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On the Nilpotent Ranks of the Principal Factors of Orientation-Preserving Transformation Semigroups

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Abstract Let X_n be a chain with n elements, and let $\mathcal{OP}_n$ be the monoid of all orientation-preserving transformations of X_n. In this paper, we investigate the nilpotent ranks of the principal factors of the semigroup $\mathcal{OP}_n$.

Keywords Orientation-preserving transformation · Rank · Idempotent rank · Nilpotent rank

Mathematics Subject Classification 20M20 · 20M10

1 Introduction and Preliminaries

Let X_n be a chain with n elements, say $X_n = \{1 < 2 < \cdots < n\}$. As usual, we denote by $\mathcal{T}_n$ the monoid of all full transformations of a finite set X_n. We say that a transformation $\alpha \in \mathcal{T}_n$ is order preserving if $x \leq y$ implies $x\alpha \leq y\alpha$, for all $x, y \in X_n$. Denote by $\mathcal{O}_n$ the submonoid of $\mathcal{T}_n$ of all full order-preserving transformations of X_n.

Let $c = (c_1, c_2, \ldots, c_t)$ be a sequence of t $(t \geq 0)$ elements from the chain X_n. We say that c is cyclic if there exists not more than one index $i \in \{1, \ldots, t\}$ such that $c_i > c_{i+1}$, where c_{t+1} denotes c_1. We say that $\alpha \in \mathcal{T}_n$ is orientation preserving if the sequence of its image $(1\alpha, 2\alpha, \ldots, n\alpha)$ is cyclic. Denote by $\mathcal{OP}_n$ the submonoid of $\mathcal{T}_n$ of all full orientation-preserving transformations of X_n.

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Let S be a semigroup. Denote by S^1 the monoid obtained from S through the adjoining of an identity if S has none and exactly S otherwise. Recall the definition of Green's equivalence relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, and $\mathcal{J}$: for all $u, v \in S$,

$$\begin{aligned} u\mathcal{R}v &\text{ if and only if } uS^1 = vS^1; \\ u\mathcal{L}v &\text{ if and only if } S^1u = S^1v; \\ u\mathcal{H}v &\text{ if and only if } u\mathcal{R}v \text{ and } u\mathcal{L}v; \\ u\mathcal{J}v &\text{ if and only if } S^1uS^1 = S^1vS^1. \end{aligned}$$

Associated to Green's relation $\mathcal{J}$, there is a quasi-order $\leq_{\mathcal{J}}$ on S defined by

$$u \leq_{\mathcal{J}} v \quad \text{if and only if} \quad S^1uS^1 \subseteq S^1vS^1,$$

for all $u, v \in S$. Notice that, for every $u, v \in S$, we have $u\,\mathcal{J}\,v$ if and only if $u \leq_{\mathcal{J}} v$ and $v \leq_{\mathcal{J}} u$. Denote by J_u^S the $\mathcal{J}$-class of the element $u \in S$. As usual, a partial order relation $\leq_{\mathcal{J}}$ is defined on the quotient set $S/\mathcal{J}$ by putting $J_u^S \leq_{\mathcal{J}} J_v^S$ if and only if $u \leq_{\mathcal{J}} v$, for all $u, v \in S$. Given a subset A of S and $u \in S$, we denote by $E(A)$ set of idempotents of S belonging to A and by L_u^S, R_u^S, and H_u^S the $\mathcal{L}$-class, $\mathcal{R}$-class, and $\mathcal{H}$-class of u, respectively. For general background on Semigroup Theory, we refer the reader to Howie's book [8].

Let $\mathcal{M}_n$ denotes any of the monoids $\mathcal{T}_n, \mathcal{O}_n$, or $\mathcal{OP}_n$. Then $\mathcal{M}_n$ is regular. The Green relations $\mathcal{L}$ and $\mathcal{R}$ of $\mathcal{M}_n$ can be characterized by $\alpha\mathcal{L}\beta$ if and only if $\operatorname{im}(\alpha) = \operatorname{im}(\beta)$, for all $\alpha, \beta \in \mathcal{M}_n$, and $\alpha\mathcal{R}\beta$ if and only if $\ker(\alpha) = \ker(\beta)$, for all $\alpha, \beta \in \mathcal{M}_n$. Regarding the Green relation $\mathcal{J}$, we have $\alpha \leq_{\mathcal{J}} \beta$ if and only if $|\operatorname{im}(\alpha)| \leq |\operatorname{im}(\beta)|$ and so $\alpha\mathcal{J}\beta$ if and only if $|\operatorname{im}(\alpha)| = |\operatorname{im}(\beta)|$, for all $\alpha, \beta \in \mathcal{M}_n$. It follows that the partial order $\leq_{\mathcal{J}}$ on the quotient $\mathcal{M}_n/\mathcal{J}$ is linear. More precisely, letting

$$J_r^{\mathcal{M}_n} = \{\alpha \in \mathcal{M}_n : |\operatorname{im}(\alpha)| = r\},$$

i.e., the $\mathcal{J}$-class of the transformations of image size r (called the *rank* of the transformations) of $\mathcal{M}_n$, for $1 \leq r \leq n$, we have

$$\mathcal{M}_n/\mathcal{J} = \{J_1^{\mathcal{M}_n} \leq_{\mathcal{J}} J_2^{\mathcal{M}_n} \leq_{\mathcal{J}} \cdots \leq_{\mathcal{J}} J_n^{\mathcal{M}_n}\}.$$

See [1,2,7,8] for more details.

Since $\mathcal{M}_n/\mathcal{J}$ is a chain, the sets

$$\mathcal{M}(n,r) = \{\alpha \in \mathcal{M}_n : |\operatorname{im}(\alpha)| \leq r\} = J_1^{\mathcal{M}_n} \cup J_2^{\mathcal{M}_n} \cup \cdots \cup J_r^{\mathcal{M}_n},$$

with $1 \leq r \leq n$, constitute all the non-null ideals of $\mathcal{M}_n$ (see [4, Note of p. 181]). Notice that $\mathcal{M}(n,r) = \mathcal{T}(n,r)$ if $\mathcal{M}_n = \mathcal{T}_n$; $\mathcal{M}(n,r) = \mathcal{O}(n,r)$ if $\mathcal{M}_n = \mathcal{O}_n$; $\mathcal{M}(n,r) = \mathcal{OP}(n,r)$ if $\mathcal{M}_n = \mathcal{OP}_n$.

Notice that every principal factor of $\mathcal{M}_n$ associated with the maximum $\mathcal{J}$-class $J_r^{\mathcal{M}_n}$ is the Rees quotient $\mathcal{M}(n,r)/\mathcal{M}(n,r-1)$ $(2 \leq r \leq n)$, which we denote by $\mathcal{P}_r^{\mathcal{M}_n}$. It is usually convenient to think of $\mathcal{P}_r^{\mathcal{M}_n}$ as $J_r^{\mathcal{M}_n} \cup \{0\}$, and the product of two elements of $\mathcal{P}_r^{\mathcal{M}_n}$ is taken to be zero if it falls in $\mathcal{M}(n, r-1)$. $\mathcal{P}_r^{\mathcal{M}_n}$ is a completely 0-simple semigroup.

As usual, the rank of a finite semigroup S is defined by rank $S = \min\{|A| : A \subseteq S, \langle A \rangle = S\}$. If S is generated by its set E of idempotents, then the idempotent rank of S is defined by idrank $S = \min\{|A| : A \subseteq E, \langle A \rangle = S\}$. If S is generated by its set N of nilpotents, then the nilpotent rank of S is defined by nilrank $S = \min\{|A| : A \subseteq N, \langle A \rangle = S\}$. Clearly, rank $S \leq$ idrank S and rank $S \leq$ nilrank S.

In [6], Gomes and Howie showed that both the rank and the idempotent rank of $\mathcal{P}_{n-1}^{\mathcal{T}_n}$ are equal to $n(n-1)/2$. This result was later generalized by Howie and McFadden [9] who showed that the rank and idempotent rank of $\mathcal{P}_r^{\mathcal{T}_n}$ are both equal to $S(n, r)$, the Stirling number of the second kind, for $2 \leq r \leq n-1$. Yang [10] showed that the nilpotent rank of $\mathcal{P}_r^{\mathcal{T}_n}$ is also $S(n, r)$, $2 \leq r \leq n-1$.

The rank and idempotent rank of $\mathcal{P}_{n-1}^{\mathcal{O}_n}$ were shown to be n and $2n-2$, respectively, by Gomes and Howie [7]. Garba [5] generalized this result by showing that both the rank and the idempotent rank of the principal factor $\mathcal{P}_r^{\mathcal{O}_n}$ are equal to $\binom{n}{r}$, for $2 \leq r \leq n-2$. Yang [11] showed that the nilpotent rank of the principal factor $\mathcal{P}_r^{\mathcal{O}_n}$ are also equal to $\binom{n}{r}$, for $3 \leq r \leq n-1$.

Regarding the semigroup $\mathcal{OP}_n$, Zhao [12] showed that both the rank and the idempotent rank of the principal factor $\mathcal{P}_{n-1}^{\mathcal{OP}_n}$ are equal to $\binom{n}{2}$. Recently, Zhao and Fernandes [13] showed that the rank and the idempotent rank of the principal factor $\mathcal{P}_r^{\mathcal{OP}_n}$ are equal to $\binom{n}{r}$, for $2 \leq r \leq n-1$. In this paper, we investigate the nilpotent rank of the principal factor $\mathcal{P}_r^{\mathcal{OP}_n}$, for $2 \leq r \leq n-1$. In Sect. 2, we characterize the structure of the minimal generating sets of $\mathcal{OP}(n, r)$. As applications, we prove that the number of distinct minimal generating sets is $r^n n!$. In Sect. 3, we show that the nilpotent rank of the principal factor $\mathcal{P}_r^{\mathcal{OP}_n}$ is equal to $\binom{n}{r}$, for $2 \leq r \leq n-1$.

Remark 1 In this paper, it will always be clear from context when additions are taken modulo n (or modulo t where t is the number of elements of any sequence).

Throughout this paper, for simplicity, we always assume that $n \geq 3$.

2 The Minimal Generating Sets of $\mathcal{OP}(n, r)$

Let $\alpha \in \mathcal{T}_n$. As usual, we write $\mathrm{im}(\alpha)$ for the image of α. The *kernel* of α is the equivalence $\ker(\alpha) = \{(x, y) \in X_n \times X_n : x\alpha = y\alpha\}$. The equivalence classes of X_n with respect to $\ker(\alpha)$ are called the *kernel classes* of α.

We denote by $[i, k]$ the set $\{i, i+1, \ldots, k-1, k\}$ for $i, k \in X_n$. A set $K \subseteq X_n$ is *convex* if K has the form $[i, i+t]$, for some $i \in X_n$ and $0 \leq t \leq n-1$. We shall refer to an equivalence π on X_n as *convex* if its classes are convex subsets of X_n, and we shall say that π is of weight r if $|X_n/\pi| = r$.

It is known that that every kernel $\ker(\alpha)$ of $\alpha \in \mathcal{OP}_n$ is convex (see [2]). Let $2 \leq r \leq n-1$. Then, given a transformation $\alpha \in \mathcal{OP}_n$ of rank r, the kernel $\ker(\alpha)$ is convex. Therefore, we may establish a one-to-one correspondence between the collection of all subsets of X_n of cardinality r (which consists of all possible images of elements of rank r of $\mathcal{OP}_n$) and the collection of all possible convex kernels of elements of rank r of $\mathcal{OP}_n$ as follows: associate to each r-set $\{a_1, a_2, \ldots, a_r\}$, with $1 \le a_1 < a_2 < \cdots < a_r \le n$, the r-partition $\{A_1, \ldots, A_r\}$ of X_n defined by

$$A_i = \{a_i, a_i + 1, \ldots, a_{i+1} - 1\}, \quad \text{for } 1 \le i \le r \tag{1}$$

(notice that $a_{r+1} = a_1$ by Remark 1). Thus, the $\mathcal{J}$-class $J_r^{\mathcal{OP}_n}$ of $\mathcal{OP}_n$ (and of $\mathcal{OP}(n, r)$) contains $\binom{n}{r}$ $\mathcal{R}$-classes and $\binom{n}{r}$ $\mathcal{L}$-classes. See [2] for more details.

Now, for $1 \le a_1 < a_2 < \cdots < a_r \le n$, define

$$\varepsilon_{a_1,a_2,\ldots,a_r} = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix},$$

where $\{A_1, \ldots, A_r\}$ is the r-partition of X_n associated to $\{a_1, a_2, \ldots, a_r\}$ as in (1). Clearly, $\varepsilon_{a_1,a_2,\ldots,a_r} \in E(J_r^{\mathcal{OP}_n})$. Moreover, the set

$$E_r = \left\{\varepsilon_{a_1,a_2,\ldots,a_r} : 1 \le a_1 < a_2 < \cdots < a_r \le n\right\}$$

contains exactly one (idempotent) element from each $\mathcal{R}$-class and from each $\mathcal{L}$-class of $\mathcal{OP}_n$ of rank r.

The following lemma was proved by Zhao and Fernandes [13, Proposition 2.4].

Lemma 2.1 *For $2 \le r \le n - 1$, the set E_r generates $\mathcal{OP}(n, r)$.*

Now, we record a well-known result, due to Miller and Clifford ([3, Theorem 2.17]).

Lemma 2.2 *For any two elements a, b in a semigroup S, $ab \in R_a \cap L_b$ if and only if $E(R_b \cap L_a) \neq \emptyset$.*

Let U be a subset of $J_r^{\mathcal{OP}_n}$. We say that U satisfies Condition ($\mathbf{R} \sim \mathbf{L}$) if U contains exactly one element from each $\mathcal{R}$-class and from each $\mathcal{L}$-class of $J_r^{\mathcal{OP}_n}$. Notice that the set E_r satisfies Condition ($\mathbf{R} \sim \mathbf{L}$). From Lemma 2.1, we know that E_r is a generating set of $\mathcal{OP}(n, r)$. In fact, we have the following.

Lemma 2.3 *Let G be a subset of $J_r^{\mathcal{OP}_n}$. If G satisfies Condition* ($\mathbf{R} \sim \mathbf{L}$)*, then $\mathcal{OP}(n, r) = \langle G \rangle$.*

Proof We shall show that $E_r \subseteq \langle G \rangle$ and so $\mathcal{OP}(n, r) = \langle E_r \rangle$ by Lemma 2.1. Let $\varepsilon \in E_r$. Since G satisfies Condition ($\mathbf{R} \sim \mathbf{L}$), there exists unique $\alpha \in G$ such that $\alpha \mathcal{R} \varepsilon$. If $(\alpha, \varepsilon) \in \mathcal{L}$, then $\alpha \mathcal{H} \varepsilon$. Notice that each $\mathcal{H}$-class of $J_r^{\mathcal{OP}_n}$ that contains an idempotent is a cyclic group of order r (by [2, Corollary 3.6]). Thus $\varepsilon = \alpha^r \in \langle G \rangle$. If $(\alpha, \varepsilon) \notin \mathcal{L}$, then since E_r satisfies Condition ($\mathbf{R} \sim \mathbf{L}$), there exists unique $\varepsilon_1 \in E_r \backslash \{\varepsilon\}$ such that $(\alpha, \varepsilon_1) \in \mathcal{L}$. Since G satisfies Condition ($\mathbf{R} \sim \mathbf{L}$), there exists unique $\alpha_1 \in G \backslash \{\alpha\}$ such that $(\alpha_1, \varepsilon_1) \in \mathcal{R}$. Notice that $\varepsilon_1 \in E(L_\alpha \cap R_{\alpha_1})$. Then, by Lemma 2.2, $\alpha\alpha_1 \in R_\alpha \cap L_{\alpha_1} = R_\varepsilon \cap L_{\alpha_1}$. If $(\alpha_1, \varepsilon) \in \mathcal{L}$, then $(\alpha\alpha_1) \mathcal{H} \varepsilon$ and so $\varepsilon = (\alpha\alpha_1)^r \in \langle G \rangle$. If $(\alpha_1, \varepsilon) \notin \mathcal{L}$, then since E_r satisfies Condition ($\mathbf{R} \sim \mathbf{L}$), there exists unique $\varepsilon_2 \in E_r \backslash \{\varepsilon, \varepsilon_1\}$ such that $(\alpha_1, \varepsilon_2) \in \mathcal{L}$. Since G satisfies Condition

($\mathbf{R} \sim \mathbf{L}$), there exists unique $\alpha_2 \in G\backslash\{\alpha, \alpha_1\}$ such that $(\alpha_2, \varepsilon_2) \in \mathcal{R}$. Notice that $\varepsilon_2 \in E(L_{\alpha_1} \cap R_{\alpha_2}) = E(L_{\alpha\alpha_1} \cap R_{\alpha_2})$. Then, by Lemma 2.2, $(\alpha\alpha_1)\alpha_2 \in R_{\alpha\alpha_1} \cap L_{\alpha_2} = R_\varepsilon \cap L_{\alpha_2}$. If $(\alpha_2, \varepsilon) \in \mathcal{L}$, then $(\alpha\alpha_1\alpha_2)\mathcal{H}\varepsilon$ and so $\varepsilon = (\alpha\alpha_1\alpha_2)^r \in \langle G\rangle$. If $(\alpha_2, \varepsilon) \notin \mathcal{L}$, then since E_r satisfies Condition ($\mathbf{R} \sim \mathbf{L}$), there exists unique $\varepsilon_3 \in E_r\backslash\{\varepsilon, \varepsilon_1, \varepsilon_2\}$ such that $(\alpha_2, \varepsilon_3) \in \mathcal{L}$. Since G satisfies Condition ($\mathbf{R} \sim \mathbf{L}$), there exists unique $\alpha_3 \in G\backslash\{\alpha, \alpha_1, \alpha_2\}$ such that $(\alpha_3, \varepsilon_3) \in \mathcal{R}$. Notice that $\varepsilon_3 \in E(L_{\alpha_2} \cap R_{\alpha_3}) = E(L_{\alpha\alpha_1\alpha_2} \cap R_{\alpha_3})$. Then, by Lemma 2.2, $(\alpha\alpha_1\alpha_2)\alpha_3 \in R_{\alpha\alpha_1\alpha_2} \cap L_{\alpha_3} = R_\varepsilon \cap L_{\alpha_3}$. Continuing this demonstration, since G and E_r satisfy Condition ($\mathbf{R} \sim \mathbf{L}$), there must exist $k \leq m$ ($m = \binom{n}{r}$) such that $\alpha_k \in G\backslash\{\alpha, \alpha_1, \alpha_{k-1}\}$, $(\alpha \ldots \alpha_{k-1})\alpha_k \in R_\varepsilon \cap L_{\alpha_k}$ and $\alpha_k\mathcal{L}\varepsilon$. Then $(\alpha\alpha_1 \ldots \alpha_k)\mathcal{H}\varepsilon$ and so $\varepsilon = (\alpha\alpha_1 \ldots \alpha_k)^r \in \langle G\rangle$. □

Since $\mathcal{OP}(n,r)$ has rank $\binom{n}{r}$ (see [13, Theorem 2.7]), a generating set of $\mathcal{OP}(n,r)$ with $\binom{n}{r}$ elements is a minimal generating set. Moreover, if α is an element of $\mathcal{OP}(n,r)$ of rank r and β and γ are two elements of $\mathcal{OP}(n,r)$ such that $\alpha = \beta\gamma$, then $\ker(\alpha) = \ker(\beta)$ and $\mathrm{im}(\alpha) = \mathrm{im}(\gamma)$. Then any generating set of $\mathcal{OP}(n,r)$ with $\binom{n}{r}$ elements be the subset having exactly one element from each $\mathcal{R}$-class and from each $\mathcal{L}$-class of rank r. These observations, together with the Lemma 2.3, prove the following result:

Theorem 2.4 *Let M be a subset of $\mathcal{OP}(n,r)$ with $\binom{n}{r}$ elements. Then M is a minimal generating set of $\mathcal{OP}(n,r)$ if and only if M be the subset having exactly one element from each $\mathcal{R}$-class and from each $\mathcal{L}$-class of $\mathcal{OP}_n$ of rank r.*

Notice that each $\mathcal{H}$-class of $J_r^{\mathcal{OP}_n}$ that contains an idempotent is a cyclic group of order r (by [2, Corollary 3.6]). Thus, we have the following corollary from Theorem 2.4:

Corollary 2.5 *Let M be a minimal generating set of $\mathcal{OP}(n,r)$. Then the number of distinct sets M is $r^n n!$.*

3 The Nilpotent Rank of $\mathcal{P}_r^{\mathcal{OP}_n}$

Recall that Zhao [12] showed that both the rank and the idempotent rank of the principal factor $\mathcal{P}_{n-1}^{\mathcal{OP}_n}$ are equal to $\binom{n}{2}$. Recently, Zhao and Fernandes [13] showed that the rank and the idempotent rank of the principal factor $\mathcal{P}_r^{\mathcal{OP}_n}$ are equal to $\binom{n}{r}$, for $2 \leq r \leq n-1$. In this section, we show that the nilpotent rank of the principal factor $\mathcal{P}_r^{\mathcal{OP}_n}$ are also equal to $\binom{n}{r}$, for $2 \leq r \leq n-1$.

Let A be a subset of X_n of cardinality r and let π be a convex equivalence of weight r on X_n. We may write $H_{(\pi,A)}$ for the $\mathcal{H}$-class of $J_r^{\mathcal{OP}_n}$, which is the intersection of $R_\pi = \{\alpha \in J_r^{\mathcal{OP}_n} : \ker(\alpha) = \pi\}$ and $L_A = \{\alpha \in J_r^{\mathcal{OP}_n} : \mathrm{im}(\alpha) = A\}$. The subset A of X_n of cardinality r is said to be a *transversal* of the convex equivalence π of weight r on X_n if each convex equivalence π-class contains exactly one element of A. The following lemma is obvious:

Lemma 3.1 *Let $\alpha \in \mathcal{P}_r^{\mathcal{OP}_n}$. Then α is nilpotent if and only if* $\mathrm{im}(\alpha)$ *is not a transversal of* $\ker(\alpha)$.

Our main result of this section is as follows:

Theorem 3.2 *Let $n \geq 3$ and $2 \leq r \leq n-1$. Then*

$$\operatorname{nilrank} \mathcal{P}_r^{\mathcal{OP}_n} = \binom{n}{r}.$$

The proof depends on the following lemma:

Lemma 3.3 *Let $A_1, A_2, \ldots, A_m$ (where $m = \binom{n}{r}$) be a list of all the subsets of X_n with cardinality r. Suppose that there exist distinct convex equivalences $\pi_1, \pi_2, \ldots, \pi_m$ of weight r on X_n with the property that A_i is not a transversal of π_i, for $1 \leq i \leq m$. Then there exist nilpotent γ_i in the $\mathcal{H}$-class $H_{(\pi_i, A_i)}$ ($1 \leq i \leq m$) such that the set $\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ is a minimal generating set of $\mathcal{P}_r^{\mathcal{OP}_n}$.*

Proof From Lemma 3.1, we know that the $\mathcal{H}$-classes $H_{(\pi_1, A_1)}, \ldots, H_{(\pi_m, A_m)}$ are non-group $\mathcal{H}$-classes, whose elements are nilpotents of $\mathcal{P}_r^{\mathcal{OP}_n}$. Put

$$\gamma_i \in H_{(\pi_i, A_i)}, \text{ for } 1 \leq i \leq m.$$

Then the set $\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ is the subset having exactly one element from each $\mathcal{R}$-class and from each $\mathcal{L}$-class of $\mathcal{OP}_n$ of rank r. It follows immediately from Theorem 2.4 that the set $\{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ is a minimal generating set of $\mathcal{P}_r^{\mathcal{OP}_n}$. □

It remains to prove that the listing of images and convex equivalences postulated in the statement of Lemma 3.3 can actually be carried out. Let $n \geq 3$ and $2 \leq r \leq n-1$, and consider the statement:

$\mathbf{P}(n, r)$: There is a way of listing all the subsets of X_n of cardinality r as $A_1, A_2, \ldots, A_m$ (with $m = \binom{n}{r}$) so that there exist distinct convex equivalences $\pi_1, \ldots, \pi_m$ of weight r on X_n with the properties that A_i is not a transversal of π_i ($i = 1, \ldots, m$).

We shall prove this by a double induction on n and r, the key step being a kind of Pascal's Triangle implication

$$\mathbf{P}(n-1, r-1) \text{ and } \mathbf{P}(n-1, r) \Rightarrow \mathbf{P}(n, r).$$

First, however, we anchor the induction with two lemmas:

Lemma 3.4 $\mathbf{P}(n, n-1)$ *holds for every $n \geq 3$.*

Proof Consider the list $A_1, A_2, \ldots, A_n$ of X_n of cardinality $n-1$, where $A_i = X_n \setminus \{i\}$. For $1 \leq i \leq n-2$, let π_i be the convex equivalence with a unique non-singleton class $\{i+1, i+2\}$ and all other classes being singletons. Let π_{n-1} have classes $\{2\}, \{3\}, \ldots, \{n-1\}, \{n, 1\}$; and let π_n have classes $\{1, 2\}, \{3\}, \ldots, \{n-1\}, \{n\}$. It is easy to verify that $A_1, A_2, \ldots, A_n$ and $\pi_1, \pi_2, \ldots, \pi_n$ have the required property. □

Lemma 3.5 $\mathbf{P}(n, 2)$ *holds for every $n \geq 3$.*

Proof We prove this by the induction n. Notice first that $P(3,2)$ holds by Lemma 3.4. For $n=4$, we arrange the subsets and convex equivalences as follows:

$$\begin{array}{cccccc} \{1,2\}, & \{1,3\}, & \{1,4\}, & \{2,3\}, & \{2,4\}, & \{3,4\} \\ 12/34, & 123/4, & 3/412, & 23/41, & 1/234, & 2/341. \end{array}$$

Then, it is easy to verify that the subsets and convex equivalences as arranged above satisfy $P(4,2)$.

Suppose inductively that $P(n-1,2)$ holds ($n \geq 5$). Thus, we have a list $A_1, A_2, \ldots, A_t$ (with $t = \binom{n-1}{2}$) of all the subsets of X_{n-1} of cardinality 2, and a list $\pi_1, \pi_2, \ldots, \pi_t$ of distinct convex equivalences of weight 2 on X_{n-1} such that A_i is not a transversal of π_i ($i = 1, 2, \ldots, t$). Notice that $t + n - 1 = \binom{n}{2}$. All subsets of X_n of cardinality 2 are

$$B_1, B_2, \ldots, B_{n-1}; A_1, A_2, \ldots, A_t,$$

where $B_j = \{j, n\}$, $1 \leq j \leq n-1$. Let π_i' ($i = 1, 2, \ldots, t$) be the convex equivalence on X_n obtained from π_i by adjoining n to the π_i-class containing $n-1$. Then $\pi_1', \pi_i', \ldots, \pi_t'$ are all distinct, and A_i is not a transversal of π_i' ($i = 1, 2, \ldots, t$).

Let $\sigma_1, \ldots, \sigma_{n-1}$ be the list of convex equivalences of weight 2 on X_n, where

$$\begin{aligned} \sigma_j &\text{ has classes } \{n, 1, \ldots, j\}, \{j+1, \ldots, n-1\}, 1 \leq j \leq n-2, \\ \sigma_{n-1} &\text{ has classes } \{n\}, \{1, 2, \ldots, n-1\}. \end{aligned}$$

Then $\sigma_1, \ldots, \sigma_{n-1}$ are all distinct, B_j is not a transversal of σ_j, for $1 \leq j \leq n-2$, and each σ_j is distinct from every π_i', since $(n-1, n) \in \pi_i'$ and $(n-1, n) \notin \sigma_j$. Notice that $t + n - 1 = \binom{n}{2}$. Hence $\{\pi_1', \ldots, \pi_t', \sigma_1, \ldots, \sigma_{n-1}, \}$ is a complete list of all the convex equivalences of weight 2 on X_n. Notice that A_1 is a subset of X_{n-1} and $(n, n-1) \in \pi_1'$. Then A_1 is not a transversal of σ_{n-1} and $B_{n-1} = \{n-1, n\}$ is not a transversal of π_1'. Arrange the subsets and the convex equivalence as follows:

$$\begin{array}{ccccccccc} A_2, & \ldots, & A_t; & B_1, & B_2, & \ldots, & B_{n-2}, & A_1, & B_{n-1}, \\ \pi_2', & \ldots, & \pi_t'; & \sigma_1, & \sigma_2, & \ldots, & \sigma_{n-2}, & \sigma_{n-1}, & \pi_1'. \end{array}$$

Then, it is easy to verify that the subsets and convex equivalences as arranged above satisfy $P(n,2)$. □

Lemma 3.6 *Let $n \geq 5$ and $3 \leq r \leq n-2$. Then* $\mathbf{P}(n-1, r-1)$ *and* $\mathbf{P}(n-1, r)$ *together imply* $\mathbf{P}(n, r)$.

Proof From the assumption $P(n-1, r)$, we have a list $A_1, \ldots, A_m$ (where $m = \binom{n-1}{r}$) of the subsets of X_{n-1} with cardinality r and a list $\sigma_1, \ldots, \sigma_m$ of the convex equivalences of weight r on X_{n-1} such that A_i is not a transversal of σ_i, for $1 \leq i \leq m$.

From the assumption $P(n-1, r-1)$, we have a list $B_1, \ldots, B_t$ (where $t = \binom{n-1}{r-1}$) of the subsets of X_{n-1} with cardinality $r-1$ and a list $\pi_1, \ldots, \pi_t$ of the convex equivalences of weight $r-1$ on X_{n-1} such that B_i is not a transversal of π_i, for $1 \leq i \leq t$.

Let σ_i' be the convex equivalence obtained from σ_i by adjoining n to the σ_i-class containing $n-1$, and define $\pi_i' = \pi_i \cup \{(n,n)\}$. Then $\sigma_1', \ldots, \sigma_m', \pi_1', \ldots, \pi_t'$ are all distinct. Notice that $m+t = \binom{n}{r}$. Hence $\{\sigma_1', \ldots, \sigma_m', \pi_1', \ldots, \pi_t'\}$ is a complete list of all the convex equivalences of weight r on X_n. Next we define

$$B_i' = B_i \cup \{n\}, \text{ for } 1 \leq i \leq t.$$

Then $A_1, \ldots, A_m, B_1', B_2', \ldots, B_t'$ are all distinct. Moreover, A_i is not a transversal of σ_i', for $1 \leq i \leq m$ and B_i' is not a transversal of π_i', for $1 \leq i \leq t$. Arrange the subsets and convex equivalences as follows:

$$\begin{array}{l} A_1, \ldots, A_m, \quad B_1', \quad B_2', \ldots, B_t' \\ \ \sigma_1', \quad \ldots, \sigma_m', \quad \pi_1', \quad \pi_2', \ldots, \pi_t' \end{array}$$

They satisfy all the properties necessary and the inductive step is complete. We have shown that $P(n,r)$ holds for all $n \geq 5$ and $3 \leq r \leq n-2$. □

The pattern of deductions is

P(4, 2)   **P**(4, 3)
↙ ↘ ↙
P(5, 2)   **P**(5, 3)   **P**(5, 4)
↙ ↘ ↙ ↘ ↙
P(6, 2)   **P**(6, 3)   **P**(6, 4)   **P**(6, 5)
↙ ↘ ↙ ↘ ↙ ↘ ↙
P(7, 2)   **P**(7, 3)   **P**(7, 4)   **P**(7, 5)   **P**(7, 6)

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