of R and $m, n \geq 1$ are positive integers.

The results we obtained are the following:

Theorem 1 *Let R be a prime ring of characteristic different from 2, C its extended centroid, $Z(R)$ its center, I a non-zero ideal of R, F, and G non-zero skew derivations of R with associated automorphism α and m, n positive integers such that*

$$[F(x), G(y)]_m = [x, y]^n \text{ for all } x, y \in I.$$

Then R is commutative.

Theorem 2 *Let R be a semiprime ring of characteristic different from 2, C its extended centroid, $Z(R)$ its center, F and G non-zero skew derivations of R with associated automorphism α, m, n positive integers such that*

$$[F(x), G(y)]_m = [x, y]^n \text{ for all } x, y \in R.$$

Then R is commutative.

2 Preliminaries

We denote the set of all skew derivations on Q by $\mathrm{SDer}(Q)$. By a skew-derivation word, we mean an additive map Δ of the form $\Delta = d_1, d_2, \ldots d_m$, with each $d_i \in \mathrm{SDer}(Q)$. A skew-differential polynomial is a generalized polynomial with coefficients in Q, of the form $\Phi(\Delta_j(x_i))$ involving non-commutative indeterminates x_i on which the derivations words Δ_j act as unary operations. The skew-differential polynomial $\Phi(\Delta_j(x_i))$ is called a skew-differential identity on a subset T of Q if it vanishes for any assignment of values from T to its indeterminates x_i.

In order to prove our result, we need to recall the following known facts:

Fact 1 *In [9], Chuang and Lee investigate polynomial identities with skew derivations. More precisely as a consequence of in* [9, Theorem 1]*, we have that if D is an outer skew derivation of R which satisfies the generalized polynomial identity $\Phi(x_i, D(x_j))$, then $\Phi(x_i, y_j)$ is also a generalized polynomial identity for R, where x_i and y_j are distinct indeterminates.*

Fact 2 *Let R be a prime ring and I a two-sided ideal of R. Then I, R, and Q satisfy the same generalized polynomial identities with coefficients in Q (see* [5]*). Furthermore, I, R, and Q satisfy the same generalized polynomial identities with automorphisms* [7, *Theorem* 1].

Fact 3 *Recall that, in case $char(R) = 0$, an automorphism α of Q is called* Frobenius *if $\alpha(x) = x$ for all $x \in C$. Moreover, in case $char(R) = p \geq 2$, an automorphism α is* Frobenius *if there exists a fixed integer t such that $\alpha(x) = x^{p^t}$ for all $x \in C$. In* [7, Theorem 2]*, Chuang proves that if $\Phi(x_i, \alpha(x_i))$ is a generalized polynomial identity for R, where R is a prime ring and $\alpha \in Aut(R)$ an automorphism of R which is not Frobenius, then R also satisfies the non-trivial generalized polynomial identity $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates.*

Fact 4 *Let R be a domain and $\alpha \in Aut(R)$ an automorphism of R which is outer. In* [13]*, Kharchenko proves that if $\Phi(x_i, \alpha(x_i))$ is a generalized polynomial identity for R, then R also satisfies the non-trivial generalized polynomial identity $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates.*

Finally, let us mention that if R is a prime ring satisfying a non-trivial generalized polynomial identity and α an automorphism of R such that $\alpha(x) = x$ for all $x \in C$, then α is an inner automorphism of R [3, Theorem 4.7.4].

Lemma 1 *Let R be a semiprime ring of characteristic different from 2, Q its symmetric Martindale quotient ring, F and G non-zero skew derivations of R, m, n positive integers such that $n \geq 2$ and*

$$[F(x), G(y)]_m = [x, y]^n \text{ for all } x, y \in R. \tag{2.1}$$

Then R is commutative.

Proof Assume first that R is prime. From the relation (2.1), we have both

$$[F(x+z), G(y)]_m = [x+z, y]^n \tag{2.2}$$

and

$$[F(x+z), G(y)]_m = [F(x), G(y)]_m + [F(z), G(y)]_m = [x, y]^n + [z, y]^n. \tag{2.3}$$

From (2.2) and (2.3), it follows that R satisfies the polynomial identity $[x+z, y]^n - [x, y]^n - [z, y]^n$. By Posner's theorem [11, Theorem 2 p. 57, Lemma 1 p. 89], $Q \subseteq M_k(E)$, the ring of $k \times k$ matrices over a field E. Moreover, Q and $M_k(E)$ satisfy the same polynomial identities. If $k \geq 2$ and for $x = e_{21}$, $y = e_{11}$, $z = e_{11}$ we have the contradiction

$$0 = [e_{21} + e_{12}, e_{11}]^n - [e_{21}, e_{11}]^n - [e_{12}, e_{11}]^n = (e_{21} - e_{12})^n \neq 0.$$

Thus, $k = 1$ and we have that Q is commutative, as well as R.

Let now R be semiprime. Since R is a semiprime ring for which $[x+z, y]^n - [x, y]^n - [z, y]^n$ is a polynomial identity, R is a subdirect product of prime rings R_α, each of which still satisfies the identity $[x+z, y]^n - [x, y]^n - [z, y]^n$. In this case, by the above argument, any R_α is commutative. Thus, we conclude that R must be commutative. □

Lemma 2 *Let R be a non-commutative prime ring of characteristic different from 2, F and G non-zero skew derivations of R, m a positive integer such that*

$$[F(x), G(y)]_m = [x, y] \text{ for all } x, y \in R.$$

Then $char(R) = p > 0$ and m is odd.

Proof For any $x, y \in R$ we have

$$[F(x), G(y+y)]_m = [x, y+y] = 2[x, y] \tag{2.4}$$

and also, by computing the m-th commutator

$$[F(x), G(2y)]_m = [F(x), 2G(y)]_m = 2^m [F(x), G(y)]_m = 2^m [x, y]. \tag{2.5}$$

Comparing (2.4) with (2.5) it follows $(2^m - 2)[x, y] = 0$ for all $x, y \in Q$. Hence, since R is not commutative and $char(R) \neq 2$, $char(R) = p \neq 0$ (since $2^m - 2 \equiv 0$, modulo p).
Moreover,

$$-[x, y] = [x, -y] = [F(x), G(-y)]_m = (-1)^m [F(x), G(y)]_m = (-1)^m [x, y]$$

which implies that m must be an odd integer. □

3 Results

Lemma 3 *Let R be a non-commutative prime ring of characteristic different from* 2*, which is isomorphic to a dense subring of the ring of linear transformations of a vector space V over a division ring D, $\alpha : R \to R$ an automorphism of R and $q, u \in R$ such that*

$$[qx, \alpha(y)u - uy]_m = [x, y]$$

for all $x, y \in R$. Then $dim_D V = 1$.

Proof By Theorem 1 in [7], R and Q satisfy the same generalized polynomial identities with automorphisms and hence $[qx, \alpha(y)u - uy]_m - [x, y]$ is a generalized polynomial identity for Q. We assume that $dim_D V \geq 2$ and prove that a contradiction follows. By [12, p. 79], there exists a semilinear automorphism $T \in End(V)$ such that $\alpha(x) = TxT^{-1}$ for all $x \in Q$. Hence, Q satisfies

$$\left[qx, TyT^{-1}u - uy\right]_m = [x, y].$$

We notice that, if for any $v \in V$ there exists $\lambda_v \in D$ such that $T^{-1}uv = v\lambda_v$, then, by a standard argument, it follows that there exists a unique $\lambda \in D$ such that $T^{-1}uv = v\lambda$, for all $v \in V$ (see for example [8, Lemma 1]). In this case,

$$\alpha(x)uv = \left(TxT^{-1}\right)uv = Txv\lambda$$

and

$$\left(\alpha(x)u - ux\right)v = T(xv\lambda) - uxv = T\left(T^{-1}uxv\right) - uxv = 0$$

which implies that $\left(\alpha(x)u - ux\right)V = (0)$. Thus $\alpha(x)u - ux = 0$ for any $x \in R$, since V is faithful, and by our assumption $0 = [qx, \alpha(y)u - uy]_m = [x, y]$ for any $x, y \in R$, which is a contradiction, since R is not commutative.

Therefore, there exists $v \in V$ such that $\{v, T^{-1}uv\}$ are linearly D-independent. By the density of Q, there exist $r, s \in Q$ such that

$$rv = 0, \quad rT^{-1}uv = v, \quad sv = T^{-1}uv, \quad sT^{-1}uv = \left(T^{-1}u\right)^2 v.$$

Hence,

$$qrv = 0, \quad \left(TsT^{-1}u - us\right) v = 0, \quad [r, s]v = v$$

and, by the main assumption, we get the contradiction

$$0 = \left(\left[qr, TsT^{-1}u - us\right]_m - [r, s]\right)v = -v \neq 0.$$

□

Lemma 4 *Let R be a non-commutative prime ring of characteristic different from* 2*, C its extended centroid, $Z(R)$ its center, I a non-zero ideal of R, F, and G non-zero skew derivations of R with associated automorphism α, m a positive integer such that*

$$[F(x), G(y)]_m = [x, y] \quad \text{for all} \quad x, y \in I.$$

Then there exist $p, q, u \in Q$, with p invertible such that $F(x) = pxp^{-1}q - qx$ and $G(x) = pxp^{-1}u - ux$, for all $x \in R$.

Proof It is known that I, R, and Q satisfy the same generalized polynomial identities with skew derivations and automorphisms, so that $[F(x), G(y)]_m = [x, y]$, for all $x, y \in Q$. Notice that if $m = 1$ then the result follows by [15]. Thus we consider $m \geq 2$.

Fix $y_0 \in R$ and denote $z_0 = G(y_0)$. Then for any $x \in Q$ we get $[F(x), z_0]_m = [x, y_0]$. In case F is an outer skew derivation of R, it is known that Q satisfies $[t, z_0]_m - [x, y_0]$, and in particular for $t = 0$, we have $[x, y_0] = 0$, for all $x \in Q$. This implies $y_0 \in C$. We may repeat this argument for any $y_0 \in Q$ and conclude that Q is commutative, a contradiction. Therefore, in the sequel, we always assume that F is an inner skew derivation of R. Thus, there exists $0 \neq q \in Q$ such that $F(x) = \alpha(x)q - qx$, for all $x \in R$.

Fix now $x_0 \in R$ and denote $z_0 = F(x_0)$. Then for any $y \in Q$ we get $[z_0, G(y)]_m = [x_0, y]$. In case G is an outer skew derivation of R, it is known that Q satisfies $[z_0, t]_m - [x_0, y]$, and in particular for $t = 0$, we have $[x_0, y] = 0$ for all $y \in Q$. This implies $x_0 \in C$. We may repeat this argument for any $x_0 \in Q$ and conclude that Q is commutative, a contradiction. Therefore, in the sequel, we always assume that G is an inner skew derivation of R. Thus, there exists $0 \neq u \in Q$ such that $G(x) = \alpha(x)u - ux$, for all $x \in R$. Hence we have that

$$[\alpha(x)q - qx, \alpha(y)u - uy]_m = [x, y] \text{ for all } x, y \in Q. \tag{3.1}$$

We assume that α is not inner. In this case

$$[tq - qx, \alpha(y)u - uy]_m - [x, y] \tag{3.2}$$

is satisfied by Q.

If α is not Frobenius, then by (3.2) it follows that Q satisfies the generalized identity

$$[tq - qx, zu - uy]_m - [x, y] \tag{3.3}$$

and in particular for $x = y = 0$ in (3.3), $[tq, zu]_m = 0$, for all $t, z \in Q$. By applying the result in [14], and since $q \neq 0$ and $u \neq 0$, it follows that either Qq is a non-zero central left ideal of Q or Qu is a non-zero central left ideal of Q. In any case, Q is commutative, a contradiction.

Thus, we consider the case when α is Frobenius. Again by (3.2), Q satisfies

$$[-qx, \alpha(y)u - uy]_m - [x, y]. \tag{3.4}$$

In case both $q \in C$ and $u \in C$, by (3.4) we have that

$$-qu^m[x, \alpha(y) - y]_m - [x, y] \tag{3.5}$$

is satisfied by Q. Then by the main Theorem in [6] and since $qu^m \neq 0$, Q satisfies a non-trivial generalized polynomial identity. On the other hand, if either $q \notin C$ or $u \notin C$, then by (3.4) and again by the main Theorem in [6], Q satisfies a non-trivial generalized polynomial identity (hence Q is a GPI-ring in any case). Therefore, by [17, Theorem 3], Q is a primitive ring and it is a dense subring of the ring of linear transformations of a vector space V over a division ring D. Moreover, Q contains non-zero linear transformations of finite rank. By Lemma 3, it follows $dim_D V = 1$, that is Q is a division algebra which is finite dimensional over C. If C is finite, then Q is finite, so that Q is a commutative field, which is a contradiction. So, we assume in all that follows that C is infinite and $char(Q) = p > 0$.

By using (3.4) in (3.2) it follows that

$$[tq, \alpha(y)u - uy]_m \tag{3.6}$$

is satisfied by Q. Let $s \geq 1$ be such that $p^s \geq m$, and $k = p^s$, then by (3.6) we have that Q satisfies

$$\Big[tq, \alpha(y)u - uy\Big]_k$$

that is

$$\left[tq, \left(\alpha(x)u - ux\right)^k\right] = 0 \text{ for all } t, x, \in Q. \tag{3.7}$$

Since α is Frobenius, $\alpha(\gamma) = \gamma^{p^h}$, for all $\gamma \in C$ and some non-zero fixed integer h. Moreover, there exists $\lambda \in C$ such that $\lambda^{p^h} \neq \lambda$, that is $\lambda^{p^h - 1} \neq 1$.

In particular, we choose $\gamma \in C$ such that $\gamma = \lambda^{p^h - 1} \neq 0$. In the relation (3.7), we replace x by λx and obtain that Q satisfies

$$\left[tq, \left(\lambda^{p^h}\alpha(x)u - \lambda ux\right)^k\right]$$

that is

$$\left[tq, \left(\gamma\alpha(x)u - ux\right)^k\right].$$

Let Φ and Ω be maps on Q, such that $\Phi(x) = -ux$ and $\Omega(x) = \alpha(x)u$, for any element x of Q. Thus it follows that $[tq, (\Phi(x) + \gamma\Omega(x))^k] = 0$ for all $x \in Q$. Expanding $(\Phi(x) + \gamma\Omega(x))^k$, we get

$$\sum_{i=0}^{n} \gamma^i \left(\sum_{(i,k-i)} \varphi_1 \cdot \varphi_2 \cdots \varphi_k \right) = 0$$

where the inside summations are taken over all permutations of $k - i$ terms of the form $\Phi(x)$ and i terms of the form $\Omega(x)$. This means that each summation inside has exactly $k - i$ terms of the form $\Phi(x)$ and i terms of the form $\Omega(x)$ but in some different order. For any $j = 0, \ldots, k$, denote $y_j = \sum_{(j,k-j)} \varphi_1 \cdot \varphi_2 \cdots \varphi_k$, then we can write

$$(\Phi(x) + \gamma\Omega(x))^k = y_0 + \gamma y_1 + \gamma^2 y_2 + \ldots + \gamma^k y_k$$

so that

$$\left[tq, y_0 + \gamma y_1 + \gamma^2 y_2 + \ldots + \gamma^k y_k\right] = 0.$$

that is

$$[tq, y_0] + \gamma[tq, y_1] + \gamma^2[tq, y_2] + \ldots + \gamma^k[tq, y_k] = 0.$$

Here, we denote by $z_i = [tq, y_i]$, for any $i = 1, \ldots, k$, then

$$z_0 + \gamma z_1 + \gamma^2 z_2 + \ldots + \gamma^k z_k = 0. \tag{3.8}$$

Replacing in the previous argument λ successively by $1, \lambda, \lambda^2, \ldots, \lambda^k$, the equation (3.8) gives the system of equations

$$\begin{aligned}
&z_0 + z_1 + z_2 + \cdots + z_k = 0\\
&z_0 + \gamma z_1 + \gamma^2 z_2 + \cdots + \gamma^k z_k = 0\\
&z_0 + \gamma^2 z_1 + \gamma^4 z_2 + \cdots + \gamma^{2k} z_k = 0\\
&z_0 + \gamma^3 z_1 + \gamma^6 z_2 + \cdots + \gamma^{3k} z_k = 0\\
&\cdots\cdots\cdots\cdots\\
&z_0 + \gamma^k z_1 + \gamma^{2k} z_2 + \cdots + \gamma^{k^2} z_k = 0.
\end{aligned} \tag{3.9}$$

Moreover, since C is infinite, there exist infinitely many $\lambda \in C$ such that $\lambda^{i(p^h-1)} \neq 1$ for $i = 1, \ldots, k$, that is there exist infinitely many $\gamma = \lambda^{p^h-1} \in C$ such that $\gamma^i \neq 1$ for $i = 1, \ldots, k$. Hence, the Vandermonde determinant (associated with the system (3.9))

$$\begin{vmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & \gamma & \gamma^2 & \cdots & \gamma^k \\ 1 & \gamma^2 & \gamma^4 & \cdots & \gamma^{2k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \gamma^k & \gamma^{2k} & \cdots & \gamma^{k^2} \end{vmatrix} = \prod_{0 \le i < j \le k} (\gamma^i - \gamma^j)$$

is not zero. Thus, we can solve the above system (3.9) and obtain $z_i = 0\,(i = 0, \ldots, k)$. In particular $z_0 = 0$, that is Q satisfies

$$\left[tq, (-ux)^k\right]. \tag{3.10}$$

Since $u \neq 0$ and $q \neq 0$, then, by (3.10) and [14], either uQ is a non-zero central right ideal of Q or Qq is a non-zero central left ideal of Q. In any case, we get the contradiction that Q is commutative.

The previous argument shows that the automorphism α must be inner, that is there exists an invertible element $p \in Q$, such that $\alpha(x) = pxp^{-1}$, for all $x \in R$, as required. □

The following result is an easy consequence of [4, Theorem 1]. It will be useful in the proof of our result:

Lemma 5 *Let R be a prime ring of characteristic different from 2, $Z(R)$ its center, C its extended centroid. Let p be an invertible element of R, d the inner derivation of R, which is induced by p, that is, $d(x) = [p, x]$ for any $x \in R$, and $\beta(x) = pxp^{-1}$ for any $x \in R$, the inner automorphism of R induced by p. Assume that F is a non-zero skew derivation of R with associated automorphism β and $0 \neq a \in R$ such that*

$$aF(x) - F(x)a = d(x) \text{ for all } x \in R.$$

Then $d = 0$, β is the identity map on R, F is an ordinary derivation of R and $a \in Z(R)$.

Proof Firstly we notice that, since F is a skew derivation of R, $F(xy) = F(x)y + \beta(x)F(y)$. Thus, F is both a right $(1, \beta)$-generalized skew derivation and a left $(1, \beta)$-generalized skew derivation of R, in the sense of [4]. Therefore, we may apply Theorem 1 in [4], and one of the following holds:

(1) (case (i) of Theorem 1 in [4]) $d(x) = apx - pxa$ for any $x \in R$. Hence $d(xy) = d(x)y + xd(y) = (apx - pxa)y + x(apy - pya)$, on the other hand $d(xy) = apxy - pxya$. Comparing the previous identities, we get

$$-px[y, a] - xapy + xpya = 0. \tag{3.11}$$

In particular, for $x = y = p^{-1}$, it follows $ap^{-1} - p^{-1}a = 0$ which implies $ap - pa = 0$. Therefore $d(x) = p[a, x]$, moreover by (3.11) we get $-px[y, a] - xpay + xpya = 0$, that is $[x, p][y, a] = 0$. As an application of [19, Theorem 3], it follows that either $p \in C$ or $a \in C$, in any case $d = 0$.

(2) (cases (ii) and (iii) of [4, Theorem 1]) There exists $q \in Q$ such that $d(x) = p[q, x]$, for all $x \in R$. Hence $d(xy) = d(x)y + xd(y) = p[q, x]y + xp[q, y]$, on the other hand $d(xy) = p[q, xy] = p[q, x]y + px[q, y]$. Comparing the previous identities, we get $[p, x][q, y] = 0$, for any $x, y \in R$, that is either $p \in C$ or $q \in C$. In any case $d = 0$.
(3) (case (iv) of [4, Theorem 1]) There exists $q \in Q$ such that $d(x) = px + p[q, x]$ for all $x \in R$. Hence $d(xy) = d(x)y + xd(y) = (px + p[q, x])y + x(py + p[q, y])$, on the other hand $d(xy) = pxy + p[q, xy] = pxy + p[q, x]y + px[q, y]$. Comparing the previous identities, we get $xpy + [p, x][q, y] = 0$ for any $x, y \in R$. In particular, for $x = p^{-1}$, it follows the contradiction $y = 0$, for all $y \in R$.

Since in any case $p \in C$, β is the identity map on R and F is an ordinary derivation of R such that $[a, F(x)] = 0$ for any $x \in R$. Hence, by the first Posner's theorem in [18], it follows $a \in Z(R)$ or $F(R) \subseteq Z(R)$. In this last case, by second Posner's theorem in [18], R is commutative. In any case we obtain that $a \in Z(R)$. □

Lemma 6 *Let $R = M_t(C)$ be the ring of $t \times t$ matrices over C, with $char(R) = l \neq 0, 2$, $m \geq 1$ be an odd integer, $p, q, u \in R$ with p invertible such that $F(x) = pxp^{-1}q - qx$, $G(x) = pxp^{-1}u - ux$ and*

$$\left[pxp^{-1}q - qx, pyp^{-1}u - uy\right]_m - [x, y] \tag{3.12}$$

for any $x, y \in R$. Then t=1.

Proof In (3.12) replace x, y by $p^{-1}x$, $p^{-1}y$, respectively, and denote $a = p^{-1}q$, $b = qp^{-1}$, $c = p^{-1}u$, $w = up^{-1}$. Therefore Q satisfies

$$\Phi(x, y) = [xa - bx, yc - wy]_m - \left[p^{-1}x, p^{-1}y\right]. \tag{3.13}$$

In case $a \in C$, then $F = 0$ and (3.12) implies $[x, y] = 0$ for any $x, y \in R$, that is R is commutative. On the other hand, if $c \in C$, then $G = 0$ and again it follows that R is commutative. Therefore, here we may assume that $t \geq 2$ and both $a \notin C$ and $c \notin C$. We prove that a number of contradiction follows.
Firstly, we notice that, for any inner automorphism φ of $M_t(C)$, we have that

$$[x\varphi(a) - \varphi(b)x, y\varphi(c) - \varphi(w)y]_m - \left[\varphi(p^{-1})x, \varphi(p^{-1})y\right] \tag{3.14}$$

is a generalized identity for R. We will make a frequent use of this fact.

As above, we denote by e_{ij} the usual matrix unit, with 1 in the (i, j) entry and zero elsewhere, and say $a = \sum_{kl} a_{kl}e_{kl}$, $b = \sum_{kl} b_{kl}e_{kl}$, $c = \sum_{kl} c_{kl}e_{kl}$ and $w = \sum_{kl} w_{kl}e_{kl}$, for $a_{kl}, b_{kl}, c_{kl}, w_{kl} \in C$.
Suppose $t \geq 3$. In (3.13), we make the following choices: $x = e_{ji}$, $y = e_{kk}$, for any i, j, k different indices; moreover, we right multiply (3.13) by e_{jj} and left multiply by e_{kk}. As a consequence we get $a_{kj}c_{ij}^m = 0$, that is $a_{kj}c_{ij} = 0$. Now, let $\varphi(x) = (1 + e_{ki})x(1 - e_{ki})$ and denote $(a'_{ij})_{t\times t}$ the entries of the matrix $\varphi(a)$, and $(c'_{ij})_{t\times t}$

the entries of the matrix $\varphi(c)$. By the above computations, we get $a'_{kj}c'_{ij} = 0$, that is $(a_{kj} + a_{ij})c_{ij} = 0$, which means $a_{ij}c_{ij} = 0$. Thus, by Proposition 1 in [10], it follows that either $a \in C$ or $c \in C$, in any case a contradiction.

Therefore, we finally consider the case $t = 2$ and $R = M_2(C)$. For $x = e_{ii}$ and $y = e_{ji}$ in (3.13), with $i \neq j$, right multiply (3.13) by e_{jj} and left multiply by e_{ii}, it follows $a_{ij}(c_{ij} + w_{ij})^m = 0$, that is

$$a_{ij}(c_{ij} + w_{ij}). \tag{3.15}$$

Let now $x = y = e_{ij}$ in (3.13), with $i \neq j$, and right multiply (3.13) by e_{ii} one has $(a_{ji} + c_{ji})^m = 0$, that is

$$a_{ji} + c_{ji} \text{ for all } i \neq j \tag{3.16}$$

which means that $a+c$ is a diagonal matrix. In this case, a standard argument shows that $a + c$ is a central matrix, say $a = \lambda - c$, for $\lambda \in Z(R)$. Analogously, for $x = y = e_{ij}$ in (3.13), and left multiply (3.13) by e_{jj} one has $(w_{ji} + b_{ji})^m = 0$, that is

$$w_{ji} + b_{ji} \text{ for all } i \neq j$$

which means that $w + b$ is a diagonal matrix and, as above, $w + b$ is a central matrix, say $b = \mu - w$, for $\mu \in Z(R)$.

In other words, $p^{-1}q = \lambda - p^{-1}u$ and $qp^{-1} = \mu - up^{-1}$. Therefore, if either $\lambda = 0$ or $\mu = 0$ then $q = -u$, $\lambda = \mu = 0$, and $F = -G$. On the other hand, if both $\lambda \neq 0$ and $\mu \neq 0$, it follows both $q = p\lambda - u$ and $q = p\mu - u$, that is $\lambda = \mu$ and easy computations show that $F = -G$ in any case.

Now, we write $v = c + w$ and let $v = \sum_{kl} v_{kl}e_{kl}$, for $v_{kl} \in C$. The next step is to prove that either v is diagonal or both a and c are diagonal matrices of R. To do this, we assume by contradiction that v is not diagonal, for example, let $v_{12} \neq 0$, and prove that a contradiction follows. In this case, by (3.15) and (3.16) we get $a_{12} = 0$ and $c_{12} = 0$. Moreover, if $v_{21} \neq 0$, then a and c are diagonal matrices and we are done. Thus we assume that $v_{21} = 0$.

Let $\varphi(x) = (1 + e_{12})x(1 - e_{12})$ and $\chi(x) = (1 - e_{12})x(1 + e_{12})$ and denote $\varphi(a) = \sum_{kl} a'_{kl}e_{kl}$, $\varphi(c) = \sum_{kl} c'_{kl}e_{kl}$, $\varphi(v) = \sum_{kl} v'_{kl}e_{kl}$, $\chi(a) = \sum_{kl} a''_{kl}e_{kl}$, $\chi(c) = \sum_{kl} c''_{kl}e_{kl}$, $\chi(v) = \sum_{kl} v''_{kl}e_{kl}$.

We notice that, if both $v'_{12} \neq 0$ and $v''_{12} \neq 0$, then $a'_{12} = 0$, $a''_{12} = 0$ and also $c'_{12} = 0$, $c''_{12} = 0$ that is $a_{22} - a_{11} - a_{21} = 0$, $-a_{22} + a_{11} - a_{21} = 0$, $c_{22} - c_{11} - c_{21} = 0$, $-c_{22} + c_{11} - c_{21} = 0$, implying $a_{21} = 0$ and $c_{21} = 0$, so that a and c are diagonal and we are done.

Thus we may assume (without loss of generality) $v''_{12} = 0$. We have proved that if $v_{12} \neq 0$ then $v_{12} = v_{22} - v_{11}$. Let $\theta(x) = (1 + e_{21})x(1 - e_{21})$ and $\theta(v) = \sum_{kl} v'''_{kl}e_{kl}$. Since $v'''_{12} = v_{12} \neq 0$, then $v'''_{12} = v'''_{22} - v'''_{11}$, that is $v_{12} = -v_{22} + v_{11}$, implying again the contradiction $v_{12} = 0$.

The previous argument says that either v is diagonal or both a and c are diagonal matrices, and as above, we may conclude that either v is central or both a and c are central matrices of R.

Since if $a \in Z(R)$ and $c \in Z(R)$, then $F = G = 0$, which is a contradiction, then we assume in what follows that $c + w = v = \nu \in Z(R)$, that is $p^{-1}q + qp^{-1} = \nu$. For $y = p$ in (3.12) it follows that

$$\left[pxp^{-1}q - qy, [p, u]\right]_m = [x, p]. \tag{3.17}$$

Assume that $[p, u]$ is an invertible matrix in $M_2(C)$, thus $0 \neq [p, u]^2 \in Z(R)$ and by computations one has

$$2^{m-1}\left(\left(pxp^{-1}q - qx\right)[p, u]^m - [p, u]\left(pxp^{-1}q - qx\right)[p, u]^{m-1}\right) = [x, p].$$

Since $[p, u]$ is an invertible matrix and $m - 1$ is even, then $[p, u]^{m-1} \in Z(R)$ and

$$2^{m-1}\left(\left(pxp^{-1}q - qx\right)[p, u]^m - [p, u]^m\left(pxp^{-1}q - qx\right)\right) = [x, p].$$

Since $F \neq 0$ and $2^{m-1}[p, u]^m \neq 0$, we may apply Lemma 5 and obtain the contradiction $p \in Z(R)$.

Therefore $[p, u]$ is not an invertible matrix in $M_2(C)$, i.e., $[p, u]^2 = 0$. Once again, for $y = p$ in (3.12) and since $m \geq 2$, we get

$$0 = \left[pxp^{-1}q - qx, [p, u]\right]_m = [x, p] \tag{3.18}$$

and as above we conclude with the contradiction $p \in Z(R)$. □

4 The proof of Theorem 1

In light of Lemma 1, we may assume $n = 1$. Since I, R, and Q satisfy the same generalized identities with skew derivations and automorphisms, we may assume

$$[F(x), G(y)]_m - [x, y] = 0 \text{ for all } x, y \in Q. \tag{4.1}$$

Moreover, by Lemma 4, there exists $p, q, u \in Q$ with p invertible such that $F(x) = pxp^{-1}q - qx$ and $G(x) = pxp^{-1}u - ux$ for all $x \in R$. Notice that, if either $F = 0$ or $G = 0$, then $[x, y] = 0$ for all $x, y \in Q$, which means that Q is commutative, as well as R. Thus we also assume both $F \neq 0$ and $G \neq 0$. Hence Q satisfies the generalized polynomial identity

$$\left[pxp^{-1}q - qx, pyp^{-1}u - uy\right]_m - [x, y]. \tag{4.2}$$

In (4.2) replace x, y by $p^{-1}x, p^{-1}y$, respectively, and denote $a = p^{-1}q$, $b = qp^{-1}$, $c = p^{-1}u$, $w = up^{-1}$. Therefore Q satisfies

$$\Phi(x, y) = [xa - bx, yc - wy]_m - \left[p^{-1}x, p^{-1}y\right]. \tag{4.3}$$

In case $a \in C$, then $F = 0$. On the other hand, if $c \in C$, then $G = 0$. Therefore, we may assume that both $a \notin C$ and $c \notin C$, in particular Q is not commutative. We will prove that a number of contradiction follows. Since both a and c are not central elements, then (4.3) is a non-trivial generalized identity for R as well as for Q. Hence, Q is a primitive ring dense of linear transformations over a vector space V over C.

Assume first that $dim_C V = t$ is a finite integer. Thus, $Q \cong M_k(C)$ and by Lemma 6 it follows the contradiction that Q is commutative.

Let now $dim_C V = \infty$. Let $y_1, y_2 \in Q$. By Litoff's theorem (see Theorem 4.3.11 in [3]) there exists an idempotent element $e \in Q$ such that

$$y_1, y_2, a, b, c, w, p \in eQe \cong M_t(C)$$

for some integer t. Of course $\Phi(x, y) = 0$ for all $x, y \in eQe$. Thus by Lemma 6, either $a \in Ce$, or $c \in Ce$ or $[y_1, y_2] = 0$. This means that either $F(eQe) = 0$ or $G(eQe) = 0$ or $[y_1, y_2] = 0$. As above, if $F(eQe) = 0$ or $G(eQe) = 0$ then eQe is commutative. Therefore, in any case we get $[y_1, y_2] = 0$. By the arbitrariness of $y_1, y_2 \in Q$, it follows that $[y_1, y_2] = 0$ for any $y_1, y_2 \in Q$, that is Q is commutative, which is a contradiction.

5 The proof of Theorem 2

Also in this case, in light of Lemma 1, we may assume $n = 1$. Let P be a prime ideal of R. Set $\overline{R} = R/P$ and write $\overline{x} = x + P \in \overline{R}$, for all $x \in R$. We start from

$$\left[\overline{F(x)}, \overline{G(y)}\right]_m = \overline{[x, y]} \text{ for all } \overline{x}, \overline{y} \in \overline{R}. \tag{5.1}$$

Case 1 $F(P) \subseteq P, \alpha(P) \nsubseteq P$.

In this case $\overline{\alpha(P)}$ is an ideal of $\overline{R}$. Moreover, for any $x \in R$, $p \in P$, $F(px) = F(p)x + \alpha(p)F(x) \in P$, so that $\alpha(p)F(x) \in P$ and $\overline{\alpha(p)F(x)} = \overline{0}$. Since $\overline{R}$ is prime and $\overline{F(x)}$ annihilates a non-zero ideal of $\overline{R}$, $\overline{F(x)} = \overline{0}$, that is $F(R) \subseteq P$. Thus $[F(x), G(y)]_m \in P$ for any $x, y \in R$ and, by (5.1), it follows $[R, R] \subseteq P$.

Case 2 $G(P) \subseteq P, \alpha(P) \nsubseteq P$.

Identical computations as in CASE 1, imply $[R, R] \subseteq P$.

Case 3 $F(P) \subseteq P, \alpha(P) \subseteq P$.

In this case $\overline{F}$ is a skew derivation of $\overline{R}$. If $G(P) \subseteq P$, then also $\overline{G}$ is a skew derivation of $\overline{R}$, and by the primeness of $\overline{R}$ and Theorem 1, we have that $\overline{R}$ is commutative, that is $[R, R] \subseteq P$.

Let now $G(P) \nsubseteq P$. Then $\overline{G(P)}$ is a non-zero ideal of $\overline{R}$. For any $x, y \in R$ and $q \in P$, we get

$$\Big[F(x), G(yq)\Big]_m = [x, yq] \text{ for all } x, y \in R$$

that is

$$\Big[F(x), G(y)q + \alpha(y)G(q)\Big]_m = [x, yq] \text{ for all } x, y \in R$$

implying that

$$\Big[F(x), \alpha(y)G(q)\Big]_m \in P \text{ for all } x, y \in R$$

and

$$\Big[\overline{F(x)}, \overline{\alpha(y)G(q)}\Big]_m = \overline{0} \text{ for all } \overline{x}, \overline{y} \in \overline{R}. \tag{5.2}$$

In particular, let $h \geq 1$ be such that $p^h \geq m$, then (5.2) is

$$\Big[\overline{F(x)}, \overline{\alpha(y)G(q)}^{p^h}\Big] = \overline{0} \text{ for all } \overline{x}, \overline{y} \in \overline{R}. \tag{5.3}$$

By the primeness of $\overline{R}$ and since $\overline{\alpha(R)G(P)}$ is a non-zero ideal of $\overline{R}$, by applying the result in [14], it follows that either $\overline{F(R)} \subseteq Z(\overline{R})$ or $\overline{\alpha(R)G(P)} \subseteq Z(\overline{R})$. In the first case, by (5.1) we get $\overline{[x, y]} = \overline{0}$. In the latter case it follows that $\overline{R}$ is commutative. In any case $[R, R] \subseteq P$.

Case 4 $G(P) \subseteq P, \alpha(P) \subseteq P$.

In light of previous cases, we may also assume that $F(P) \nsubseteq P$. In this case, $\overline{G}$ is a skew derivation of $\overline{R}$, moreover $\overline{F(P)}$ is a non-zero ideal of $\overline{R}$. For any $x, y \in R$ and $q \in P$, we get

$$\Big[F(qx), G(y)\Big]_m = [qx, y], \quad \forall\, x, y \in R$$

that is

$$\Big[F(q)x + \alpha(q)F(x), G(y)\Big]_m \in P, \quad \forall\, x, y \in R$$

implying that

$$\Big[F(q)x, G(y)\Big]_m \in P, \quad \forall\, x, y \in R$$

and

$$\Big[\overline{F(P)R}, \overline{G(R)}\Big]_m = \overline{0}. \tag{5.4}$$

Since $\overline{R}$ is prime and $\overline{F(P)R}$ is a non-zero ideal of $\overline{R}$, $\overline{F(P)R}$, and $\overline{R}$ satisfy the same generalized identities with skew derivation $\overline{G}$ and automorphism $\overline{\alpha}$. Therefore, by (5.4) we get

$$\Big[\overline{R}, \overline{G(R)}\Big]_m = \overline{0}. \tag{5.5}$$

From (5.5) and (5.1) it follows again $\overline{[x, y]} = \overline{0}$ for any $x, y \in R$, that is $[R, R] \subseteq P$.

Case 5 $F(P) \nsubseteq P$, $G(P) \nsubseteq P$.

For any $x, y \in R$ and $q, u \in P$, by (5.1) we get

$$\Big[F(xq), G(yu)\Big]_m = [xq, yu] \in P \text{ for all } x, y \in R$$

that is

$$\Big[F(x)q + \alpha(x)F(q), G(y)u + \alpha(y)G(u)\Big]_m \in P \text{ for all } x, y \in R$$

implying that

$$\Big[\alpha(x)F(q), \alpha(y)G(u)\Big]_m \in P \text{ for all } x, y \in R$$

and

$$\Big[\overline{\alpha(R)F(P)}, \overline{\alpha(R)G(P)}\Big]_m = \overline{0}.$$

Assume first that $\alpha(P) \subseteq P$, so that both $\overline{F(P)}$ and $\overline{G(P)}$ are non-zero ideals of $\overline{R}$. On the other hand, if $\alpha(P) \nsubseteq P$ then $\overline{\alpha(R)F(P)}$ and $\overline{\alpha(R)G(P)}$ are left ideals of $\overline{R}$. In any case, for any $\overline{x} \in \overline{\alpha(R)F(P)}$ and $\overline{y} \in \overline{\alpha(R)G(P)}$, it follows $[\overline{x}, \overline{y}] = \overline{0}$. Since $\overline{R}$ is prime, we apply again a reduced version of main result in [14], and conclude that either $\overline{\alpha(R)F(P)} \subseteq Z(\overline{R})$ or $\overline{\alpha(R)G(P)} \subseteq Z(\overline{R})$. In any case $\overline{R}$ contains a non-zero central ideal (either left or two-sided), so that $\overline{R}$ is commutative, i.e., $\overline{[x, y]} = \overline{0}$, for any $x, y \in R$ and $[R, R] \subseteq P$.

Therefore in any case $[R, R] \subseteq P$, for any prime ideal P of R. Then $[R, R] \subseteq \bigcap_i P_i = (0)$ (where P_i are all prime ideals of R), that is R is commutative.

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