Bull. Malays. Math. Sci. Soc. (2018) 41:1759–1771
https://doi.org/10.1007/s40840-016-0423-2

BULLETIN *of the* MALAYSIAN MATHEMATICAL SCIENCES SOCIETY
http://www.springer.com/mathematics/journal/40840



# Surfaces in $\mathbb{S}^3$ of $L_1$-2-Type

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Received: 14 January 2016 / Revised: 19 September 2016 / Published online: 27 September 2016


**Abstract** In this paper, we show that an $L_1$-2-type surface $M^2 \subset \mathbb{S}^3$ is either an open portion of a standard Riemannian product $\mathbb{S}^1(a) \times \mathbb{S}^1(b)$, of any radii, or it has non-constant mean curvature $H$, non-constant Gaussian curvature $K$, and non-constant principal curvatures $\kappa_1$ and $\kappa_2$.

**Keywords** Spherical surface · Cheng–Yau operator $L_1$ · $L_1$-finite-type surface · $L_1$-biharmonic surface · Newton transformation

**Mathematics Subject Classification** 53C40 · 53A05 · 53B25

## 1 Introduction

Submanifolds of finite-type $M$ (i.e., submanifolds whose isometric immersion in the Euclidean space is constructed by using eigenfunctions of their Laplacian) were introduced by Chen during the late 1970s, and the first results on this subject were collected in his book [5]. In subsequent papers, Chen has provided a detailed account of recent development on problems and conjectures about finite-type submanifolds, [6,7]. It is

Communicated by Rosihan M. Ali.

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well known that the Laplacian operator $\Delta$ can be seen as the first one of a sequence of operators $L_0 = \Delta, L_1, \ldots, L_{n-1}$, $n = \dim(M)$, where $L_k$ stands for the linearized operator of the first variation of the $(k+1)$-th mean curvature arising from normal variations (see, for instance, [13]). $L_1$ is nothing but the differential operator $\Box$ introduced by Cheng and Yau [8].

The notion of finite-type submanifold can be defined for any operator $L_k$, [10], and then it is natural to try to obtain new results and compare them with the classical ones. For example, it is well known that the only 2-type surfaces in the unit 3-sphere $\mathbb{S}^3$ are open portions of the product of two circles $\mathbb{S}^1(a) \times \mathbb{S}^1(b)$ of different radii, [4,5,9].

In the present article, we study the same problem for the operator $L_1$, that is, we study isometric immersions $\psi : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ of $L_1$-2-type. These surfaces are characterized by the following spectral decomposition of the position vector $\psi$:

$$\psi = a + \psi_1 + \psi_2, \quad L_1\psi_1 = \lambda_1\psi_1, \quad L_1\psi_2 = \lambda_2\psi_2, \quad \lambda_1 \neq \lambda_2,\ \lambda_i \in \mathbb{R},$$

where $a$ is a constant vector in $\mathbb{R}^4$, and $\psi_1, \psi_2$ are $\mathbb{R}^4$-valued non-constant differentiable functions on $M^2$. It is easy to see that open portions of the product of two circles $\mathbb{S}^1(a) \times \mathbb{S}^1(b)$, of any radii, are surfaces of $L_1$-2-type (see the example 2). Our main theorem is the following local result:

**Theorem** *Let $\psi : M^2 \to \mathbb{S}^3$ be an orientable surface of $L_1$-2-type. Then either $M^2$ is an open portion of a standard Riemannian product $\mathbb{S}^1(a) \times \mathbb{S}^1(b)$ of any radii, or $M^2$ has non-constant mean curvature $H$, non-constant Gaussian curvature $K$, and non-constant principal curvatures $\kappa_1$ and $\kappa_2$.*

## 2 Preliminaries

Let $\psi : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an isometric immersion in the unit 3-sphere $\mathbb{S}^3$ (centered at the origin of $\mathbb{R}^4$) of a connected orientable surface $M^2$, with Gauss map $N$. We denote by $\nabla^0$, $\overline{\nabla}$, and $\nabla$ the Levi-Civita connections on $\mathbb{R}^4$, $\mathbb{S}^3$, and $M^2$, respectively. Then the Gauss and Weingarten formulas are given by

$$\nabla^0_X Y = \nabla_X Y + \langle SX, Y\rangle N - \langle X, Y\rangle \psi, \tag{1}$$

$$SX = -\overline{\nabla}_X N = -\nabla^0_X N, \tag{2}$$

for all tangent vector fields $X, Y \in \mathfrak{X}(M^2)$, where $S : \mathfrak{X}(M^2) \to \mathfrak{X}(M^2)$ stands for the shape operator (or Weingarten endomorphism) of $M^2$, with respect to the chosen orientation $N$. The mean curvature $H$ and the scalar curvature $H_2$ (also called the extrinsic curvature) of $M^2$ are defined by $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ and $H_2 = \kappa_1\kappa_2$, respectively, $\kappa_1$ and $\kappa_2$ being the eigenvalues of $S$ (i.e., the principal curvatures of the surface). From the Gauss equation, we know that the Gaussian curvature $K$ is given by $K = 1 + \det(S) = 1 + H_2$.

The Newton transformation of $M^2$ is the operator $P : \mathfrak{X}(M^2) \to \mathfrak{X}(M^2)$ defined by

$$P = 2HI - S. \tag{3}$$

Note that by the Cayley–Hamilton theorem, we have $S \circ P = H_2 I$. Observe also that, at any point $p \in M^2$, $S(p)$ and $P(p)$ can be simultaneously diagonalized: if $\{e_1, e_2\}$ are the eigenvectors of $S(p)$ corresponding to the eigenvalues $\kappa_1(p)$ and $\kappa_2(p)$, respectively, then they are also the eigenvectors of $P(p)$ with corresponding eigenvalues $\kappa_2(p)$ and $\kappa_1(p)$, respectively.

According to [12, p. 86], for a tensor $T$, the contraction of the new covariant slot in its covariant differential $\nabla T$ with one of its original slots is called a divergence of $T$. Hence the divergence of a vector field $X$ is the differential function defined by

$$\mathrm{div}\,(X) = C(\nabla X) = \left\langle \nabla_{E_1} X, E_1 \right\rangle + \left\langle \nabla_{E_2} X, E_2 \right\rangle,$$

$\{E_1, E_2\}$ being any local orthonormal frame of tangent vectors fields. For an operator $T : \mathfrak{X}(M^2) \to \mathfrak{X}(M^2)$, the divergence associated to the metric contraction $C_{12}$ will be the vector field $\mathrm{div}\,(T) \in \mathfrak{X}(M^2)$ defined as

$$\mathrm{div}\,(T) = C_{12}(\nabla T) = (\nabla_{E_1} T)E_1 + (\nabla_{E_2} T)E_2.$$

We have the following properties of $P$. The first three claims are direct computations; for a proof of claims (d) and (e), see e.g., [1].

**Lemma 1** *The Newton transformation $P$ satisfies the following:*

*(a)* $tr\,(P) = 2H$.
*(b)* $tr\,(S \circ P) = 2H_2$.
*(c)* $tr\,(S^2 \circ P) = 2HH_2$.
*(d)* $tr\,(\nabla_X S \circ P) = \langle \nabla H_2, X \rangle$, *where $\nabla H_2$ stands for the gradient of $H_2$.*
*(e)* $div\,(P) = 0$.

Associated to the Newton transformation $P$, we can define a second-order linear differential operator $L_1 : \mathcal{C}^\infty(M^2) \to \mathcal{C}^\infty(M^2)$ by

$$L_1(f) = \mathrm{tr}\left(P \circ \nabla^2 f\right), \tag{4}$$

where $\nabla^2 f : \mathfrak{X}(M^2) \to \mathfrak{X}(M^2)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of $f$, given by $\left\langle \nabla^2 f(X), Y \right\rangle = \langle \nabla_X(\nabla f), Y \rangle$. An interesting property of $L_1$ is the following. For every couple of differentiable functions $f, g \in C^\infty(M^2)$, we have

$$L_1(fg) = gL_1(f) + fL_1(g) + 2\,\langle P(\nabla f), \nabla g \rangle\,. \tag{5}$$

The operator $L_1$ can be extended to vector functions as follows: If $F = (f_1, f_2, f_3, f_4) : M^2 \to \mathbb{R}^4$, $f_i \in \mathcal{C}^\infty(M^2)$, then $L_1 F := (L_1 f_1, L_1 f_2, L_1 f_3, L_1 f_4)$.

## 3 First Results

Let $a \in \mathbb{R}^4$ be an arbitrary fixed vector. A direct computation shows that the gradient of the function $\langle \psi, a \rangle$ is given by

$$\nabla \langle \psi, a \rangle = a^\top = a - \langle N, a \rangle N - \langle \psi, a \rangle \psi, \tag{6}$$

where $a^\top \in \mathfrak{X}(M^2)$ denotes the tangential component of $a$. Taking covariant derivative in (6), and using the Gauss and Weingarten formulas, we obtain

$$\nabla_X \nabla \langle \psi, a \rangle = \nabla_X a^\top = \langle N, a \rangle SX - \langle \psi, a \rangle X, \tag{7}$$

for every vector field $X \in \mathfrak{X}(M^2)$. Finally, by using (4) and Lemma 1, we find that

$$\begin{aligned} L_1 \langle \psi, a \rangle &= \langle N, a \rangle \operatorname{tr}(S \circ P) - \langle \psi, a \rangle \operatorname{tr}(P) \\ &= 2H_2 \langle N, a \rangle - 2H \langle \psi, a \rangle . \end{aligned} \tag{8}$$

Then $L_1\psi$ can be computed as

$$L_1\psi = 2H_2N - 2H\psi. \tag{9}$$

A straightforward computation yields

$$\nabla \langle N, a \rangle = -Sa^\top .$$

From the Weingarten formula and (7), we find that

$$\begin{aligned} \nabla_X \nabla \langle N, a \rangle &= -(\nabla_X S)a^\top - S(\nabla_X a^\top) \\ &= -(\nabla_{a^\top} S)X - \langle N, a \rangle S^2 X + \langle \psi, a \rangle SX, \end{aligned}$$

for every tangent vector field $X$. This equation, jointly with (4) and Lemma 1, yields

$$\begin{aligned} L_1 \langle N, a \rangle &= -\operatorname{tr}(\nabla_{a^\top} S \circ P) - \langle N, a \rangle \operatorname{tr}(S^2 \circ P) + \langle \psi, a \rangle \operatorname{tr}(S \circ P) \\ &= -\langle \nabla H_2, a \rangle - 2HH_2 \langle N, a \rangle + 2H_2 \langle \psi, a \rangle . \end{aligned} \tag{10}$$

In other words,

$$L_1 N = -\nabla H_2 - 2HH_2N + 2H_2\psi. \tag{11}$$

From (9), (11), and (5), we obtain the following result.

**Lemma 2** *For any $f \in \mathcal{C}^\infty(M^2)$, we have*

$$\begin{aligned} L_1(f\psi) &= 2P(\nabla f) + 2fH_2N + (L_1 f - 2Hf)\psi, \\ L_1(fN) &= -(f\nabla H_2 + 2H_2\nabla f) + (L_1 f - 2HH_2 f)N + 2H_2 f\psi. \end{aligned}$$

On the other hand, Eqs. (5), (8), and (10) lead to

$$\begin{aligned} L_1^2 \langle \psi, a \rangle &= 2H_2 L_1 \langle N, a \rangle + 2L_1(H_2) \langle N, a \rangle + 4\big\langle P(\nabla H_2), \nabla \langle N, a \rangle \big\rangle \\ &\quad - 2HL_1 \langle \psi, a \rangle - 2L_1(H) \langle \psi, a \rangle - 4\big\langle P(\nabla H), \nabla \langle \psi, a \rangle \big\rangle, \\ &= -2H_2 \langle \nabla H_2, a \rangle - 4 \langle (S \circ P)(\nabla H_2), a \rangle - 4 \langle P(\nabla H), a \rangle \\ &\quad + \left[2L_1 H_2 - 4HH_2(H_2+1)\right] \langle N, a \rangle \\ &\quad + \left[4H_2^2 + 4H^2 - 2L_1 H\right] \langle \psi, a \rangle . \end{aligned}$$

Finally, we get

$$\begin{aligned} L_1^2 \psi &= -4P(\nabla H) - 3\nabla H_2^2 \\ &\quad + 2\big[L_1 H_2 - 2HH_2\big(H_2+1\big)\big]N \\ &\quad + 2\big[2H_2^2 + 2H^2 - L_1 H\big]\psi. \end{aligned} \tag{12}$$

### 3.1 $L_1$-Biharmonic Surfaces

An isometric immersion $x : M^n \to \mathbb{R}^m$ is said to be *biharmonic* if $\Delta \mathbf{H} = 0$, where $\Delta$ and $\mathbf{H}$ are the rough Laplacian on the submanifold $M^n$ and the mean curvature vector field of the immersion, respectively (see e.g., [6]). From the Beltrami formula $\Delta x = n\mathbf{H}$, we know that the submanifold $M^n$ is biharmonic if and only if $\Delta^2 x = 0$. The following definition appears in a natural way (see [3] and [11]).

**Definition 3** An isometric immersion $\psi : M^2 \to \mathbb{R}^4$ is said to be $L_1$-biharmonic if $L_1^2\psi = 0$. In the case $L_1^2\psi = 0$ and $L_1\psi \neq 0$, we will say that $\psi$ is a proper $L_1$-biharmonic surface.

If $M^2$ is a totally geodesic surface of $\mathbb{S}^3$, then Eq. (9) implies $L_1\psi = 0$, and hence $M^2$ is a (trivial) $L_1$-biharmonic surface in $\mathbb{R}^4$.

Let $\psi : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an $L_1$-biharmonic surface. Then (12) yields

$$4P(\nabla H) + 3\nabla H_2^2 = 0, \tag{13}$$

$$L_1 H_2 - 2HH_2(H_2+1) = 0, \tag{14}$$

$$L_1 H - 2(H^2 + H_2^2) = 0. \tag{15}$$

If $H$ is constant, then (15) yields $H = H_2 = 0$, i.e., $M^2$ is a totally geodesic surface in $\mathbb{S}^3$; in other words, $M^2$ is an open portion of a unit 2-sphere $\mathbb{S}^2$. If $K$ is constant (and so $H_2$ also is), by taking divergence in (13) we get $L_1 H = 0$. Then from (15) we also deduce that $M^2$ is an open portion of a unit 2-sphere $\mathbb{S}^2$. We have obtained the following result.

**Proposition 4** *Let $\psi : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an $L_1$-biharmonic surface. Then either $M^2$ is an open portion of a unit 2-sphere $\mathbb{S}^2$ or $M^2$ has non-constant curvatures $H$ and $K$.*

This result can be improved as follows: If $H$ is an $L_1$-harmonic function (i.e., $L_1 H = 0$), then (15) implies again that $M^2$ is an open portion of a unit 2-sphere $\mathbb{S}^2$. The same conclusion is also reached when $H_2$ (or $K$) is an $L_1$-harmonic function. In this case, (14) yields

$$H H_2(H_2 + 1) = 0.$$

Let us assume that $H$ is non-constant (otherwise, there is nothing to prove) and take the non-empty set $\mathcal{U} = \{p \in M^2 \mid \nabla H^2(p) \neq 0\}$. On this set, we have $H_2(H_2 + 1) = 0$, and then $H_2$ is constant on $\mathcal{U}$. Hence Proposition 4 implies that $\mathcal{U}$ is an open portion of a unit 2-sphere $\mathbb{S}^2$, but then the mean curvature $H$ is constant. This is a contradiction. The following result has been proved.

**Proposition 5** *Let $\psi : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an $L_1$-biharmonic surface. Then either $M^2$ is an open portion of a unit 2-sphere $\mathbb{S}^2$ or the curvatures $H$ and $K$ are not $L_1$-harmonic.*

When $M^2$ is a closed surface, we can improve that result. By taking divergence in (13), we get

$$L_1 H = -\frac{3}{4}\Delta H_2^2.$$

From here and (15), and by using the divergence theorem, we obtain

$$0 = \int_M L_1 H \, dv = 2\int_M (H^2 + H_2^2)\, dv.$$

This implies $H = H_2 = 0$. We have proved the following result.

**Proposition 6** *Let $\psi : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ be a closed surface. Then $M^2$ is an $L_1$-biharmonic surface if and only if it is a unit 2-sphere $\mathbb{S}^2$.*

### 3.2 Equations Characterizing the $L_1$-2-Type Surfaces

Let us suppose that $M^2$ is of $L_1$-2-type in $\mathbb{R}^4$, that is, the position vector $\psi$ of $M^2$ in $\mathbb{R}^4$ can be written as follows:

$$\psi = a + \psi_1 + \psi_2, \quad L_1\psi_1 = \lambda_1\psi_1, \quad L_1\psi_2 = \lambda_2\psi_2, \quad \lambda_1 \neq \lambda_2,\ \lambda_i \in \mathbb{R},$$

where $a$ is a constant vector in $\mathbb{R}^4$, and $\psi_1, \psi_2$ are $\mathbb{R}^4$-valued non-constant differentiable functions on $M^2$.

Since $L_1\psi = \lambda_1\psi_1 + \lambda_2\psi_2$ and $L_1^2\psi = \lambda_1^2\psi_1 + \lambda_2^2\psi_2$, an easy computation shows that

$$L_1^2\psi = (\lambda_1 + \lambda_2)L_1\psi - \lambda_1\lambda_2(\psi - a),$$

and by using (9), we obtain

$$L_1^2\psi = \lambda_1\lambda_2 a^\top + \big[2(\lambda_1+\lambda_2)H_2 + \lambda_1\lambda_2\langle N,a\rangle\big]N - \big[2(\lambda_1+\lambda_2)H + \lambda_1\lambda_2 - \lambda_1\lambda_2\langle\psi,a\rangle\big]\psi.$$

This equation, jointly with (12), yields the following equations that characterize the $L_1$-2-type surfaces in $\mathbb{S}^3$:

$$\lambda_1\lambda_2 a^\top = -3\nabla H_2^2 - 4P(\nabla H), \tag{16}$$

$$\lambda_1\lambda_2\langle N,a\rangle = 2L_1H_2 - 2H_2\big(2HH_2 + 2H + \lambda_1+\lambda_2\big), \tag{17}$$

$$\lambda_1\lambda_2\langle\psi,a\rangle = 4H_2^2 + 4H^2 + 2(\lambda_1+\lambda_2)H + \lambda_1\lambda_2 - 2L_1H. \tag{18}$$

*Example 1* **(Surfaces of $L_1$-1-type)** Totally umbilical surfaces in $\mathbb{S}^3$ are of $L_1$-1-type. Indeed, let $M^2\subset\mathbb{S}^3$ be a totally umbilical surface, then its shape operator $S$ is given by $S = HI$. We know that $H$ and $H_2$ are constants. By taking covariant derivative, we get

$$\nabla_X^0(N + H\psi) = 0,$$

for all $X\in\mathfrak{X}(M^2)$, and then $N + H\psi = b$, for a constant vector $b$. By using this in (9), we deduce

$$L_1\psi = 2H_2 b + \lambda\psi, \quad \lambda = -2H(1+H_2).$$

If $\lambda\neq 0$, then we write

$$\psi = a + \psi_1, \quad a = -\frac{2H_2}{\lambda}b, \quad \psi_1 = \psi + \frac{2H_2}{\lambda}b,$$

with $L_1\psi_1 = \lambda\psi_1$, i.e., $M^2$ is of $L_1$-1-type.

In the case $\lambda = 0$, the surface $M^2$ is totally geodesic ($H = H_2 = 0$) and then (9) yields $L_1\psi = 0$, showing that $M^2$ is of $L_1$-1-type.

By using [2], we easily deduce the following proposition.

**Proposition 7** *Let $\psi : M^2\to\mathbb{S}^3\subset\mathbb{R}^4$ be an isometric immersion. Then $\psi$ is of $L_1$-1-type if and only if $M^2$ is an open portion of a 2-sphere $\mathbb{S}^2(r)$.*

*Example 2* **(Surfaces of $L_1$-2-type)** We will see that the standard Riemannian product $M_r^2 = \mathbb{S}^1(\sqrt{1-r^2})\times\mathbb{S}^1(r)\subset\mathbb{S}^3$, $0<r<1$, is of $L_1$-2-type in $\mathbb{R}^4$. Let us consider

$$M^2 = \{x = (x_1,x_2,x_3,x_4)\in\mathbb{S}^3 \mid x_3^2 + x_4^2 = r^2\}.$$

In this case, the Gauss map on $M^2$ is given by

$$N(x) = \left(\frac{-r}{\sqrt{1-r^2}}x_1, \frac{-r}{\sqrt{1-r^2}}x_2, \frac{\sqrt{1-r^2}}{r}x_3, \frac{\sqrt{1-r^2}}{r}x_4\right),$$

and its principal curvatures are

$$\kappa_1 = \frac{r}{\sqrt{1-r^2}} \quad \text{and} \quad \kappa_2 = \frac{-\sqrt{1-r^2}}{r}.$$

If we put $\psi_1 = (x_1, x_2, 0, 0)$ and $\psi_2 = (0, 0, x_3, x_4)$, it is easy to see that $\psi = \psi_1 + \psi_2$, and by using 9, we get that

$$L_1\psi_1 = \lambda_1\psi_1 \quad \text{and} \quad L_1\psi_2 = \lambda_2\psi_2, \quad \text{with} \quad \lambda_1 = \frac{1}{r\sqrt{1-r^2}} \quad \text{and} \quad \lambda_2 = -\lambda_1.$$

Therefore, $M^2$ is of $L_1$-2-type in $\mathbb{R}^4$.

## 4 Main Results

**Theorem 8** *Let $\psi : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an orientable surface of $L_1$-2-type. Then $M^2$ has constant mean curvature if and only if $M^2$ is an open portion of a standard Riemannian product $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^1(r)$, $0 < r < 1$.*

*Proof* Let $M^2$ be a surface of $L_1$-2-type with constant mean curvature. Our goal is to prove that the scalar curvature $H_2$ of $M^2$ is constant. Otherwise, let us consider the non-empty open set

$$\mathcal{U}_2 = \{p \in M^2 \mid \nabla H_2^2(p) \neq 0\}.$$

By taking covariant derivative in (18), we have $\lambda_1\lambda_2 a^\top = 4\nabla H_2^2$. Using this in (16), we deduce $H_2 = 0$, which is a contradiction.

Therefore, $M^2$ is an isoparametric surface in $\mathbb{S}^3$, and then either $M^2$ is an open portion of a 2-sphere $\mathbb{S}^2(r)$, $0 < r \leq 1$, or $M^2$ is an open portion of a Riemannian product $M_r^2$, $0 < r < 1$. Since the totally umbilical surfaces are of $L_1$-1-type, the result follows. □

**Theorem 9** *Let $\psi : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an orientable surface of $L_1$-2-type. Then $M^2$ has constant Gaussian curvature if and only if $M^2$ is an open portion of a standard Riemannian product $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^1(r)$, $0 < r < 1$.*

*Proof* Let $M^2$ be a surface of $L_1$-2-type with constant Gaussian curvature $K$, and consider the open set

$$\mathcal{U} = \{p \in M^2 \mid \nabla H^2(p) \neq 0\}.$$

Our goal is to show that $\mathcal{U}$ is empty. Suppose it is not empty.

By taking covariant derivative in (17), and using that $H_2$ is constant, we obtain

$$\lambda_1\lambda_2 S a^\top = 4H_2(H_2+1)\nabla H.$$

From (16) and bearing in mind that $S\circ P = H_2 I$, we have $\lambda_1\lambda_2 S a^\top = -4H_2\nabla H$, and therefore

$$H_2(H_2+2)\nabla H = 0.$$

Consequently, on $\mathcal{U}$ we have either $H_2 = -2$ or $H_2 = 0$. We will study each case separately.

*Case 1*: $H_2 = -2$. By applying the operator $L_1$ on both sides of (17) and using (18) we get

$$\lambda_1\lambda_2 L_1 \langle N, a\rangle = 4\big[\lambda_1\lambda_2 \langle \psi, a\rangle - 4H^2 - 2(\lambda_1+\lambda_2)H - \lambda_1\lambda_2 - 16\big].$$

On the other hand, (10) leads to

$$\lambda_1\lambda_2 \langle N, a\rangle H - \lambda_1\lambda_2 \langle \psi, a\rangle = \lambda_1\lambda_2 \langle a, \psi\rangle - 4H^2 - 2(\lambda_1+\lambda_2)H - \lambda_1\lambda_2 - 16,$$

and using (17) we find that

$$\lambda_1\lambda_2 \langle \psi, a\rangle = -2H^2 + 3(\lambda_1+\lambda_2)H + \frac{1}{2}(\lambda_1\lambda_2 + 16). \tag{19}$$

Taking gradients in (19), and using (16) and (3), we obtain

$$\big[-4H + 3(\lambda_1+\lambda_2)\big]\nabla H = -4P_1(\nabla H) = -8H\nabla H + 4S(\nabla H), \tag{20}$$

that is,

$$S(\nabla H) = \frac{4H + 3(\lambda_1+\lambda_2)}{4}\nabla H.$$

Now, by applying the operator $S$ on both sides of the first equality of (20), and bearing in mind that $S\circ P = -2I$, we obtain

$$S(\nabla H) = \frac{8}{-4H + 3(\lambda_1+\lambda_2)}\nabla H.$$

The last two equations for $S(\nabla H)$ imply that $H$ is constant on $\mathcal{U}$, which is a contradiction.

*Case 2*: $H_2 = 0$. Let us suppose $\kappa_1 = 0$ and $\kappa_2 = 2H \neq 0$ (otherwise, $M^2$ would be a totally geodesic surface and then of $L_1$-1-type). Let $\{E_1, E_2\}$ be a local orthonormal frame of principal directions of $S$ such that $SE_i = \kappa_i E_i$. From Codazzi's equation, we easily obtain

$$\begin{aligned} &\nabla_{E_1} E_1 = 0, && \nabla_{E_1} E_2 = 0, \\ &\nabla_{E_2} E_1 = -\tfrac{\alpha}{H} E_2, && \nabla_{E_2} E_2 = \tfrac{\alpha}{H} E_1 \quad [E_1, E_2] = \tfrac{\alpha}{H} E_2, \end{aligned}$$

where $\alpha = E_1(H)$. Now, from the definition of curvature tensor, we get

$$\begin{aligned} R(E_1, E_2)E_1 &= \nabla_{[E_1,E_2]} E_1 - \nabla_{E_1} \nabla_{E_2} E_1 + \nabla_{E_2} \nabla_{E_1} E_1 \\ &= \Big[ E_1\big(\tfrac{\alpha}{H}\big) - \big(\tfrac{\alpha}{H}\big)^2 \Big] E_2, \end{aligned}$$

and from the Gauss equation we have $R(E_1, E_2)E_1 = E_2$. By equating the last two equations, we deduce

$$H E_1(\alpha) = H^2 + 2\alpha^2. \tag{21}$$

On the other hand, from the definition of $L_1$, see (4), and after a little calculation, we obtain

$$L_1 H = \kappa_2 \left\langle E_1, \nabla_{E_1} \nabla H \right\rangle + \kappa_1 \left\langle E_2, \nabla_{E_2} \nabla H \right\rangle = 2H E_1(\alpha). \tag{22}$$

By using (21) and (22), (18) can be rewritten as

$$\lambda_1 \lambda_2 \langle \psi, a \rangle = 2(\lambda_1 + \lambda_2)H + \lambda_1 \lambda_2 - 8\alpha^2. \tag{23}$$

Taking covariant derivative along $E_1$ here, we have

$$E_1(\lambda_1 \lambda_2 \langle \psi, a \rangle) = 2(\lambda_1 + \lambda_2)\alpha - 16\alpha E_1(\alpha). \tag{24}$$

On the other hand, from (18), we get $\lambda_1 \lambda_2 a^\top = -8H\alpha E_1$, and therefore

$$E_1(\lambda_1 \lambda_2 \langle \psi, a \rangle) = \left\langle \lambda_1 \lambda_2 a^\top, E_1 \right\rangle = -8H\alpha.$$

This equation, jointly with (24), implies that $(\lambda_1 + \lambda_2)\alpha - 8\alpha E_1(\alpha) = -4H\alpha$. Since $\alpha \neq 0$, see (21), we deduce

$$8E_1(\alpha) = 4H + \lambda_1 + \lambda_2. \tag{25}$$

From here and using (22) we get $4L_1 H = 4H^2 + (\lambda_1 + \lambda_2)H$. By using this in (18), we find

$$\lambda_1 \lambda_2 \langle \psi, a \rangle = 2H^2 + \frac{3}{2}(\lambda_1 + \lambda_2)H + \lambda_1 \lambda_2. \tag{26}$$

Taking gradient here, and using (16) and (3), we obtain

$$\left[4H + \frac{3}{2}(\lambda_1 + \lambda_2)\right]\nabla H = -4P(\nabla H) = -8H\nabla H + 4S(\nabla H), \tag{27}$$

that is,

$$S(\nabla H) = \left(3H + \frac{3}{8}(\lambda_1 + \lambda_2)\right)\nabla H.$$

On the other hand, by applying the operator $S$ on both sides of the first equality of (27), and bearing in mind that $S \circ P = 0$, we obtain

$$\left[4H + \frac{3}{2}(\lambda_1 + \lambda_2)\right]S(\nabla H) = 0.$$

The last two equations imply that $H$ is constant on $\mathcal{U}$, which is a contradiction.

We have proved that if $M^2$ is a $L_1$-2-type surface with constant Gaussian curvature, then its mean curvature is constant. Then reasoning as in the proof of Theorem 8 we deduce that $M^2$ is an open portion of a Riemannian product $M_r^2$, $0 < r < 1$. This finishes the proof of Theorem 9. □

A surface in $\mathbb{S}^3$ is said to have a *constant principal curvature* if one of its principal curvatures is constant.

**Theorem 10** *Let $\psi : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an orientable surface of $L_1$-2-type. Then $M^2$ has a constant principal curvature if and only if $M^2$ is an open portion of a standard Riemannian product $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^1(r)$, $0 < r < 1$.*

*Proof* Let $M^2$ be a surface of $L_1$-2-type and assume that $\kappa_1$ is a non-zero constant (otherwise, $H_2 = 0$ and Theorem 9 applies). Consider the open set

$$\mathcal{U} = \left\{p \in M^2 \mid \nabla\kappa_2^2(p) \neq 0\right\}.$$

Our goal is to show that $\mathcal{U}$ is empty.

Otherwise, Eqs. (16)–(18) of $L_1$-2-type can be rewritten in terms of $\kappa_2$ as follows:

$$\lambda_1\lambda_2 a^\top = [-6\kappa_1^2\kappa_2 - 2(\kappa_1 + \kappa_2)]\nabla\kappa_2 + 2S(\nabla\kappa_2), \tag{28}$$

$$\lambda_1\lambda_2 \langle N, a\rangle = 2\kappa_1 L_1\kappa_2 - 2\kappa_1\kappa_2\big[(\kappa_1 + \kappa_2)(\kappa_1\kappa_2 + 1) + \lambda_1 + \lambda_2\big], \tag{29}$$

$$\lambda_1\lambda_2 \langle \psi, a\rangle = 4\kappa_1^2\kappa_2^2 + (\kappa_1 + \kappa_2)^2 + (\lambda_1 + \lambda_2)(\kappa_1 + \kappa_2) + \lambda_1\lambda_2 - L_1\kappa_2. \tag{30}$$

From (29) and (30), we find

$$\lambda_1\lambda_2 \langle N, a\rangle = -2\kappa_1\lambda_1\lambda_2 \langle \psi, a\rangle + 2\kappa_1 \times \left[3\kappa_1^2\kappa_2^2 + \kappa_1^2 + \kappa_1\kappa_2 + (\lambda_1 + \lambda_2)\kappa_1 + \lambda_1\lambda_2 - \kappa_1\kappa_2^3\right].$$

By taking gradient here, we obtain

$$-\lambda_1\lambda_2 S a^\top = -2\kappa_1\lambda_1\lambda_2 a^\top + 2\kappa_1^2\Big[1 + 6\kappa_1\kappa_2 - 3\kappa_2^2\Big]\nabla\kappa_2. \tag{31}$$

On the other hand, by using $S \circ P = H_2 I$, we get

$$\lambda_1\lambda_2 S a^\top = -6\kappa_1^2\kappa_2 S(\nabla\kappa_2) - 2\kappa_1\kappa_2\nabla\kappa_2. \tag{32}$$

Now, from (28), (31), and (32), we deduce

$$(3\kappa_1\kappa_2 + 2)S(\nabla\kappa_2) = (-3\kappa_1\kappa_2^2 + (12\kappa_1^2 + 1)\kappa_2 + 3\kappa_1)\nabla\kappa_2.$$

Since $3\kappa_1\kappa_2 + 2 \neq 0$ (otherwise, $\kappa_2$ would be constant), we deduce

$$S(\nabla\kappa_2) = f(\kappa_1,\kappa_2)\nabla\kappa_2,\ f(\kappa_1,\kappa_2) = \frac{-3\kappa_1\kappa_2^2 + (12\kappa_1^2 + 1)\kappa_2 + 3\kappa_1}{(3\kappa_1\kappa_2 + 2)}.$$

This equation implies that either $f(\kappa_1,\kappa_2) = \kappa_1$ or $f(\kappa_1,\kappa_2) = \kappa_2$. In any case it follows that $\kappa_2$ is constant on $\mathcal{U}$, and this is a contradiction. This finishes the proof of Theorem 10. □

As a consequence of Theorems 8, 9, and 10, we have the following characterization of $L_1$-2-type surfaces in $\mathbb{S}^3$.

**Theorem 11** *Let $\psi : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an orientable surface of $L_1$-2-type. Then either $M^2$ is an open portion of a standard Riemannian product $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^1(r)$, $0 < r < 1$, or $M^2$ has non-constant mean curvature $H$, non-constant Gaussian curvature $K$, and non-constant principal curvatures $\kappa_1$, and $\kappa_2$.*

**Acknowledgements** This work has been partially supported by MINECO (Ministerio de Economía y Competitividad) and FEDER (Fondo Europeo de Desarrollo Regional), Project MTM2015-65430-P, and by Fundación Séneca (Región de Murcia) Project 19901/GERM/15, Spain

## References

1. Alías, L.J., Gürbüz, N.: An extension of Takahashi theorem for the linearized operators of the higher order mean curvatures. Geom. Dedic. **121**, 113–127 (2006)
2. Alías, L.J., Kashani, M.B.: Hypersurfaces in space forms satisfying the condition $L_k\psi = A\psi + b$. Taiwan. J. Math. **14**, 1957–1978 (2010)
3. Aminian, M., Kashani, S.M.B.: $L_k$-biharmonic hypersurfaces in the Euclidean space. Taiwan. J. Math. **19**, 861–874 (2015)
4. Barros, M., Garay, O.J.: 2-type surfaces in $S^3$. Geom. Dedic. **24**, 329–336 (1987)
5. Chen, B.Y.: Total Mean Curvature and Submanifolds of Finite Type. World Scientific Publisher, Singapore (1984)
6. Chen, B.Y.: A report on submanifolds of finite type. Soochow J. Math. **22**, 117–337 (1996)
7. Chen, B.Y.: Some open problems and conjectures on submanifolds of finite type: recent development. Tamkang J. Math. **45**, 87–108 (2014)
8. Cheng, S.Y., Yau, S.T.: Hypersurfaces with constant scalar curvature. Math. Ann. **225**, 195–204 (1977)

9. Hasanis, Th, Vlachos, Th: A local classification of 2-type surfaces in $S^3$. Proc. Am. Math. Soc. **112**, 533–538 (1991)
10. Kashani, S.M.B.: On some $L_1$-finite type (hyper)surfaces in $\mathbb{R}^{n+1}$. Bull. Korean Math. Soc. **46**, 35–43 (2009)
11. Mahammadpouri, A., Pashaie, F.: $L_1$-biharmonic hypersurfaces with three distinct principal curvatures in Euclidean 5-space. Funct. Anal. Approx. Comput. **7**, 67–75 (2015)
12. O'Neill, B.: Semi-Riemannian Geometry With Applications to Relativity. Academic Press, New York (1983)
13. Reilly, R.: Variational properties of functions of the mean curvatures for hypersurfaces in space forms. J. Differ. Geom. **8**, 465–477 (1973)