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On α-Tall Modules

M. Davoudian[1] · **N. Shirali**[1]

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Abstract In this article, we introduce and study α-tall modules. We show that an α-tall module, where $\alpha \geq 0$, is a tall module, i.e. M contains a submodule N such that N and $\frac{M}{N}$ are both non-Noetherian. We observe that every submodule of α-tall modules is countably generated, where α is countable. It is shown that if M is a β-atomic module, where $\beta = \alpha + 2$, for some ordinal α, then M is α-tall. It is also proved that if M is an α-atomic module, where α is a limit ordinal, then M is both an α-tall and α-short module.

Keywords α-Tall module · Tall module · Short module · α-Short module · Noetherian dimension

Mathematics Subject Classification Primary 16P60 · Secondary 16P20

1 Introduction

Lemonnier [20] introduced the concept of deviation and codeviation of an arbitrary poset, which in particular, when applied to the lattice of all submodules of a module M_R give the concepts of Krull dimension (in the sense of Rentschler and Gabriel, see [6,10,19]) and dual Krull dimension of M, respectively. Later, Chambless in [8] studied dual Krull dimension and called it N-dimension. Karamzadeh also extensively studied the latter dimension in his Ph.D. thesis [12] and called it Noetherian dimension.

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✉ M. Davoudian
m.davoudian@scu.ac.ir; davoudian_mm@yahoo.com

1 Department of Mathematics, Shahid Chamran University of Ahvaz, Ahvaz, Iran

Roberts in [23] calls this dual dimension again Krull dimension. The latter dimension is also called dual Krull dimension in some other articles, see for example [1–5]. The Noetherian dimension of an R-module M is denoted by n-dim M and by k-dim M; we denote the Krull dimension of M, see [10–16,18,19,21,22] for more details. It is convenient, when we are dealing with the latter dimensions, to begin our list of ordinals with -1. If an R-module M has Noetherian dimension and α is an ordinal number, then M is called α-atomic if $n\text{-dim}\, M = \alpha$ and $n\text{-dim}\, N < \alpha$, for all proper submodule N of M. An R-module M is called atomic if it is α-atomic for some ordinal α (note, atomic modules are also called conotable, critical and N-critical in some other articles for example [3,8,20]). In [24], Sarath defines an R-module M to be tall if M contains a submodule N such that N and $\frac{M}{N}$ are both non-Noetherian. Bilhan and Smith [7] introduced short modules. In [9], Davoudian, Karamzadeh and Shirali introduced α-short modules, (a 0-short module is just a short module, i.e. for each submodule N of M either N or $\frac{M}{N}$ is Noetherian). They show that if M is an α-short module, then the Noetherian dimension of M is either α or $\alpha + 1$. In this article, we introduce and study α-tall modules. We show that an α-tall module where $\alpha \geq 0$ is tall. Our aim of studying the concept of α-tall modules, in this article, is twofold. We aim to extend the concept of tall modules and at the same time provide a dual to the concept of α-short modules. Tall modules are not necessarily Noetherian and one dose not know how far they are from being Noetherian. Although α-tall modules for $\alpha > -1$ are similarly non-Noetherian, but the ordinal α measures how these modules deviate from being Noetherian (note, it is observed that if M is α-tall, then $\alpha < n\text{-dim}\, M \leq \alpha + 2$). This is an advantage for α-tall modules over tall modules. We observe that if M is an α-tall module, then each non-zero submodule (each non-zero factor module) of M which is not simple is β-tall for some $\beta \leq \alpha$. All modules in this paper are assumed to be unital modules over an associative ring with unit.

2 Preliminaries

In this section, we recall some useful facts on modules with Noetherian dimension [8,10,12,20].

We recall that the Noetherian dimension of an R-module M is defined by transfinite recursion as follows: If $M = 0$, then $n\text{-dim}\, M = -1$. If α is an ordinal and $n\text{-dim} \not< \alpha$, then $n\text{-dim}\, M = \alpha$, provided there is no infinite ascending chain $M_1 \subseteq M_2 \subseteq \ldots$ of submodules of M such that $n\text{-dim}\, \frac{M_{i+1}}{M_i} \not< \alpha$ for each $i \geq 1$. It is possible that there is no ordinal α such that $n\text{-dim}\, M = \alpha$. In that case, we say M dose not have Noetherian dimension

Lemma 2.1 *If M is an R-module and for each submodule N of M, either N or $\frac{M}{N}$ has Noetherian dimension, then so does M.*

Proposition 2.2 *If M is an R-module, then we have $n\text{-dim}\, M = \sup\{n\text{-dim}\, \frac{M}{N} :$ N is a non-zero proper submodule of $M\}$, if either side exists.*

Proposition 2.3 *If M is an R-module, then we have $n\text{-dim}\, M \leq \sup\{n\text{-dim}\, N :$ N is a proper submodule of $M\}+1$, if either side exists.*

Lemma 2.4 *If N is a submodule of an R-module M, then* $n\text{-dim}\, M = \sup\{n\text{-dim}\, \frac{M}{N}, n\text{-dim}\, N\}$*, if either side exists.*

Remark 2.5 An R-module M is called α-short, if for each submodule N of M, either $n\text{-dim}\, N \leq \alpha$ or $n\text{-dim}\, \frac{M}{N} \leq \alpha$ and α is the least ordinal number with this property, see [9]. Clearly each 0-short module is just a short module, we recall that an R-module M is called a short module if for each submodule N of M, either N or $\frac{M}{N}$ is Noetherian, see [7].

We cite the following facts from [9].

Remark 2.6 If M is an R-module with $n\text{-dim}\, M = \alpha$, then M is β-short for some $\beta \leq \alpha$.

In view of Remark 2.5 and Lemma 2.1, the following corollary is now evident.

Corollary 2.7 *Let M be an α-short module. Then M has Noetherian dimension and* $n\text{-dim}\, M \geq \alpha$.

The following is also in [9, Proposition 1.12]. We give the proof for completeness.

Proposition 2.8 *If M is an α-short R-module, then either* $n\text{-dim}\, M = \alpha$ *or* $n\text{-dim}\, M = \alpha + 1$.

Proof Clearly in view of Remark 2.6 and Corollary 2.7, we have $n\text{-dim}\, M \geq \alpha$. If $n\text{-dim}\, M \neq \alpha$, then $n\text{-dim}\, M \geq \alpha + 1$. Now let $M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots$ be any ascending chain of submodules of M. If there exists some k such that $n\text{-dim}\, \frac{M}{M_k} \leq \alpha$, then $n\text{-dim}\, \frac{M_{i+1}}{M_i} \leq n\text{-dim}\, \frac{M}{M_i} = n\text{-dim}\, \frac{M/M_k}{M_i/M_k} \leq n\text{-dim}\, \frac{M}{M_k} \leq \alpha$ for each $i \geq k$. Otherwise $n\text{-dim}\, M_i \leq \alpha$ (note, M is α-short) for each i, hence $n\text{-dim}\, \frac{M_i}{M_{i+1}} \leq \alpha$ for each i. Thus in any case, there exists an integer k such that for each $i \geq k$, $n\text{-dim}\, \frac{M_{i+1}}{M_i} \leq \alpha$. This shows that $n\text{-dim}\, M \leq \alpha + 1$, i.e. $n\text{-dim}\, M = \alpha + 1$. □

3 α-Tall Modules

In [24], Sarath defines an R-module M to be tall if M contains a submodule N such that N and $\frac{M}{N}$ are both non-Noetherian. In this section, we introduce and study α-tall modules. We observe that an α-tall module, where $\alpha \geq 0$, is tall. We show that if M is an α-tall module, then each non-zero submodule (non-zero factor module) of M which is not simple is β-tall for some $\beta \leq \alpha$. In particular, we show that each α-tall module has Noetherian dimension and its Noetherian dimension is either $\alpha + 1$ or $\alpha + 2$. We observe that if M is an α-tall module, where α is a countable ordinal, then every submodule of M is countably generated. If an R-module M is not tall, then it is a short module and by Davoudian et al. [9, Proposition 1.12], and M has Noetherian dimension and $n\text{-dim}\, M \leq 1$.

Next, we give our definition of α-tall modules.

Definition 3.1 Let M be an R-module and β be an ordinal number. Put $\psi_M = \{\beta :$ *there exists a submodule N of M such that* $n\text{-dim}\, N > \beta$ *and* $n\text{-dim}\, \frac{M}{N} > \beta\}$. If $\psi_M \neq \emptyset$ and $\alpha = \sup \psi_M$, then M is called α-tall.

Remark 3.2 A simple module M is not α-tall for any ordinal number α, and every non-zero Noetherian module M which is not simple, is -1-tall.

The following results provide a criterion for non-simple α-tall modules.

Lemma 3.3 *Let M be a non-zero R-module which is not simple. Then $\psi_M \neq \emptyset$ if and only if M has Noetherian dimension.*

Proof If $\psi_M \neq \emptyset$, then there exists a proper non-zero submodule N of M such that N and $\frac{M}{N}$ both have Noetherian dimension. Thus M has Noetherian dimension, see Lemma 2.4. Conversely, since M is not simple, it has a non-zero proper submodule, N say. By our assumption M has Noetherian dimension, thus both N and $\frac{M}{N}$ have Noetherian dimension, see Lemma 2.4. This implies that $\psi_M \neq \emptyset$. □

In view of Lemma 3.3, we have the following results.

Corollary 3.4 *Let M be a non-zero R-module which is not simple. Then $\psi_M = \emptyset$ if and only if M does not have Noetherian dimension.*

Corollary 3.5 *Let M be a non-zero R-module which is not simple. Then M has Noetherian dimension if and only if it is α-tall for some ordinal number α.*

Lemma 3.6 *Let M be a non-zero R-module which is not simple. Then M is tall if and only if either $\psi_M = \emptyset$ or M is α-tall for some ordinal number $\alpha \geq 0$.*

Proof If M is α-tall for some ordinal number $\alpha \geq 0$, then there exists a submodule N of M such that $n\text{-dim}\, N > 0$ and $n\text{-dim}\, \frac{M}{N} > 0$, therefore N and $\frac{M}{N}$ both are non-Noetherian and M is a tall module. Now let $\psi_M = \emptyset$, then by Corollary 3.4, M dose not have Noetherian dimension, hence M is a tall module (note, if M is not tall, then M is a short module and by Corollary 2.7, M has Noetherian dimension, which is a contradiction). Conversely, let M be a tall module, it is clear that M is not simple. We consider two cases. If M does not have Noetherian dimension, then by Corollary 3.4, $\psi_M = \emptyset$. Otherwise, M has Noetherian dimension. Since M is a tall module, there exists a submodule N of M, such that N and $\frac{M}{N}$ are both non-Noetherian. Thus $n\text{-dim}\, N > 0$ and $n\text{-dim}\, \frac{M}{N} > 0$, see Lemma 2.4. Therefore $0 \in \psi_M$ and hence $\psi_M \neq \emptyset$ and M is α-tall for some ordinal number $\alpha \geq 0$. □

Proposition 3.7 *If M is an α-tall R-module, then either $n\text{-dim}\, M = \alpha + 1$ or $n\text{-dim}\, M = \alpha + 2$.*

Proof For each submodule N of M, we get either $n\text{-dim}\, N \leq \alpha + 1$ or $n\text{-dim}\, \frac{M}{N} \leq \alpha + 1$ (note, if there exists a submodule N of M such that $n\text{-dim}\, N > \alpha + 1$ and $n\text{-dim}\, \frac{M}{N} > \alpha + 1$, then $\alpha + 1 \in \psi_M$ and M is γ-tall for some $\gamma \geq \alpha + 1$, which is a contradiction). Thus M is $\alpha + 1$-short and by Proposition 2.8, M has Noetherian dimension and $n\text{-dim}\, M = \alpha + 1$ or $n\text{-dim}\, M = \alpha + 2$. □

Here are some elementary properties of α-tall modules.

Corollary 3.8 *If M is a -1-tall module, then either $n\text{-dim}\, M = 1$ or M is a non-zero Noetherian module which is not simple.*

Corollary 3.9 *Every* -1*-tall module is a short module and if* M *is a non-zero short module which is not simple, then it is* -1*-tall.*

It is well known that any module with Noetherian dimension has finite uniform dimension. In view of Proposition 3.7, the following corollary is now evident.

Corollary 3.10 *Every* α*-tall module has finite uniform dimension.*

The following corollary shows that there exists a tall module which is not α-tall.

Corollary 3.11 *Let* M *be an* R*-module. If* M *has infinite uniform dimension, then it is tall.*

Proof Since M has infinite uniform dimension, we infer that M contains an infinite direct sum such as $X = \sum_{i\in\mathbb{N}} \oplus M_i$. Now we put $N = \sum_{i=2k} \oplus M_i$. It is clear that N and $\frac{M}{N}$ both are non-Noetherian. Hence M is tall. □

In view of Corollaries 3.10, 3.11 and Lemma 3.6, we have the following results.

Corollary 3.12 *Let* M *be an* R*-module. If* M *has infinite uniform dimension, then it is a tall module which is not* α*-tall, i.e.* $\psi_M = \emptyset$.

Corollary 3.13 *If* M *is an* α*-tall module, where* α *is a countable ordinal number, then every submodule of* M *is countably generated.*

Proof By Proposition 3.7, n-dim $M = \alpha+1$ or n-dim $M = \alpha+2$. We know that every submodule of a module with countable Noetherian dimension is countably generated, see [17, Corollary 1.8]. Hence we are through. □

For the atomic modules, the following facts are important.

Lemma 3.14 *Let* M *be an* $\alpha+2$*-atomic module. Then* M *is an* α*-tall module.*

Proof Since M is $\alpha+2$-atomic, for each proper submodule N of M, we get n-dim $N \leq \alpha+1$ and n-dim $\frac{M}{N} = \alpha+2$. If for each proper submodule N of M we have n-dim $N \leq \alpha$, then n-dim $M \leq \sup\{n\text{-dim } N : N \subsetneq M\} + 1 \leq \alpha + 1$ and this is a contradiction. This shows that there exists a proper submodule N of M such that n-dim $N > \alpha$. But n-dim $\frac{M}{N} = \alpha + 2 > \alpha$, therefore $\alpha \in \psi_M$. For each proper submodule N of M, n-dim $N \leq \alpha + 1$, therefore $\alpha + 1 \notin \psi_M$. Hence M is an α-tall module. □

Lemma 3.15 *Let* M *be an* α*-atomic* R*-module, where* α *is a limit ordinal number. Then* M *is an* α*-tall module.*

Proof Let $\beta < \alpha$ be an ordinal number. If for each proper submodule N of M we have n-dim $N \leq \beta$, then n-dim $M \leq \sup\{n\text{-dim } N : N \subseteq M\} + 1 \leq \beta + 1 < \alpha$ and it is a contradiction. Thus there exists a proper submodule N of M such that n-dim $N > \beta$. Since M is α-atomic, we must have n-dim $\frac{M}{N} = \alpha > \beta$. This shows that $\beta \in \psi_M$. If $\gamma \geq \alpha$ is an ordinal number, then for each submodule N of M we have n-dim $N < \gamma$, thus $\gamma \notin \psi_M$. Therefore $\sup \psi_M = \alpha$, i.e. M is α-tall. □

In view of previous Lemma and [9, Proposition 1.16], we have the following corollary.

Corollary 3.16 *Let M be an α-atomic R-module, where α is a limit ordinal number. Then M is both α-tall and α-short.*

The previous corollary will rise the natural question, namely, what are α-tall modules which are also α-short, in general?

The following result is evident.

Corollary 3.17 *Let M be a tall module. If M has Noetherian dimension, then $n\text{-dim}\, M \geq 1$.*

Proof There exists a submodule N of M such that N and $\frac{M}{N}$ both are non-Noetherian. Therefore $n\text{-dim}\, N > 0$ and $n\text{-dim}\, \frac{M}{N} > 0$, see Lemma 2.4. Now by Lemma 2.4, we have $n\text{-dim}\, M > 0$, i.e. $n\text{-dim}\, M \geq 1$ and we are done. □

Lemma 3.18 *If M is an α-tall module, then each non-zero submodule (non-zero factor module) of M which is not simple is β-tall for some $\beta \leq \alpha$.*

Proof By Proposition 3.7, $n\text{-dim}\, M = \alpha + 1$ or $n\text{-dim}\, M = \alpha + 2$. Thus for each submodule N of M, N and $\frac{M}{N}$ both have Noetherian dimension, see Lemma 2.4. Now let N and $\frac{M}{N}$ are non-zero R-modules which are not simple, therefore N and $\frac{M}{N}$ both are β-tall for some ordinal number β, see Corollary 3.5. Hence in view of Lemma 2.4 and Proposition 3.7, we infer that $\beta \leq \alpha+1$. We claim that $\beta \neq \alpha+1$. Since if $\beta = \alpha+1$, then there exists a non-zero proper submodule N_1 of N (similarly there exists a non-zero proper submodule $\frac{N_1}{N}$ of $\frac{M}{N}$) such that $n\text{-dim}\, N_1 > \alpha + 1$ and $n\text{-dim}\, \frac{N}{N_1} > \alpha + 1$ (similarly $n\text{-dim}\, \frac{N_1}{N} > \alpha + 1$ and $n\text{-dim}\, \frac{M}{N_1} > \alpha + 1$). Therefore $n\text{-dim}\, N_1 > \alpha + 1$ and $n\text{-dim}\, \frac{M}{N_1} > \alpha + 1$ (similarly $n\text{-dim}\, \frac{M}{N_1} > \alpha + 1$ and $n\text{-dim}\, N_1 > \alpha + 1$). Thus M is γ-tall for some $\gamma \geq \alpha + 1$, which is a contradiction. □

Proposition 3.19 *Let N be a submodule of an R-module M such that N is α-tall and $\frac{M}{N}$ is β-tall. Let $\mu = \sup\{\alpha, \beta\}$, then M is γ-tall such that $\mu \leq \gamma \leq \mu + 1$.*

Proof Since N is α-tall, by Proposition 3.7, $n\text{-dim}\, N = \alpha + 1$ or $n\text{-dim}\, N = \alpha + 2$. Similarly since $\frac{M}{N}$ is β-tall, either $n\text{-dim}\, \frac{M}{N} = \beta+1$ or $n\text{-dim}\, \frac{M}{N} = \beta+2$. We infer that M has Noetherian dimension and $n\text{-dim}\, M = \sup\left\{n\text{-dim}\, N, n\text{-dim}\, \frac{M}{N}\right\}$, see Lemma 2.4. Therefore $\mu + 1 \leq n\text{-dim}\, M \leq \mu + 2$. By Corollary 3.5, M is γ-tall for some ordinal number γ and by Proposition 3.7, $\gamma + 1 \leq n\text{-dim}\, M \leq \gamma + 2$. This shows that $\gamma = \mu$ or $\gamma = \mu + 1$ (note, by Lemma 3.18, we always have $\mu \leq \gamma$) and we are done. □

We also have the following two facts.

Lemma 3.20 *Let N be a simple submodule of an R-module M. If $\frac{M}{N}$ is β-tall, then M is a γ-tall module for some $\beta \leq \gamma \leq \beta + 1$.*

Lemma 3.21 *Let N be a maximal submodule of an R-module M. If N is an α-tall module, then M is γ-tall for some $\alpha \leq \gamma \leq \alpha + 1$.*

Proposition 3.22 *Let M be a non-zero R-module which is not simple and α be an ordinal number. If every non-zero proper factor module of M which is not simple, is γ-tall for some ordinal number $\gamma \leq \alpha$, then M is μ-tall where $\mu \leq \alpha$.*

Proof It is clear that for each simple factor module $\frac{M}{N}$ of M, n-dim $\frac{M}{N} = 0 \leq \alpha + 2$. Now let $0 \neq N \subsetneq M$ be any submodule such that $\frac{M}{N}$ is γ-tall for some ordinal number γ with $\gamma \leq \alpha$. We infer that n-dim $\frac{M}{N} \leq \gamma + 2 \leq \alpha + 2$, by Proposition 3.7. Thus we have n-dim $M = \sup\{n\text{-dim } \frac{M}{N} : N \neq 0\}$, by Proposition 2.2. This shows that n-dim $M \leq \alpha + 2$. If n-dim $M \leq \alpha + 1$, then it is clear that M is μ-tall for some $\mu \leq \alpha$. Hence we may suppose that n-dim $M = \alpha + 2$. If $0 \neq N \subsetneq M$ is a submodule of M, then we are to show that either n-dim $\frac{M}{N} \leq \alpha + 1$ or n-dim $N \leq \alpha + 1$ (note, this implies that $\alpha + 1 \notin \psi_M$ and hence $\sup \psi_M \leq \alpha$). To this end, let us suppose that n-dim $\frac{M}{N} = \alpha + 2$ and show that n-dim $N \leq \alpha + 1$. Now let $0 \neq N' \subsetneq N \subsetneq M$. Since n-dim $\frac{M/N'}{N/N'} = n$-dim $\frac{M}{N} = \alpha + 2$, we must have n-dim $\frac{N}{N'} \leq \alpha + 1$ (note, $\frac{M}{N'}$ is γ-tall for some $\gamma \leq \alpha$). But n-dim $N = \sup\{n\text{-dim } \frac{N}{N'} : 0 \neq N' \subseteq N\} \leq \alpha + 1$ and we are through. □

Proposition 3.23 *Let α be an ordinal number and M be a non-zero R-module which is not simple . If every non-zero proper submodule of M which is not simple is γ-tall for some ordinal number $\gamma \leq \alpha$. Then either* n-dim $M = \alpha + 2$ *or M is μ-tall for some ordinal number $\mu \leq \alpha$. In particular, M is μ-tall for some ordinal number $\mu \leq \alpha + 1$.*

Proof If M has no non-simple proper submodule, than M is -1-tall. Let N be a simple proper submodule of M, then n-dim $N = 0 \leq \alpha + 2$. Now let $0 \neq N \subsetneq M$ be any non-simple submodule of M. Since N is γ-tall for some ordinal number $\gamma \leq \alpha$, we infer that n-dim $N \leq \gamma + 2 \leq \alpha + 2$, by Proposition 3.7. In view of Proposition 2.3, we infer that n-dim $M \leq \alpha + 3$. If n-dim $M \leq \alpha + 2$, then we are through. Hence we may suppose that n-dim $M = \alpha + 3$ and M is not μ-tall for any $\mu \leq \alpha$ and seek a contradiction. Since M is not μ-tall for any $\mu \leq \alpha$, we infer that there must exists a proper submodule K of M such that n-dim $K \geq \alpha + 2$. But we have already observed that n-dim $K \leq \alpha + 2$, therefore n-dim $K = \alpha + 2$. We now claim that n-dim $\frac{M}{K} \leq \alpha + 2$ which trivially implies that n-dim $M = \alpha + 2$ and this is a contradiction (note, n-dim $M = \alpha + 3$). Let $K \subset N' \subset M$. Since n-dim $K = \alpha + 2$ and N' is γ-tall for some $\gamma \leq \alpha$, we get n-dim $\frac{N'}{K} \leq \alpha + 1$. But n-dim $\frac{M}{K} \leq \sup\left\{n\text{-dim } \frac{N'}{K} : \frac{N'}{K} \subset \frac{M}{K}\right\} + 1 \leq \alpha + 2$ and we are done. The final part is now evident. □

Now we have the following definition.

Definition 3.24 Let M be an R-module with Noetherian dimension. For each ordinal α, we put:

$$G_\alpha(M) = \cap\left\{N : n\text{-dim } \frac{M}{N} \leq \alpha,\, N \subseteq M\right\}$$
$$H_\alpha(M) = \cap\{N : n\text{-dim } N > \alpha,\, N \subseteq M\}.$$

We recall that a submodule N of M is called α-coatomic, where α is an ordinal number, if $\frac{M}{N}$ is α atomic. An R-module N is called coatomic if it is α-coatomic for some ordinal α.

Remark 3.25 If n-dim $M > \alpha$ and N be a submodule of M such that n-dim $\frac{M}{N} \leq \alpha$, then clearly n-dim $N > \alpha$. This shows that $H_\alpha(M) \subseteq G_\alpha(M)$, where n-dim $M > \alpha$. If N is an α-coatomic submodule of M, then n-dim $\frac{M}{N} = \alpha$, thus $G_\alpha(M) \subseteq N$.

The following lemma is now immediate.

Lemma 3.26 *Let M be an R-module with Noetherian dimension and α be an ordinal number. If n-dim $\frac{M}{G_\alpha(M)} \leq \alpha$ and $H_\alpha(G_\alpha(M)) \neq G_\alpha(M)$, then M is γ-tall for some $\gamma \geq \alpha$.*

Proof Since $H_\alpha(G_\alpha(M)) \neq G_\alpha(M)$, we infer that $G_\alpha(M) \neq 0$ and n-dim $M \neq \alpha$. Thus there exists $P \subsetneq G_\alpha(M)$ such that n-dim $P > \alpha$. Since n-dim $\frac{M}{G_\alpha(M)} \leq \alpha$, we get n-dim $\frac{M/P}{G_\alpha(M)/P} = n$-dim $\frac{M}{G_\alpha(M)} \leq \alpha$. If n-dim $\frac{G_\alpha(M)}{P} \leq \alpha$, then n-dim $\frac{M}{P} \leq \alpha$, see Proposition 2.4. This shows that $G_\alpha(M) = P$ and this is a contradiction. Thus n-dim $\frac{G_\alpha(M)}{P} > \alpha$. This shows that n-dim $\frac{M}{P} > \alpha$, see Proposition 2.4. Hence M is γ-tall for some ordinal number $\gamma \geq \alpha$. □

Now in view of [9, Proposition 2.21], we observe the following result.

Proposition 3.27 *The following statements are equivalent for a commutative ring R:*

(1) *Every Artinian R-module is Noetherian.*
(2) *Every m-short module is both Artinian and Noetherian for all integers $m \geq -1$.*
(3) *Every α-short module is both Artinian and Noetherian for all ordinal α.*
(4) *Every m-tall module is both Artinian and Noetherian for all integers $m \geq -1$.*
(5) *Every α-tall module is both Artinian and Noetherian for all ordinal α.*
(6) *No homomorphic image of R can be isomorphic to a dense subring of a complete local domain of Krull dimension 1.*

Proof By Proposition 3.7 and [9, Proposition 2.21], we are through. □

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