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Coefficient Bounds for Certain Analytic Functions

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Abstract Let φ be an analytic function with the positive real parts, $\varphi(0) = 1$ and $\varphi'(0) > 0$. Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ be an analytic function satisfying the subordination $\alpha f'(z) + (1-\alpha)zf'(z)/f(z) \prec \varphi(z)$, $(f'(z))^{\alpha}(zf'(z)/f(z))^{(1-\alpha)} \prec \varphi(z)$, $(f'(z))^{\alpha}(1+zf''(z)/f'(z))^{(1-\alpha)} \prec \varphi(z)$, $(f(z)/z)^{\alpha}(zf'(z)/f(z))^{(1-\alpha)} \prec \varphi(z)$, or $(f(z)/z)^{\alpha}(1+zf''(z)/f'(z))^{(1-\alpha)} \prec \varphi(z)$. For these functions, the bounds for the second Hankel determinant $a_2a_4 - a_3^2$ as well as the Fekete–Szegö coefficient functional are obtained. Our results include some previously known results.

Keywords Second Hankel determinant · Fekete–Szegö coefficient functional · Subordination · Functions with positive real part

Mathematics Subject Classification 30C80 · 30C45

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1 Introduction

Let $\mathcal{A}$ be the class of all normalized analytic functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{D}$. The qth Hankel determinant (denoted by $H_q(n)$) for $q = 1, 2, \ldots$ and $n = 1, 2, 3, \ldots$ of the function f is given by

$$H_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

In 1996, Pommerenke [36] investigated the Hankel determinant of areally p-valent functions, univalent functions and starlike functions. Further, the Hankel determinants of areally mean p-valent functions were investigated in [29–31]. Noor studied the bounds on Hankel determinant of close-to-convex functions in [32–34]. Pommerenke [37] also established that Hankel determinant of univalent functions satisfying the following relation: $|H_q(n)| < Kn^{-(\frac{1}{2}+\beta)q+\frac{3}{2}}$ $(n = 1, 2, \ldots; q = 1, 2, \ldots)$, where $\beta > 1/40{,}000$ and K depend on q. Later, Hayman [14] proved that $|H_2(n)| < An^{1/2}(n = 1, 2, \ldots; A$ is an absolute constant) for areally mean univalent functions. In 1986, Elhosh [9,10] investigated the Hankel determinant for univalent functions with a positive Hayman index α and k-fold symmetric and close-to-convex functions. The second Hankel determinant $H_2(2) := a_2a_4 - a_3^2$ for the class of functions derivative of which has positive real part, the classes of starlike and convex functions, close-to-starlike, and close-to-convex functions with respect to symmetric points have been studied in [13,16], respectively. Lee et al. [26] obtained bounds on second Hankel determinant for the classes of Ma–Minda starlike and convex functions with respect to φ and two other similar subclasses. One may refer to the survey given by Liu et al. [27] for the other works done in the research of Hankel determinant for univalent functions. They also obtained the second Hankel determinant for some other subclasses of analytic functions. Hankel determinants have been studied by several other authors for various classes of analytic functions and can be referred to in [1,4,23].

In 1933, Fekete and Szegö proved that

$$|a_3 - \mu a_2^2| \leq \begin{cases} 4\mu - 3, & (\mu \geq 1); \\ 1 + \exp\left(\frac{-2\mu}{1-\mu}\right), & (0 \leq \mu \leq 1); \\ 3 - 4\mu, & (\mu \leq 0). \end{cases}$$

holds for functions $f \in \mathcal{S}$, and the result is sharp. Keogh and Merkes [17], in 1969, obtained the sharp upper bound of the Fekete–Szegö functional $|a_3 - \mu a_2^2|$ for functions in some subclasses of $\mathcal{S}$. The Fekete–Szegö functional problem for close-to-convex functions was investigated among others by Koepf [21], Kim et al. [6,20] and Cho et al. [5]. The problem for starlike and convex functions were investigated in a more general settings by Ma and Minda [28]. For other general classes of p-valent functions, the Fekete–Szegö functional problem was investigated by Ali et al. [2] and

Ali et al. [3]. For classes defined by quasi-subordination, see Mohd and Darus [12]. Jakubowski and Zyskowska [15] obtained the estimate for $|a_2 - ca_2^2| + c|a_2|^n$ for $c \in \mathbb{R}$, $f \in \mathcal{S}$. Kiepiela et al. [19] obtained bounds for certain combinations of initial coefficients of bounded functions; these results were used later for estimating fourth coefficients of many classes [2,18]. Other results related to this functional can be seen in [2,3,7,22,24,25,35].

An analytic function f is said to be subordinate to F, written $f \prec F$ or $f(z) \prec F(z)$, $(z \in \mathbb{D})$ if there exists an analytic function $w : \mathbb{D} \to \mathbb{D}$ satisfying $w(0) = 0$ and $f(z) = F(w(z))$ in $\mathbb{D}$. If F is univalent in $\mathbb{D}$, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(\mathbb{D}) \subseteq F(\mathbb{D})$. Let φ be a univalent function with positive real part, $\varphi(0) = 1$ and $\varphi'(0) > 0$. In this paper, we determine the bounds on the second Hankel determinant $H_2(2)$ and the Fekete–Szegö functional for the functions f for which $\alpha f'(z) + (1-\alpha)zf'(z)/f(z)$, $(f'(z))^{\alpha}(zf'(z)/f(z))^{(1-\alpha)}$, $(f'(z))^{\alpha}(1 + zf''(z)/f'(z))^{(1-\alpha)}$, $(f(z)/z)^{\alpha}(zf'(z)/f(z))^{(1-\alpha)}$ or $(f(z)/z)^{\alpha}(1 + zf''(z)/f'(z))^{(1-\alpha)}$ is subordinate to φ. Our results include some previously known results.

2 Main Results

Let $\varphi : \mathbb{D} \to \mathbb{C}$ be a function with positive real part with

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0;\ B_1, B_2, B_3 \in \mathbb{R}). \tag{2.1}$$

For $0 \le \alpha \le 1$, the class $\mathcal{V}_\alpha(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfying the following subordination:

$$\alpha f'(z) + (1-\alpha)\frac{zf'(z)}{f(z)} \prec \varphi(z).$$

Note that

$$\mathcal{S}^*(\varphi) = \mathcal{V}_0(\varphi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}$$

is the class of Ma–Minda starlike functions and

$$\mathcal{R}(\varphi) = \mathcal{V}_1(\varphi) = \left\{ f \in \mathcal{A} : f'(z) \prec \varphi(z) \right\}$$

is a subclass of close-to-convex function. Thus, our class provides a continuous passage from a subclass of starlike functions to the subclass of close-to-convex functions when α varies from 0 to 1.

For functions in the class $\mathcal{V}_\alpha(\varphi)$, we have the following result:

Theorem 2.1 *Let the function $f \in \mathcal{V}_\alpha(\varphi)$ be given by $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$.*

(1) *If B_1, B_2 and B_3 satisfy the conditions*

$$\alpha(1-\alpha)B_1^2 + 2(1+\alpha)|B_2| \le (1+\alpha)\left(\alpha^2 + 4\alpha + 2\right) B_1,$$

and

$$\Big|(\alpha-1)\left(3\alpha^2+5\alpha+1\right)B_1^4-(1+\alpha)^4(3+\alpha)B_2^2+(1+\alpha)^3(2+\alpha)^2B_1B_3$$
$$+\alpha(1-\alpha)(1+\alpha)^2B_1^2B_2\Big|\le(1+\alpha)^4(3+\alpha)B_1^2,$$

then the second Hankel determinant satisfies

$$|a_2a_4-a_3^2|\le\frac{B_1^2}{(2+\alpha)^2}.$$

(2) *If* B_1, B_2 *and* B_3 *satisfy the conditions*

$$\alpha(1-\alpha)B_1^2+2(1+\alpha)|B_2|\ge(1+\alpha)\left(\alpha^2+4\alpha+2\right)B_1,$$

and

$$2\Big|(\alpha-1)\left(3\alpha^2+5\alpha+1\right)B_1^4-(1+\alpha)^4(3+\alpha)B_2^2+(1+\alpha)^3(2+\alpha)^2B_1B_3$$
$$+\ \alpha(1-\alpha)(1+\alpha)^2B_1^2B_2\Big|\ge(1+\alpha)^2\Big[\alpha(1-\alpha)B_1^3+2(1+\alpha)|B_2|B_1$$
$$+(1+\alpha)(2+\alpha)^2B_1^2\Big],$$

or the conditions

$$\alpha(1-\alpha)B_1^2+2(1+\alpha)|B_2|\le(1+\alpha)\left(\alpha^2+4\alpha+2\right)B_1,$$

and

$$\Big|(\alpha-1)\left(3\alpha^2+5\alpha+1\right)B_1^4-(1+\alpha)^4(3+\alpha)B_2^2+(1+\alpha)^3(\alpha+2)^2B_1B_3$$
$$+\alpha(1-\alpha)(1+\alpha)^2B_1^2B_2\Big|\ge(1+\alpha)^4(3+\alpha)B_1^2,$$

then the second Hankel determinant satisfies

$$|a_2a_4-a_3^2|\le\frac{1}{(1+\alpha)^4(2+\alpha)^2(3+\alpha)}\Big|(\alpha-1)\left(3\alpha^2+5\alpha+1\right)B_1^4$$
$$-\ (1+\alpha)^4(3+\alpha)B_2^2+(1+\alpha)^3(2+\alpha)^2B_1B_3$$
$$+\alpha(1-\alpha)(1+\alpha)^2B_1^2B_2\Big|.$$

(3) *If* B_1, B_2 *and* B_3 *satisfy the conditions*

$$\alpha(1-\alpha)B_1^2+2(1+\alpha)|B_2|\ >(1+\alpha)\left(\alpha^2+4\alpha+2\right)B_1,$$

and

$$2\Big|(\alpha-1)\left(3\alpha^2+5\alpha+1\right)B_1^4-(1+\alpha)^4(3+\alpha)B_2^2+(1+\alpha)^3(2+\alpha)^2B_1B_3 \\ +\alpha(1-\alpha)(1+\alpha)^2B_1^2B_2\Big| \le (1+\alpha)^2\Big[\alpha(1-\alpha)B_1^3+2(1+\alpha)|B_2|B_1 \\ +(1+\alpha)(2+\alpha)^2B_1^2\Big],$$

then the second Hankel determinant satisfies

$$|a_2a_4-a_3^2| \le \frac{B_1^2M}{4(2+\alpha)^2(3+\alpha)N},$$

where

$$M=4(3+\alpha)\Big|(\alpha-1)\left(3\alpha^2+5\alpha+1\right)B_1^4-(1+\alpha)^4(3+\alpha)B_2^2 \\ +(1+\alpha)^3(2+\alpha)^2B_1B_3+\alpha(1-\alpha)(1+\alpha)^2B_1^2B_2\Big| \\ +\Big[2(1+\alpha)|B_2|-(1+\alpha)\left(\alpha^2+4\alpha+2\right)B_1+\alpha(1-\alpha)B_1^2\Big]^2$$

and

$$N=\Big|(\alpha-1)(3\alpha^2+5\alpha+1)B_1^4-(1+\alpha)^4(3+\alpha)B_2^2+(1+\alpha)^3(2+\alpha)^2B_1B_3 \\ +\alpha(1-\alpha)(1+\alpha)^2B_1^2B_2\Big|-\alpha(1-\alpha)(1+\alpha)^2B_1^3-(1+\alpha)^3\left(2B_1|B_2|+B_1^2\right).$$

Remark 2.2 When $\varphi(z)=(1+(1-2\lambda)z)/(1-z)$, $(0\le\lambda<1)$, $\sqrt{1+z}$, $1+2/\pi^2(\log((1+\sqrt{z})/(1-\sqrt{z})))^2$ and $((1+z)/(1-z))^\beta$, $0<\beta\le 1$, the class $\mathcal{S}^*(\varphi)$ becomes the class $\mathcal{S}^*(\gamma)$ of starlike functions of order γ, the class $\mathcal{S}_L^*$ of lemniscate starlike functions, the class $\mathcal{S}_{\mathrm{p}}^*$ of parabolic starlike functions and the class $\mathcal{S}_\beta^*$ of strongly starlike functions of order β, respectively. When $\alpha=1$ and $\varphi=(1+z)/(1-z)$, Theorem 2.1 reduces to [16, Theorem 3.1]. When $\alpha=0$, Theorem 2.1 reduces to [26, Theorem 1].

In particular, we get the following corollary:

Corollary 2.3 [26, Theorem 1]

(1) *If* $f\in\mathcal{S}^*(\gamma)$, *then* $|a_2a_4-a_3^2|\le(1-\gamma)^2$.
(2) *If* $f\in\mathcal{S}_L^*$, *then* $|a_2a_4-a_3^2|\le 1/16=0.0625$.
(3) *If* $f\in\mathcal{S}_p^*$, *then* $|a_2a_4-a_3^2|\le 16/\pi^4\approx 0.164255$.
(4) *If* $f\in\mathcal{S}_\beta^*$, *then* $|a_2a_4-a_3^2|\le\beta^2$.

Theorem 2.1 is proved by expressing the coefficient of the function f in terms of the coefficient of a function with positive real part. Recall that the class $\mathcal{P}$ of functions with positive real part consists of all analytic functions $p(z)=1+\sum_{n=0}^\infty c_nz^n$ with

Re $p(z) > 0$ for $z \in \mathbb{D}$. Let Ω be the class of all analytic functions $w : \mathbb{D} \to \mathbb{D}$ of the form $w(z) = w_1 z + w_2 z^2 + \cdots$. To prove our results, we need the following lemmas:

Lemma 2.4 [8] *If the function*

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \tag{2.2}$$

is in $\mathcal{P}$, then the following sharp estimate holds: $|c_n| \le 2 \quad (n = 1, 2, 3, \ldots)$.

Lemma 2.5 [11] *If the function given by* (2.2) *is in $\mathcal{P}$, then,*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right), \tag{2.3}$$

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)y, \tag{2.4}$$

for some x, y with $|x| \le 1$ and $|y| \le 1$.

Lemma 2.6 [28] *Let $p \in \mathcal{P}$ be given by* (2.2). *Then,*

$$|c_2 - \nu c_1^2| \le \begin{cases} -4\nu + 2, & \text{if } \nu \le 0; \\ 2, & \text{if } 0 \le \nu \le 1; \\ 4\nu - 2, & \text{if } \nu \ge 1. \end{cases}$$

Proof of Theorem 2.1 Since $f \in \mathcal{V}_\alpha(\varphi)$, there is an analytic function $w(z) = w_1 z + w_2 z^2 + \cdots \in \Omega$, such that

$$\alpha f'(z) + (1 - \alpha)\frac{zf'(z)}{f(z)} = \varphi(w(z)). \tag{2.5}$$

Define $p_1(z)$ by

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots.$$

Then

$$w(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \cdots\right).$$

Clearly p_1 is analytic in $\mathbb{D}$ with $p_1(0) = 1$ and $p_1 \in \mathcal{P}$. Since $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, we get

$$\varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) = 1 + \frac{1}{2}B_1 c_1 z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right)z^2 + \cdots. \tag{2.6}$$

Also, the Taylor series expansion of f gives

$$\alpha f'(z)+(1-\alpha)z\frac{f'(z)}{f(z)}=1+a_2(1+\alpha)z+\Big((2+\alpha)a_3-(1-\alpha)a_2^2\Big)z^2 + \Big((3+\alpha)a_4-(1-\alpha)\left(3a_2a_3-a_2^3\right)\Big)z^3+\cdots. \tag{2.7}$$

Then from (2.5), (2.6), and (2.7), we get

$$a_2=\frac{B_1c_1}{2(1+\alpha)}. \tag{2.8}$$

$$a_3=\frac{1}{4(2+\alpha)}\left[2B_1c_2+\left(\frac{(1-\alpha)}{(1+\alpha)^2}B_1^2+(B_2-B_1)\right)c_1^2\right]. \tag{2.9}$$

$$a_4=\frac{1}{8(3+\alpha)}\Bigg[4B_1c_3+\Bigg(B_1-\frac{3(1-\alpha)}{(1+\alpha)(2+\alpha)}B_1^2+\frac{(1-\alpha)(1-4\alpha)}{(1+\alpha)^3(2+\alpha)}B_1^3-2B_2 + \frac{3(1-\alpha)}{(1+\alpha)(2+\alpha)}B_1B_2+B_3\Bigg)c_1^3+\Bigg(-4B_1+\frac{6(1-\alpha)}{(1+\alpha)(2+\alpha)}B_1^2+4B_2\Bigg)c_1c_2\Bigg] \tag{2.10}$$

Therefore,

$$\begin{aligned}a_2a_4-a_3^2=\frac{1}{16}B_1\Bigg[\Bigg\{&\frac{(\alpha-1)\left(3\alpha^2+5\alpha+1\right)}{(1+\alpha)^4(2+\alpha)^2(3+\alpha)}B_1^3\\&+\frac{\alpha(1-\alpha)}{(1+\alpha)^2(2+\alpha)^2(3+\alpha)}B_1(B_2-B_1)\\&+\frac{1}{(2+\alpha)^2}\frac{B_2^2}{B_1}+\frac{1}{(1+\alpha)(3+\alpha)}B_3+\frac{1}{(1+\alpha)(2+\alpha)^2(3+\alpha)}(B_1-2B_2)\Bigg\}c_1^4\\&+\Bigg\{\frac{4}{(1+\alpha)(2+\alpha)^2(3+\alpha)}(B_2-B_1)+\frac{2\alpha(1-\alpha)}{(1+\alpha)^2(2+\alpha)^2(3+\alpha)}B_1^2\Bigg\}c_2c_1^2\\&-\frac{4}{(2+\alpha)^2}B_1c_2^2+\frac{4}{(1+\alpha)(3+\alpha)}B_1c_1c_3\Bigg].\end{aligned}$$

Since the function $p(e^{i\theta}z)$ $(\theta\in\mathbb{R})$ is in the class $\mathcal{P}$ for any $p\in\mathcal{P}$, without loss of generality, we can assume that $c_1=c>0$. Substituting the values of c_2 and c_3 from (2.3) and (2.4) in the above expression, we get

$$\begin{aligned}|a_2a_4-a_3^2|=\frac{1}{16}B_1\Bigg|\Bigg\{&\frac{(\alpha-1)\left(3\alpha^2+5\alpha+1\right)}{(1+\alpha)^4(2+\alpha)^2(3+\alpha)}B_1^3+\frac{\alpha(1-\alpha)}{(1+\alpha)^2(2+\alpha)^2(3+\alpha)}B_1B_2\\&-\frac{1}{(2+\alpha)^2}\frac{B_2^2}{B_1}+\frac{1}{(1+\alpha)(3+\alpha)}B_3\Bigg\}c^4+\Bigg\{\frac{2}{(1+\alpha)(2+\alpha)^2(3+\alpha)}B_2\\&+\frac{\alpha(1-\alpha)}{(1+\alpha)^2(2+\alpha)^2(3+\alpha)}B_1^2\Bigg\}c^2\left(4-c^2\right)x-\Bigg\{\frac{1}{(1+\alpha)(2+\alpha)^2(3+\alpha)}c^2\end{aligned}$$

$$+\frac{4}{(2+\alpha)^2}\Bigg\}B_1\left(4-c^2\right)x^2+\frac{2}{(1+\alpha)(3+\alpha)}B_1c\left(4-c^2\right)\left(1-|x|^2\right)y\Bigg|.$$

Replacing $|x|$ by μ and by making use of the triangle inequality and the fact that $|y| \leq 1$ in the above expression, we get

$$\begin{aligned}|a_2a_4-a_3^2| \leq \frac{1}{16}B_1\Bigg[\Bigg|\Bigg\{&\frac{(\alpha-1)\left(3\alpha^2+5\alpha+1\right)}{(1+\alpha)^4(2+\alpha)^2(3+\alpha)}B_1^3-\frac{1}{(2+\alpha)^2}\frac{B_2^2}{B_1}\\ &+\frac{1}{(1+\alpha)(3+\alpha)}B_3+\frac{\alpha(1-\alpha)}{(1+\alpha)^2(2+\alpha)^2(3+\alpha)}B_1B_2\Bigg\}\Bigg|c^4\\ &+\frac{2c}{(1+\alpha)(3+\alpha)}B_1\left(4-c^2\right)+\Bigg\{\frac{\alpha(1-\alpha)}{(1+\alpha)^2(2+\alpha)^2(3+\alpha)}B_1^2\\ &+\frac{2}{(1+\alpha)(2+\alpha)^2(3+\alpha)}|B_2|\Bigg\}c^2\left(4-c^2\right)\mu\\ &+\Bigg\{\frac{c^2}{(1+\alpha)(2+\alpha)^2(3+\alpha)}-\frac{2c}{(1+\alpha)(3+\alpha)}+\frac{4}{(2+\alpha)^2}\Bigg\}\\ &\times B_1\left(4-c^2\right)\mu^2\Bigg]=F(c,\mu).\end{aligned} \tag{2.11}$$

We shall now maximize $F(c, \mu)$ for $(c, \mu) \in [0, 2] \times [0, 1]$. Differentiating $F(c, \mu)$ in (2.11) partially with respect to μ, we get

$$\begin{aligned}\frac{\partial F}{\partial \mu}=\frac{1}{16}B_1\Bigg[&\frac{\alpha(1-\alpha)}{(1+\alpha)^2(2+\alpha)^2(3+\alpha)}c^2\left(4-c^2\right)B_1^2\\ &+\frac{2}{(1+\alpha)(2+\alpha)^2(3+\alpha)}c^2\left(4-c^2\right)|B_2|\\ &+2\mu B_1\left(4-c^2\right)\Bigg(\frac{1}{(1+\alpha)(2+\alpha)^2(3+\alpha)}c^2\\ &-\frac{2}{(1+\alpha)(3+\alpha)}c+\frac{4}{(2+\alpha)^2}\Bigg)\Bigg].\end{aligned} \tag{2.12}$$

For $0 < \mu < 1$, and for any fixed $c \in [0, 2]$, we observe that $\partial F/\partial\mu > 0$. Thus, $F(c, \mu)$ is an increasing function of μ, and for $c \in [0, 2]$, $F(c, \mu)$ has a maximum value at $\mu = 1$. Thus, we have

$$\max F(c,\mu)=F(c,1)=G(c). \tag{2.13}$$

The Eqs. (2.11) and (2.13), upon a little simplification, yield

$$\begin{aligned}G(c)=\frac{1}{16}B_1\Bigg[\Bigg\{\Bigg|&\frac{(\alpha-1)\left(3\alpha^2+5\alpha+1\right)}{(1+\alpha)^4(2+\alpha)^2(3+\alpha)}B_1^3-\frac{1}{(2+\alpha)^2}\frac{B_2^2}{B_1}+\frac{1}{(1+\alpha)(3+\alpha)}B_3\\ &+\frac{\alpha(1-\alpha)}{(1+\alpha)^2(2+\alpha)^2(3+\alpha)}B_1B_2\Bigg|-\frac{\alpha(1-\alpha)}{(1+\alpha)^2(2+\alpha)^2(3+\alpha)}B_1^2\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{(1+\alpha)(2+\alpha)^2(3+\alpha)}(2|B_2|+B_1)\Big\}c^4 + 4\Big\{\frac{\alpha(1-\alpha)}{(1+\alpha)^2(2+\alpha)^2(3+\alpha)}B_1^2 \\
&+ \frac{2}{(1+\alpha)(2+\alpha)^2(3+\alpha)}|B_2| - \frac{\alpha^2+4\alpha+2}{(1+\alpha)(2+\alpha)^2(3+\alpha)}B_1\Big\}c^2 \\
&+ \frac{16}{(2+\alpha)^2}B_1\Big] \\
&= \frac{B_1}{16}\left(Pc^4 + Qc^2 + R\right), \qquad (2.14)
\end{aligned}
$$

where

$$
\begin{aligned}
P &= (1+\alpha)^{-4}(2+\alpha)^{-2}(3+\alpha)^{-1}\Big(\Big|(\alpha-1)\left(3\alpha^2+5\alpha+1\right)B_1^3 \\
&\quad - (1+\alpha)^4(3+\alpha)\frac{B_2^2}{B_1} + (1+\alpha)^3(2+\alpha)^2B_3 + \alpha(1-\alpha)(1+\alpha)^2B_1B_2\Big| \\
&\quad - \alpha(1-\alpha)(1+\alpha)^2B_1^2 - (1+\alpha)^3(2|B_2|+B_1)\Big), \qquad (2.15) \\
Q &= 4(1+\alpha)^{-2}(2+\alpha)^{-2}(3+\alpha)^{-1}\Big(\alpha(1-\alpha)B_1^2 + 2(1+\alpha)|B_2| \\
&\quad - (1+\alpha)(\alpha^2+4\alpha+2)B_1\Big), \qquad (2.16)
\end{aligned}
$$

and

$$
R = 16(2+\alpha)^{-2}B_1. \qquad (2.17)
$$

We know that

$$
\max_{0\le t\le 4}\left(Pt^2+Qt+R\right) = \begin{cases} R, & Q\le 0,\ P\le -\frac{Q}{4}; \\ 16P+4Q+R, & Q\ge 0,\ P\ge -\frac{Q}{8} \ \text{ or } \ Q\le 0,\ P\ge -\frac{Q}{4}; \\ \frac{4PR-Q^2}{4P}, & Q>0,\ P\le -\frac{Q}{8}. \end{cases} \qquad (2.18)
$$

Thus, we have, from (2.14),

$$
|a_2a_4 - a_3^2| \le \frac{B_1}{16}\begin{cases} R, & Q\le 0,\ P\le -\frac{Q}{4}; \\ 16P+4Q+R, & Q\ge 0,\ P\ge -\frac{Q}{8} \ \text{ or } \ Q\le 0,\ P\ge -\frac{Q}{4}; \\ \frac{4PR-Q^2}{4P}, & Q>0,\ P\le -\frac{Q}{8}. \end{cases}
$$

where P, Q, and R are given by (2.15), (2.16), and (2.17), respectively. A simple computation will give the results stated in the theorem. □

Our next theorem gives the Fekete–Szegö inequality for functions in the class $\mathcal{V}_\alpha(\varphi)$.

Theorem 2.7 *Let φ be defined as in (2.1) and let the function $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{V}_\alpha(\varphi)$. Then we have the following:*

(1) *If B_1, B_2 and λ satisfy the condition*

$$(2+\alpha)B_1^2\lambda \le (1-\alpha)B_1^2 + (B_2 - B_1)(1+\alpha)^2,$$

then

$$|a_3 - \lambda a_2^2| \le \frac{1}{(2+\alpha)}\left[B_2 + \frac{(1-\alpha)}{(1+\alpha)^2}B_1^2 - \frac{(2+\alpha)\lambda}{(1+\alpha)^2}B_1^2\right].$$

(2) *If B_1, B_2 and λ satisfy the condition*

$$\begin{aligned}(1-\alpha)B_1^2 + (B_2 - B_1)(1+\alpha)^2 &\le (2+\alpha)B_1^2\lambda \\ &\le (1-\alpha)B_1^2 + (B_2 + B_1)(1+\alpha)^2,\end{aligned}$$

then

$$|a_3 - \lambda a_2^2| \le \frac{B_1}{2+\alpha}.$$

(3) *If B_1, B_2 and λ satisfy the condition*

$$(1-\alpha)B_1^2 + (B_2 + B_1)(1+\alpha)^2 \le (2+\alpha)B_1^2\lambda,$$

then

$$|a_3 - \lambda a_2^2| \le \frac{1}{2+\alpha}\left[-B_2 - \frac{1-\alpha}{(1+\alpha)^2}B_1^2 + \frac{(2+\alpha)\lambda}{(1+\alpha)^2}B_1^2\right].$$

Proof By using (2.8) and (2.9), we get

$$\begin{aligned}a_3 - \lambda a_2^2 &= \frac{1}{2+\alpha}\left[\frac{c_2B_1}{2} - \frac{c_1^2B_1}{4} + \frac{c_1^2B_2}{4} - \frac{(1-\alpha)c_1^2B_1^2}{4(1+\alpha)^2} - \frac{\lambda B_1^2c_1^2}{4(1+\alpha)^2}\right] \\ &= \frac{B_1}{2(2+\alpha)}\left(c_2 - \nu c_1^2\right),\end{aligned}$$

where

$$\nu = \frac{1}{2}\left(1 - \frac{B_2}{B_1} - \frac{(1-\alpha)B_1}{(1+\alpha)^2} + \frac{(2+\alpha)B_1}{(1+\alpha)^2}\lambda\right).$$

Now, by using Lemma 2.6, we get the desired result. □

Remark 2.8 Bounds for the second and the third coefficients for f in $\mathcal{V}_\alpha(\varphi)$ can be directly obtained from Eq. (2.8) and Theorem 2.7 as follows:

$$|a_2| \le \frac{B_1}{1+\alpha},$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_2 + \frac{1-\alpha}{(1+\alpha)^2} B_1^2}{2+\alpha}, & (1-\alpha)B_1^2 + (B_2 - B_1)(1+\alpha)^2 \geq 0; \\ \dfrac{B_1}{2+\alpha}, & (1-\alpha)B_1^2 + (B_2 - B_1)(1+\alpha)^2 \leq 0 \quad \text{or} \\ & (1-\alpha)B_1^2 + (B_2 + B_1)(1+\alpha)^2 \geq 0; \\ \dfrac{-B_2 - \frac{1-\alpha}{(1+\alpha)^2} B_1^2}{2+\alpha}, & (1-\alpha)B_1^2 + (B_2 + B_1)(1+\alpha)^2 \leq 0. \end{cases}$$

Let $\varphi : \mathbb{D} \to \mathbb{C}$ be an analytic function given by (2.1). For $0 \leq \alpha \leq 1$, the class $\mathcal{M}_\alpha(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfying the following subordination

$$\left(f'(z)\right)^\alpha \left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} \prec \varphi(z).$$

We see that $\mathcal{M}_0(\varphi) = \mathcal{S}^*(\varphi)$ is the class of Ma–Minda starlike functions and

$$\mathcal{M}_1(\varphi) = \mathcal{R}(\varphi) = \left\{f \in \mathcal{A} : f'(z) \prec \varphi(z)\right\}$$

is a subclass of close-to-convex function. Thus, this class also provides a passage from a subclass of starlike functions to the subclass of close-to-convex functions when α varies from 0 to 1. Also, for different functions of φ, we get different subclasses of starlike functions as stated earlier.

Theorem 2.9 *Let the function $f \in \mathcal{M}_\alpha(\varphi)$ be given by $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then,*

(1) *If B_1, B_2 and B_3 satisfy the conditions*

$$2|B_2| \leq ((1+\alpha)(3+\alpha) - 1)B_1,$$

and

$$\left|-(2+\alpha)^2(3+\alpha)B_1^4 - 12(1+\alpha)^3(3+\alpha)B_2^2 + 12(1+\alpha)^2(2+\alpha)^2 B_1 B_3\right| - 12(1+\alpha)^3(3+\alpha)B_1^2 \leq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{(2+\alpha)^2}.$$

(2) *If B_1, B_2 and B_3 satisfy the conditions*

$$2|B_2| \geq ((1+\alpha)(3+\alpha) - 1)B_1,$$

and

$$\Big|-(1-\alpha)(2+\alpha)^2(3+\alpha)B_1^4-12(1+\alpha)^3(3+\alpha)B_2^2 + 12(1+\alpha)^2(2+\alpha)^2B_1B_3\Big|-12(1+\alpha)^2|B_2|B_1 - 6(1+\alpha)^2((1+\alpha)(3+\alpha)+1)B_1^2 \geq 0,$$

or the conditions

$$2|B_2| \leq ((1+\alpha)(3+\alpha)-1)B_1,$$

and

$$\Big|-(2+\alpha)^2(3+\alpha)B_1^4-12(1+\alpha)^3(3+\alpha)B_2^2+12(1+\alpha)^2(2+\alpha)^2B_1B_3\Big| - 12(1+\alpha)^3(3+\alpha)B_1^2 \geq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4-a_3^2| \leq \frac{1}{12(1+\alpha)^3(2+\alpha)^2(3+\alpha)}\Big|-12(1+\alpha)^3(3+\alpha)B_2^2 - (2+\alpha)^2(3+\alpha)B_1^4+12(1+\alpha)^2(2+\alpha)^2B_1B_3\Big|.$$

(3) *If* B_1, B_2 *and* B_3 *satisfy the conditions*

$$2|B_2| > ((1+\alpha)(3+\alpha)-1)\,B_1,$$

and

$$\Big|-(1-\alpha)(2+\alpha)^2(3+\alpha)B_1^4-12(1+\alpha)^3(3+\alpha)B_2^2 + 12(1+\alpha)^2(2+\alpha)^2B_1B_3\Big|-12(1+\alpha)^2|B_2|B_1 - 6(1+\alpha)^2((1+\alpha)(3+\alpha)+1)B_1^2 \leq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4-a_3^2| \leq \frac{B_1^2M}{((2+\alpha)^2(3+\alpha))N},$$

where

$$M=\Big|-(1-\alpha)(2+\alpha)^2(3+\alpha)^2B_1^4-12(1+\alpha)^3(3+\alpha)^2B_2^2 + 12(1+\alpha)^2(2+\alpha)^2(3+\alpha)B_1B_3\Big|-12(1+\alpha)(2+\alpha)^2B_1|B_2| - 12(1+\alpha)B_2^2-3(1+\alpha)(2+\alpha)^4B_1^2,$$

and

$$N = \Big|-(1-\alpha)(2+\alpha)^2(3+\alpha)B_1^4 - 12(1+\alpha)^3(3+\alpha)B_2^2 + 12(1+\alpha)^2(2+\alpha)^2 B_1 B_3\Big| - 12(1+\alpha)^2 B_1^2 - 24(1+\alpha)^2 B_1|B_2|.$$

Proof Since $f \in \mathcal{M}_\alpha(\varphi)$, there is an analytic function $w(z) = w_1 z + w_2 z^2 + \cdots \in \Omega$, such that

$$(f'(z))^\alpha \left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} = \varphi(w(z)). \tag{2.19}$$

Define $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots,$$

then this implies

$$w(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \cdots\right).$$

Clearly p_1 is analytic in $\mathbb{D}$ with $p_1(0) = 1$ and $p_1 \in \mathcal{P}$. Then, since $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, we get

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1 c_1 z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right)z^2 + \cdots. \tag{2.20}$$

Also, the Taylor series expansion of f gives

$$(f'(z))^\alpha \left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} = 1 + a_2(1+\alpha)z + \frac{1}{2}\Big((2+\alpha)(2a_3 - (1-\alpha)a_2^2\Big)z^2 + \frac{1}{6}(3+\alpha)(6a_4 - 6(1-\alpha)a_2 a_3 + (1-\alpha)(2-\alpha)a_2^3))z^3 + \cdots. \tag{2.21}$$

Then from (2.19), (2.20) and (2.21), we get

$$a_2 = \frac{B_1 c_1}{2(1+\alpha)}. \tag{2.22}$$

$$a_3 = \frac{1}{2(2+\alpha)}\left[B_1 c_2 - \left(\frac{2(1+\alpha)^2(B_1 - B_2) - (1-\alpha)(2+\alpha)B_1^2}{4(1+\alpha)^2}\right)c_1^2\right], \tag{2.23}$$

$$a_4 = \frac{1}{2(3+\alpha)}\Bigg[B_1 c_3 + \Bigg(\frac{B_1}{4} - \frac{(1-\alpha)(3+\alpha)}{4(1+\alpha)(2+\alpha)}B_1^2 + \frac{(1-\alpha)(1-2\alpha)(3+\alpha)}{24(1+\alpha)^3}B_1^3$$

$$+\frac{B_2}{2}+\frac{(1-\alpha)(3+\alpha)}{4(1+\alpha)(2+\alpha)}B_1B_2+\frac{B_3}{4}\Bigg)c_1^3+\Bigg(-B_1+\frac{(1-\alpha)(3+\alpha)}{2(1+\alpha)(2+\alpha)}B_1^2$$
$$+B_2\Bigg)c_1c_2\Bigg]. \tag{2.24}$$

Thus,

$$a_2a_4-a_3^2=\frac{1}{16}B_1\Bigg[\Bigg\{\frac{-(1-\alpha)}{12(1+\alpha)^3}B_1^3-\frac{1}{(2+\alpha)^2}\frac{B_2^2}{B_1}+\frac{1}{(1+\alpha)^3(3+\alpha)}B_3$$
$$+\frac{1}{(1+\alpha)(2+\alpha)^2(3+\alpha)}(B_1-2B_2)\Bigg\}c_1^4$$
$$+\frac{4}{(1+\alpha)(2+\alpha)^2(3+\alpha)}(B_2-B_1)c_2c_1^2$$
$$-\frac{4}{(2+\alpha)^2}B_1c_2^2+\frac{4}{(1+\alpha)(3+\alpha)}B_1c_1c_3\Bigg].$$

Since the function $p(e^{i\theta}z)$ $(\theta\in\mathbb{R})$ is in the class $\mathcal{P}$ for any $p\in\mathcal{P}$, without loss of generality, we can assume that $c_1=c>0$. Substituting the values of c_2 and c_3 from (2.3) and (2.4) in the above expression, we get

$$|a_2a_4-a_3^2|=\frac{1}{16}B_1\Bigg|\Bigg\{\frac{\alpha-1}{12(1+\alpha)^3}B_1^3-\frac{1}{(2+\alpha)^2}\frac{B_2^2}{B_1}+\frac{1}{(1+\alpha)(3+\alpha)}B_3\Bigg\}c^4$$
$$+\Bigg\{\frac{2}{(1+\alpha)(2+\alpha)^2(3+\alpha)}B_2\Bigg\}c^2\left(4-c^2\right)x-\Bigg\{\frac{1}{(1+\alpha)(2+\alpha)^2(3+\alpha)}c^2$$
$$+\frac{4}{(2+\alpha)^2}\Bigg\}B_1\left(4-c^2\right)x^2+\frac{2}{(1+\alpha)(3+\alpha)}B_1c\left(4-c^2\right)\left(1-|x|^2\right)y\Bigg|.$$

Replacing $|x|$ by μ and by making use of the triangle inequality and the fact that $|y|\le 1$ in the above expression, we get

$$|a_2a_4-a_3^2|\le\frac{1}{16}B_1\Bigg[\Bigg|\Bigg\{\frac{\alpha-1}{12(1+\alpha)^3}B_1^3-\frac{1}{(2+\alpha)^2}\frac{B_2^2}{B_1}+\frac{1}{(1+\alpha)(3+\alpha)}B_3\Bigg\}\Bigg|c^4$$
$$+\frac{2c}{(1+\alpha)(3+\alpha)}B_1\left(4-c^2\right)+\frac{2}{(1+\alpha)(2+\alpha)^2(3+\alpha)}|B_2|c^2\left(4-c^2\right)\mu$$
$$+\Bigg\{\frac{c^2}{(1+\alpha)(2+\alpha)^2(3+\alpha)}-\frac{2c}{(1+\alpha)(3+\alpha)}+\frac{4}{(2+\alpha)^2}\Bigg\}B_1\left(4-c^2\right)\mu^2\Bigg]$$
$$=F(c,\mu). \tag{2.25}$$

We shall now maximize $F(c,\mu)$ for $(c,\mu)\in[0,2]\times[0,1]$. Differentiating $F(c,\mu)$ in (2.11) partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = \frac{1}{16}B_1\left[\frac{2}{(1+\alpha)(2+\alpha)^2(3+\alpha)}c^2\left(4-c^2\right)|B_2| + 2\mu B_1\left(4-c^2\right)\left\{\frac{1}{(1+\alpha)(2+\alpha)^2(3+\alpha)}c^2 - \frac{2}{(1+\alpha)(3+\alpha)}c + \frac{4}{(2+\alpha)^2}\right\}\right]. \tag{2.26}$$

For $0 < \mu < 1$, and for any fixed $c \in [0, 2]$, we observe that $\partial F/\partial\mu > 0$. Thus $F(c, \mu)$ is an increasing function of μ, and for $c \in [0, 2]$, $F(c, \mu)$ has a maximum value at $\mu = 1$. Thus, we have

$$\max F(c, \mu) = F(c, 1) = G(c). \tag{2.27}$$

The Eqs. (2.25) and (2.27), upon a little simplification, yield

$$\begin{aligned} G(c) = \frac{B_1}{16}\Bigg[& c^4\left\{\left|-\frac{1}{(2+\alpha)^2}\frac{B_2^2}{B_1} - \frac{1-\alpha}{12(1+\alpha)^3}B_1^3 + \frac{1}{(1+\alpha)(3+\alpha)}B_3\right| + \frac{1}{(1+\alpha)(2+\alpha)^2(3+\alpha)}(2|B_2|+B_1)\right\} \\ & + c^2\left\{\frac{1}{(1+\alpha)(2+\alpha)^2(3+\alpha)}4(B_1+2|B_2|) - \frac{4}{(2+\alpha)^2}B_1\right\} + \frac{16}{(2+\alpha)^2}B_1\Bigg] \\ = \frac{B_1}{16}\left(Pc^4 + Qc^2 + R\right), \end{aligned} \tag{2.28}$$

where

$$P = \frac{1}{12}(1+\alpha)^{-3}(2+\alpha)^{-2}(3+\alpha)^{-1}\Big(\Big|12(1+\alpha)^3(3+\alpha)\frac{-B_2^2}{B_1} - (1-\alpha)(2+\alpha)^2(3+\alpha)B_1^3 + 12(1+\alpha)^2(2+\alpha)^2B_3\Big| - 12(1+\alpha)^2(B_1+2|B_2|)\Big), \tag{2.29}$$

$$Q = 4(1+\alpha)^{-1}(2+\alpha)^{-2}(3+\alpha)^{-1}\Big((B_1+2|B_2|) - (1+\alpha)(3+\alpha)B_1\Big), \tag{2.30}$$

$$R = 16(2+\alpha)^{-2}B_1. \tag{2.31}$$

Thus, using (2.18) and (2.28) we get,

$$|a_2a_4 - a_3^2| \le \frac{B_1}{16}\begin{cases} R, & Q \le 0,\ P \le -\frac{Q}{4}; \\ 16P + 4Q + R, & Q \ge 0,\ P \ge -\frac{Q}{8} \quad \text{or} \quad Q \le 0,\ P \ge -\frac{Q}{4}; \\ \frac{4PR - Q^2}{4P}, & Q > 0,\ P \le -\frac{Q}{8}. \end{cases}$$

where P, Q, $and R$ are given by (2.29), (2.30), and (2.31), respectively. □

Remark 2.10 When $\alpha = 1$ and $\varphi = (1+z)/(1-z)$, Theorem 2.9 reduces to [16, Theorem 3.1]. When $\alpha = 0$, Theorem 2.9 reduces to [26, Theorem 1]. Therefore, Corollary 2.3 follows as a particular case.

Our next theorem gives the Fekete–Szegö inequality for functions in the class $\mathcal{M}_\alpha(\varphi)$.

Theorem 2.11 *Let φ be defined as in (2.1) and let the function $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{M}_\alpha(\varphi)$. Then we have the following:*

(1) *If B_1, B_2 and μ satisfy the condition*

$$2(2+\alpha)B_1^2\mu \leq (2+\alpha)(1-\alpha)B_1^2 + 2(B_2 - B_1)(1+\alpha)^2,$$

then

$$|a_3 - \mu a_2^2| \leq \frac{1}{(2+\alpha)}\left(B_2 + \frac{(2+\alpha)(1-\alpha)}{2(1+\alpha)^2}B_1^2 - \frac{(2+\alpha)\mu}{(1+\alpha)^2}B_1^2\right).$$

(2) *If B_1, B_2 and μ satisfy the condition*

$$(1-\alpha)(2+\alpha)B_1^2 + 2(B_2 - B_1)(1+\alpha)^2 \leq 2(2+\alpha)B_1^2\mu \\ \leq (1-\alpha)(2+\alpha)B_1^2 + 2(B_2 + B_1)(1+\alpha)^2,$$

then

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2+\alpha}.$$

(3) *If B_1, B_2 and μ satisfy the condition*

$$(1-\alpha)(2+\alpha)B_1^2 + 2(B_2 + B_1)(1+\alpha)^2 \leq 2(2+\alpha)B_1^2\mu,$$

then

$$|a_3 - \mu a_2^2| \leq \frac{1}{2+\alpha}\left(-B_2 - \frac{(1-\alpha)(2+\alpha)}{2(1+\alpha)^2}B_1^2 + \frac{(2+\alpha)\mu}{(1+\alpha)^2}B_1^2\right).$$

Proof Using (2.22) and (2.23), we get,

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{(2+\alpha)}\Bigg(\frac{c_2B_1}{2} - \frac{c_1^2B_1}{4} + \frac{c_1^2B_2}{4} + \frac{(1-\alpha)(2+\alpha)}{4(1+\alpha)^2}c_1^2B_1^2 \\ &\quad - \frac{\mu(2+\alpha)B_1^2c_1^2}{4(1+\alpha)^2}\Bigg) \\ &= \frac{B_1}{2(2+\alpha)}\left(c_2 - c_1^2\left(\frac{1}{2}\left(1 - \frac{B_2}{B_1}\right) - \frac{(1-\alpha)(2+\alpha)}{4(1+\alpha)^2}B_1 + \frac{(2+\alpha)}{2(1+\alpha)^2}\mu B_1\right)\right) \\ &= \frac{B_1}{2(2+\alpha)}(c_2 - \nu c_1^2), \end{aligned}$$

where $\nu = \frac{1}{2}\left(1 - \frac{B_2}{B_1} - \frac{(1-\alpha)(2+\alpha)}{2(1+\alpha)^2}B_1 + \frac{\mu(2+\alpha)}{(1+\alpha)^2}B_1\right)$. Using Lemma 2.6, we get the desired result. □

Remark 2.12 Bounds for the second and the third coefficients for f can be directly obtained from Theorem 2.11 as follows:

$$|a_2| \le \frac{B_1}{1+\alpha},$$

and

$$|a_3| \le \begin{cases} \dfrac{B_2 + \frac{(1-\alpha)(2+\alpha)}{2(1+\alpha)^2}B_1^2}{2+\alpha}, & (1-\alpha)(2+\alpha)B_1^2 + 2(B_2 - B_1)(1+\alpha)^2 \ge 0; \\ \dfrac{B_1}{2+\alpha}, & (1-\alpha)(2+\alpha)B_1^2 + 2(B_2 - B_1)(1+\alpha)^2 \le 0 \quad \text{or} \\ & (1-\alpha)(2+\alpha)B_1^2 + 2(B_2 + B_1)(1+\alpha)^2 \ge 0; \\ \dfrac{-B_2 - \frac{(1-\alpha)(2+\alpha)}{2(1+\alpha)^2}B_1^2}{2+\alpha}, & (1-\alpha)(2+\alpha)B_1^2 + 2(B_2 + B_1)(1+\alpha)^2 \le 0. \end{cases}$$

Let $\varphi : \mathbb{D} \to \mathbb{C}$ be an analytic function given by (2.1). For $0 \le \alpha \le 1$, the class $\mathcal{L}_\alpha(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfying the following subordination:

$$(f'(z))^\alpha \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z).$$

We see that $\mathcal{L}_0(\varphi) = \mathcal{K}(\varphi)$ is the class of Ma–Minda convex functions and

$$\mathcal{L}_1(\varphi) = \mathbb{R}(\varphi) = \{f \in \mathcal{A} : f'(z) \prec \varphi(z)\}$$

is a subclass of close-to-convex function. Thus, this class also provides a continuous passage from a subclass of convex functions to the subclass of close-to-convex functions when α varies from 0 to 1.

Theorem 2.13 *Let the function $f \in \mathcal{L}_\alpha(\varphi)$ be given by $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then,*

(1) *If B_1, B_2 and B_3 satisfy the conditions*

$$2|B_2|(9\alpha^2 - 20\alpha + 12) + B_1^2(1-\alpha)(6+5\alpha) \le \left(-9\alpha^2 + 4\alpha + 12\right) B_1,$$

and

$$\Big|(1-\alpha)(2\alpha^2 - 5\alpha - 6)B_1^4 - 8(3-2\alpha)B_2^2 + 9\,(2-\alpha)^2\, B_1B_3 + (1-\alpha)(6+5\alpha)B_1^2B_2\Big| - 8(3-2\alpha)B_1^2 \le 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{9(2-\alpha)^2}.$$

(2) *If B_1, B_2 and B_3 satisfy the conditions*

$$2|B_2|(9\alpha^2 - 20\alpha + 12) + B_1^2(1-\alpha)(6+5\alpha) \geq \left(-9\alpha^2 + 4\alpha + 12\right) B_1,$$

and

$$\begin{aligned} 2\Big|(1-\alpha)\left(2\alpha^2 - 5\alpha - 6\right) B_1^4 - 8(3-2\alpha)B_2^2 + 9\,(2-\alpha)^2\, B_1B_3 \\ + (1-\alpha)(6+5\alpha)B_1^2B_2\Big| - 2(9\alpha^2 - 20\alpha + 12)|B_2|B_1 \\ - (1-\alpha)(6+5\alpha)B_1^3 - 9(2-\alpha)^2B_1^2 \geq 0, \end{aligned}$$

or the conditions

$$2|B_2|(9\alpha^2 - 20\alpha + 12) + B_1^2(1-\alpha)(6+5\alpha) \leq \left(-9\alpha^2 + 4\alpha + 12\right) B_1,$$

and

$$\begin{aligned} \Big|(1-\alpha)(2\alpha^2 - 5\alpha - 6)B_1^4 - 8(3-2\alpha)B_2^2 + 9\,(2-\alpha)^2\, B_1B_3 \\ + (1-\alpha)(6+5\alpha)B_1^2B_2\Big| - 8(3-2\alpha)B_1^2 \geq 0, \end{aligned}$$

then the second Hankel determinant satisfies

$$\begin{aligned} |a_2a_4 - a_3^2| \leq \frac{1}{72(2-\alpha)^2(3-2\alpha)}\Big|-8(3-2\alpha)B_2^2 + (1-\alpha)\left(2\alpha^2 - 5\alpha - 6\right)B_1^4 \\ + 9(2-\alpha)^2B_1B_3 + (1-\alpha)(6+5\alpha)B_1^2B_2\Big|. \end{aligned}$$

(3) *If B_1, B_2 and B_3 satisfy the conditions*

$$2|B_2|(9\alpha^2 - 20\alpha + 12) + B_1^2(1-\alpha)(6+5\alpha) > \left(-9\alpha^2 + 4\alpha + 12\right) B_1,$$

and

$$\begin{aligned} 2\Big|(1-\alpha)\left(2\alpha^2 - 5\alpha - 6\right) B_1^4 - 8(3-2\alpha)B_2^2 + 9\,(2-\alpha)^2\, B_1B_3 \\ + (1-\alpha)(6+5\alpha)B_1^2B_2\Big| - 2(9\alpha^2 - 20\alpha + 12)|B_2|B_1 \\ - (1-\alpha)(6+5\alpha)B_1^3 - 9(2-\alpha)^2B_1^2 \leq 0, \end{aligned}$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \le \frac{B_1^2 M}{1152(2-\alpha)^2(3-2\alpha)N},$$

where

$$\begin{aligned}M = 4\Big(&\Big|32(1-\alpha)(3-2\alpha)\left(2\alpha^2-5\alpha-6\right)B_1^4 - 256(3-2\alpha)^2B_2^2\\&+ 288(2-\alpha)^2(3-2\alpha)B_1B_3 + 32(1-\alpha)(6+5\alpha)(3-2\alpha)B_1^2B_2\Big|\\&- 18(1-\alpha)(2-\alpha)^2(6+5\alpha)B_1^3 - 36(2-\alpha)^2\left(12-20\alpha+9\alpha^2\right)B_1|B_2|\\&- 81(2-\alpha)^4B_1^2 - (1-\alpha)^2(6+5\alpha)^2B_1^4 - 4\left(12-20\alpha+9\alpha^2\right)^2B_2^2\\&-4(1-\alpha)(6+5\alpha)\left(12-20\alpha+9\alpha^2\right)B_1^2|B_2|\Big)\end{aligned}$$

and

$$\begin{aligned}N = &\Big|(1-\alpha)(2\alpha^2-5\alpha-6)B_1^4 - 8(3-2\alpha)B_2^2 + 9(2-\alpha)^2B_1B_3\\&+ (1-\alpha)(6+5\alpha)B_1^2B_2\Big| - (1-\alpha)(6+5\alpha)B_1^3 - 2(9\alpha^2-20\alpha+12)B_1|B_2|\\&- (9\alpha^2-20\alpha+12)B_1^2.\end{aligned}$$

Proof Since $f \in \mathcal{L}_\alpha(\varphi)$, there is an analytic function $w(z) = w_1z + w_2z^2 + \cdots \in \Omega$, such that

$$(f'(z))^\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha} = \varphi(w(z)). \tag{2.32}$$

Define $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1z + c_2z^2 + \cdots,$$

which then implies

$$w(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left(c_1z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \cdots\right).$$

Clearly p_1 is analytic in $\mathbb{D}$ with $p_1(0) = 1$ and $p_1 \in \mathcal{P}$. Then, since $\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots$, we get

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1c_1z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right)z^2 + \cdots. \tag{2.33}$$

Also, the Taylor series expansion of f gives

$$(f'(z))^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha} = 1+2a_2z+(3(2-\alpha)a_3-4(1-\alpha)a_2^2)z^2 + \left(4(3-2\alpha)a_4-18(1-\alpha)a_2a_3+8(1-\alpha)a_2^3\right)z^3+\cdots. \tag{2.34}$$

Then from (2.32), (2.33), and (2.34), we get

$$a_2 = \frac{B_1c_1}{4}, \tag{2.35}$$

$$a_3 = \frac{1}{6(2+\alpha)}\left[B_1c_2-\left(\frac{(B_1-B_2)-(1-\alpha)B_1^2}{2}\right)c_1^2\right], \tag{2.36}$$

$$a_4 = \frac{1}{8(3-2\alpha)}\left[B_1c_3+\left(\frac{B_1}{4}-\frac{3(1-\alpha)}{4(2-\alpha)}B_1^2+\frac{(1-\alpha)(1-2\alpha)}{4(2-\alpha)}B_1^3-\frac{B_2}{2} + \frac{3(1-\alpha)}{4(2-\alpha)}B_1B_2+\frac{B_3}{4}\right)c_1^3+\left(-B_1+\frac{3(1-\alpha)}{2(2-\alpha)}B_1^2+B_2\right)c_1c_2\right]. \tag{2.37}$$

Thus,

$$\begin{aligned}a_2a_4-a_3^2 = \frac{1}{1152}B_1\Bigg[\Bigg\{&\frac{(1-\alpha)(2\alpha^2-5\alpha-6)}{(2-\alpha)^2(3-2\alpha)}B_1^3-\frac{8}{(2-\alpha)^2}\frac{B_2^2}{B_1}+\frac{9}{3-2\alpha}B_3\\ &-\frac{9\alpha^2-20\alpha+12}{(2-\alpha)^2(3-2\alpha)}(B_1-2B_2)+\frac{(1-\alpha)(6+5\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1(B_2-B_1)\Bigg\}c_1^4\\ &-\left\{\frac{4(9\alpha^2-20\alpha+12)}{(2-\alpha)^2(3-2\alpha)}(B_2-B_1)+\frac{2(1-\alpha)(6+5\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1^2\right\}c_2c_1^2\\ &-\frac{32}{(2-\alpha)^2}B_1c_2^2+\frac{36}{3-2\alpha}B_1c_1c_3\Bigg].\end{aligned}$$

Since the function $p(\mathrm{e}^{i\theta}z)$ $(\theta\in\mathbb{R})$ is in the class $\mathcal{P}$ for any $p\in\mathcal{P}$, without loss of generality, we can assume that $c_1=c>0$. Substituting the values of c_2 and c_3 from (2.3) and (2.4) in the above expression, we get

$$\begin{aligned}|a_2a_4-a_3^2| = \frac{1}{1152}B_1\Bigg|\Bigg\{&\frac{(1-\alpha)\left(2\alpha^2-5\alpha-6\right)}{(2-\alpha)^2(3-2\alpha)}B_1^3+\frac{(1-\alpha)(5\alpha+6)}{(2-\alpha)^2(3-2\alpha)}B_1B_2\\ &-\frac{8}{(2-\alpha)^2}\frac{B_2^2}{B_1}+\frac{8}{(3-2\alpha)}B_3\Bigg\}c^4+\Bigg\{\frac{2(9\alpha^2-20\alpha+12)}{(2-\alpha)^2(3-2\alpha)}B_2\\ &+\frac{(1-\alpha)(6+5\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1^2\Bigg\}c^2\left(4-c^2\right)x-\Bigg\{\frac{9\alpha^2-20\alpha+12}{(2-\alpha)^2(3-2\alpha)}c^2\\ &+\frac{32}{(2-\alpha)^2}\Bigg\}B_1\left(4-c^2\right)x^2+\frac{18}{3-2\alpha}B_1c\left(4-c^2\right)\left(1-|x|^2\right)y\Bigg|.\end{aligned}$$

Replacing $|x|$ by μ and by making use of the triangle inequality and the fact that $|y| \leq 1$ in the above expression, we get

$$\begin{aligned}|a_2a_4 - a_3^2| \leq \frac{1}{1152} B_1 \Bigg[\Bigg| \Bigg\{ &\frac{(1-\alpha)\left(2\alpha^2 - 5\alpha - 6\right)}{(2-\alpha)^2(3-2\alpha)} B_1^3 - \frac{8}{(2-\alpha)^2} \frac{B_2^2}{B_1} + \frac{8}{3-2\alpha} B_3 \\ &+ \frac{(1-\alpha)(5\alpha+6)}{(2-\alpha)^2(3-2\alpha)} B_1 B_2 \Bigg\} \Bigg| c^4 + \frac{18c}{3-2\alpha} B_1 \left(4-c^2\right) \\ &+ \left\{ \frac{(1-\alpha)(6+5\alpha)}{(2-\alpha)^2(3-2\alpha)} B_1^2 + \frac{2(9\alpha^2 - 20\alpha + 12)}{(2-\alpha)^2(3-2\alpha)} |B_2| \right\} c^2 \left(4-c^2\right) \mu \\ &+ \left\{ \frac{(9\alpha^2 - 20\alpha + 12)c^2}{(2-\alpha)^2(3-2\alpha)} - \frac{18c}{3-2\alpha} + \frac{32}{(2-\alpha)^2} \right\} B_1 \left(4-c^2\right) \mu^2 \Bigg] \\ = F(c,\mu). \end{aligned} \tag{2.38}$$

We shall now maximize $F(c, \mu)$ for $(c, \mu) \in [0, 2] \times [0, 1]$. Differentiating $F(c, \mu)$ in (2.11) partially with respect to μ, we get

$$\begin{aligned}\frac{\partial F}{\partial \mu} = &\frac{1}{1152} B_1 \left[\frac{(1-\alpha)(6+5\alpha)}{(2-\alpha)^2(3-2\alpha)} c^2 \left(4-c^2\right) B_1^2 \right. \\ &+ \frac{2(9\alpha^2 - 20\alpha + 12)}{(2-\alpha)^2(3-2\alpha)} c^2 \left(4-c^2\right) |B_2| - 2\mu B_1 \left(4-c^2\right) \\ &\left. \times \left(\frac{9\alpha^2 - 20\alpha + 12}{(2-\alpha)^2(3-2\alpha)} c^2 - \frac{18}{3-2\alpha} c + \frac{32}{(2-\alpha)^2} \right) \right]. \end{aligned} \tag{2.39}$$

For $0 < \mu < 1$, and for any fixed $c \in [0, 2]$, we observe that $\partial F / \partial \mu > 0$. Thus $F(c, \mu)$ is an increasing function of μ, and for $c \in [0, 2]$, $F(c, \mu)$ has a maximum value at $\mu = 1$. Thus, we have

$$\max F(c, \mu) = F(c, 1) = G(c). \tag{2.40}$$

The Eqs. (2.38) and (2.40), upon a little simplification, yield

$$\begin{aligned}G(c) = \frac{B_1}{1152} \Bigg[c^4 \Bigg\{ \Bigg| &-\frac{8}{(2-\alpha)^2} \frac{B_2}{B_1} + \frac{(1-\alpha)\left(2\alpha^2 - 5\alpha - 6\right)}{(2-\alpha)^2(3-2\alpha)} B_1^3 + \frac{9}{3-2\alpha} B_3 \\ &+ \frac{(1-\alpha)(6+5\alpha)}{(2-\alpha)^2(3-2\alpha)} B_1 B_2 \Bigg| - \frac{(1-\alpha)(6+5\alpha)}{(2-\alpha)^2(3-2\alpha)} B_1^2 - \frac{9\alpha^2 - 20\alpha + 12}{(2-\alpha)^2(3-2\alpha)} (B_1 + 2|B_2|) \Bigg\} \\ &+ 4c^2 \left\{ \frac{2(9\alpha^2 - 20\alpha + 12)}{(2-\alpha)^2(3-2\alpha)} |B_2| + \frac{(1-\alpha)(6+5\alpha)}{(2-\alpha)^2(3-2\alpha)} B_1^2 + \frac{9\alpha^2 - 4\alpha - 12}{(2-\alpha)^2(3-2\alpha)} B_1 \right\} \\ &+ \frac{128}{(2-\alpha)^2} B_1 \Bigg] \\ = &\frac{B_1}{1152} (Pc^4 + Qc^2 + R), \end{aligned} \tag{2.41}$$

where

$$P=(2-\alpha)^{-2}(3-2\alpha)^{-1}\Big(\Big|-8(3-2\alpha)\frac{B_2^2}{B_1}-(1-\alpha)\left(2\alpha^2-5\alpha-6\right)B_1^3$$
$$+9(2-\alpha)^2B_3+(1-\alpha)(6+5\alpha)B_1B_2\Big|-2(9\alpha^2-20\alpha+12)|B_2|$$
$$-\ (1-\alpha)(6+5\alpha)B_1^2-(9\alpha^2-20\alpha+12)B_1\Big), \tag{2.42}$$

$$Q=(2-\alpha)^{-2}(3-2\alpha)^{-1}\Big(4(1-\alpha)(6+5\alpha)B_1^2+4(9\alpha^2-4\alpha-12)(B_1+2|B_2|)\Big), \tag{2.43}$$

$$R=128(2-\alpha)^{-2}B_1. \tag{2.44}$$

Thus using (2.18) and (2.41) we get,

$$|a_2a_4-a_3^2|\le\frac{B_1}{1152}\begin{cases}R, & Q\le 0,\ P\le-\frac{Q}{4};\\ 16P+4Q+R, & Q\ge 0,\ P\ge-\frac{Q}{8}\ \text{ or }\ Q\le 0,\ P\ge-\frac{Q}{4};\\ \frac{4PR-Q^2}{4P}, & Q>0,\ P\le-\frac{Q}{8}.\end{cases}$$

where P, Q, R are given by (2.42), (2.43) and (2.44), respectively. □

Remark 2.14 When $\alpha=1$ and $\varphi=(1+z)/(1-z)$, Theorem 2.13 reduces to [16, Theorem 3.1]. When $\alpha=0$, Theorem 2.13 reduces to [26, Theorem 2].

Our next theorem gives the Fekete–Szegö inequality for functions in the class $\mathcal{L}_\alpha(\varphi)$.

Theorem 2.15 *Let φ be defined as in (2.1) and let the function $f(z)=z+a_2z^2+a_3z^3+\cdots\in\mathcal{L}_\alpha(\varphi)$. Then we have the following:*

(1) *If B_1, B_2 and μ satisfy the condition*

$$3(2-\alpha)B_1^2\mu\le 4(1-\alpha)B_1^2+4(B_2-B_1),$$

then

$$|a_3-\mu a_2^2|\le\frac{1}{3(2-\alpha)}\left(B_2+(1-\alpha)B_1^2-\frac{3(2-\alpha)\mu}{4}B_1^2\right).$$

(2) *If B_1, B_2 and μ satisfy the condition*

$$4(1-\alpha)B_1^2+4(B_2-B_1)\le 3(2-\alpha)B_1^2\mu\le 4(1-\alpha)B_1^2+4(B_2+B_1),$$

then

$$|a_3-\mu a_2^2|\le\frac{B_1}{3(2+\alpha)}.$$

(3) *If B_1, B_2 and μ satisfy the condition*

$$4(1-\alpha)B_1^2 + 4(B_2+B_1) \le 3(2-\alpha)B_1^2\mu,$$

then

$$|a_3 - \mu a_2^2| \le \frac{1}{3(2+\alpha)}\Big(-B_2 - (1-\alpha)B_1^2 + \frac{3(2-\alpha)\mu}{4}B_1^2\Big).$$

Proof Using (2.35) and (2.36) we get,

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{(2-\alpha)}\left(\frac{c_2 B_1}{6} - \frac{c_1^2 B_1}{12} + \frac{c_1^2 B_2}{12} + \frac{1-\alpha}{12}c_1^2 B_1^2 - \frac{\mu(2-\alpha)B_1^2 c_1^2}{16}\right) \\ &= \frac{B_1}{6(2-\alpha)}\left(c_2 - \frac{c_1^2}{2}\left(\left(1-\frac{B_2}{B_1}\right) - (1-\alpha)B_1 + \frac{3(2-\alpha)}{4}\mu B_1\right)\right) \\ &= \frac{B_1}{6(2+\alpha)}(c_2 - \nu c_1^2), \end{aligned}$$

where $\nu = \dfrac{1}{2}\left(1 - \dfrac{B_2}{B_1} - (1-\alpha)B_1 + \dfrac{3\mu(2-\alpha)}{4}B_1\right)$. Using Lemma 2.6 we get the desired result. □

Remark 2.16 Bounds for the second and the third coefficient for f can be directly obtained from Theorem 2.13 as follows:

$$|a_2| \le \frac{B_1}{2},$$

and

$$|a_3| \le \begin{cases} \dfrac{B_2 + (1-\alpha)B_1^2}{3(2-\alpha)}, & (1-\alpha)B_1^2 + (B_2 - B_1) \ge 0; \\ \dfrac{B_1}{3(2-\alpha)}, & (1-\alpha)B_1^2 + (B_2 - B_1) \le 0 \quad \text{or} \\ & (1-\alpha)B_1^2 + (B_2 + B_1) \ge 0; \\ \dfrac{-B_2 - (1-\alpha)B_1^2}{3(2-\alpha)}, & (1-\alpha)B_1^2 + (B_2 + B_1) \le 0. \end{cases}$$

Let $\varphi : \mathbb{D} \to \mathbb{C}$ be an analytic function given by (2.1). For $0 \le \alpha \le 1$, the class $\mathcal{K}_\alpha(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfying the following subordination:

$$\left(\frac{f(z)}{z}\right)^{\alpha}\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} \prec \varphi(z).$$

We see that $\mathcal{K}_0(\varphi) = \mathcal{S}^*(\varphi)$ is the class of Ma–Minda starlike functions.

Theorem 2.17 *Let the function $f \in \mathcal{K}_\alpha(\varphi)$ be given by $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. Then,*

(1) *If B_1, B_2 and B_3 satisfy the conditions*

$$2(1-\alpha)^2|B_2| + (1-\alpha)\alpha B_1^2 \leq \left(2-\alpha^2\right) B_1,$$

and

$$\Big| -(1-\alpha)B_1^4 - (3-2\alpha)B_2^2 + (2-\alpha)^2 B_1B_3 + \alpha(1-\alpha)B_1^2B_2\Big| - (3-2\alpha)B_1^2 \leq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{(2-\alpha)^2}.$$

(2) *If B_1, B_2 and B_3 satisfy the conditions*

$$2(1-\alpha)^2|B_2| + (1-\alpha)\alpha B_1^2 \geq \left(2-\alpha^2\right) B_1,$$

and

$$\Big|-2(1-\alpha)B_1^4 - 2(3-2\alpha)B_2^2 + 2(2-\alpha)^2 B_1B_3 + 2\alpha(1-\alpha)B_1^2B_2\Big| - 2(1-\alpha)^2|B_2|B_1 - (1-\alpha)\alpha B_1^3 - (2-\alpha)^2B_1^2 \geq 0,$$

or the conditions

$$2(1-\alpha)^2|B_2| + (1-\alpha)\alpha B_1^2 \leq \left(2-\alpha^2\right) B_1,$$

and

$$\Big| -(1-\alpha)B_1^4 - (3-2\alpha)B_2^2 + (2-\alpha)^2 B_1B_3 + \alpha(1-\alpha)B_1^2B_2\Big| - (3-2\alpha)B_1^2 \geq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{1}{(2-\alpha)^2(3-2\alpha)}\Big|-(3-2\alpha)B_2^2 - (1-\alpha)B_1^4 + (2-\alpha)^2B_1B_3 + \alpha(1-\alpha)B_1^2B_2\Big|.$$

(3) *If B_1, B_2 and B_3 satisfy the conditions*

$$2(1-\alpha)^2|B_2| + (1-\alpha)\alpha B_1^2 > \left(2-\alpha^2\right) B_1,$$

and

$$\left|-2(1-\alpha)B_1^4 - 2(3-2\alpha)B_2^2 + 2\,(2-\alpha)^2\,B_1B_3 + 2\alpha(1-\alpha)B_1^2B_2\right|$$
$$-\,2(1-\alpha)^2|B_2|B_1 - (1-\alpha)\alpha B_1^3 - (2-\alpha)^2B_1^2 \leq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2M}{(2-\alpha)^2(3-2\alpha)N},$$

where

$$M = \Big|-(1-\alpha)(3-2\alpha)B_1^4 - (3-2\alpha)^2B_2^2 + (2-\alpha)^2(3-2\alpha)B_1B_3$$
$$+\,\alpha(1-\alpha)(3-2\alpha)B_1^2B_2\Big| - \frac{\alpha}{2}(1-\alpha)(2-\alpha)^2B_1^3 - (1-\alpha)^2(2-\alpha)^2B_1|B_2|$$
$$-\,\frac{(2-\alpha)^4}{4}B_1^2 - \frac{\alpha^2}{4}(1-\alpha)^2B_1^4 - (1-\alpha)^4B_2^2 - \alpha(1-\alpha)^3B_1^2|B_2|$$

and

$$N = \left|-(1-\alpha)B_1^4 - (3-2\alpha)B_2^2 + (2-\alpha)^2B_1B_3 + \alpha(1-\alpha)B_1^2B_2\right|$$
$$-\,\alpha(1-\alpha)B_1^3 - 2(1-\alpha)^2B_1|B_2| - (1-\alpha)^2B_1^2.$$

Proof Since $f \in \mathcal{K}_\alpha(\varphi)$, there is an analytic function $w(z) = w_1z + w_2z^2 + \cdots \in \Omega$, such that

$$\left(\frac{f(z)}{z}\right)^{\alpha}\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} = \varphi(w(z)). \tag{2.45}$$

Define $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1z + c_2z^2 + \cdots,$$

which implies

$$w(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left(c_1z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \cdots\right).$$

Clearly p_1 is analytic in $\mathbb{D}$ with $p_1(0) = 1$ and $p_1 \in \mathcal{P}$. Then, since $\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots$, we get

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1c_1z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right)z^2 + \cdots. \tag{2.46}$$

Also, the Taylor series expansion of f gives

$$\left(\frac{f(z)}{z}\right)^{\alpha}\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} = 1 + a_2 z + \Big((2-\alpha)a_3 - (1-\alpha)a_2^2\Big)z^2 + \Big((3-2\alpha)a_4 - 3(1-\alpha)a_2a_3 + (1-\alpha)a_2^3\Big)z^3 + \cdots . \tag{2.47}$$

Then from (2.45), (2.46) and (2.47), we get

$$a_2 = \frac{B_1c_1}{2}, \tag{2.48}$$

$$a_3 = \frac{1}{4(2-\alpha)}\Big[2B_1c_2 - \Big((B_1 - B_2) - (1-\alpha)B_1^2\Big)c_1^2\Big], \tag{2.49}$$

$$a_4 = \frac{1}{8(3-2\alpha)}\Big[4B_1c_3 + \Big(B_1 - \frac{3(1-\alpha)}{2-\alpha}B_1^2 + \frac{(1-\alpha)(1-2\alpha)}{2-\alpha}B_1^3 - 2B_2 + \frac{3(1-\alpha)}{2-\alpha}B_1B_2 + B_3\Big)c_1^3 + \Big(-4B_1 + \frac{6(1-\alpha)}{2-\alpha}B_1^2 + 4B_2\Big)c_1c_2\Big]. \tag{2.50}$$

Thus,

$$\begin{aligned} a_2a_4 - a_3^2 = \frac{1}{16}B_1\Bigg[\Bigg\{&-\frac{1-\alpha}{(2-\alpha)^2(3-2\alpha)}B_1^3 - \frac{1}{(2-\alpha)^2}\frac{B_2^2}{B_1} + \frac{1}{3-2\alpha}B_3 \\ &+ \frac{(1-\alpha)^2}{(2-\alpha)^2(3-2\alpha)}(B_1 - 2B_2) + \frac{\alpha(1-\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1(B_2 - B_1)\Bigg\}c_1^4 \\ &+ \Bigg\{\frac{4(1-\alpha)^2}{(2-\alpha)^2(3-2\alpha)}(B_2 - B_1) + \frac{2\alpha(1-\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1^2\Bigg\}c_2c_1^2 \\ &- \frac{4}{(2-\alpha)^2}B_1c_2^2 - \frac{4}{3-2\alpha}B_1c_1c_3\Bigg]. \end{aligned}$$

Proceeding similarly as in the proof of Theorem 2.1, we would see that $|a_2a_4 - a_3^2|$ will be bounded by

$$\begin{aligned} G(c) = \frac{B_1}{16}\Bigg[c^4\Bigg\{\Bigg|&-\frac{1}{(2-\alpha)^2}\frac{B_2^2}{B_1} - \frac{1-\alpha}{(2-\alpha)^2(3-2\alpha)}B_1^3 + \frac{1}{3-2\alpha}B_3 \\ &+ \frac{\alpha(1-\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1B_2\Bigg| - \frac{\alpha(1-\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1^2 \\ &- \frac{(1-\alpha)^2}{(2-\alpha)^2(3-2\alpha)}(B_1 + 2|B_2|)\Bigg\} + 4c^2\Bigg\{\frac{2(1-\alpha)^2}{(2-\alpha)^2(3-2\alpha)}|B_2| \end{aligned}$$

$$+\frac{\alpha(1-\alpha)}{(2-\alpha)^2(3-2\alpha)}B_1^2-\frac{\left(2-\alpha^2\right)}{(2-\alpha)^2(3-2\alpha)}B_1\Bigg\}+\frac{16}{(2-\alpha)^2}B_1\Bigg]$$

$$=\frac{B_1}{16}(Pc^4+Qc^2+R), \tag{2.51}$$

where

$$P=(2-\alpha)^{-2}(3-2\alpha)^{-1}\Bigg(\Bigg|-(3-2\alpha)\frac{B_2^2}{B_1}-(1-\alpha)B_1^3+(2-\alpha)^2B_3$$
$$+\alpha(1-\alpha)B_1B_2\Bigg|-2(1-\alpha)^2|B_2|-\alpha(1-\alpha)B_1^2+(1-\alpha)^2B_1\Bigg), \tag{2.52}$$

$$Q=4(2-\alpha)^{-2}(3-2\alpha)^{-1}\left(2(1-\alpha)^2|B_2|+\alpha(1-\alpha)B_1^2-\left(2-\alpha^2\right)B_1\right) \tag{2.53}$$

$$R=16(2-\alpha)^{-2}B_1. \tag{2.54}$$

Thus, using (2.18) and (2.51), we get,

$$|a_2a_4-a_3^2|\le\frac{B_1}{16}\begin{cases}R, & Q\le 0,\ P\le-\frac{Q}{4};\\ 16P+4Q+R, & Q\ge 0,\ P\ge-\frac{Q}{8}\ \text{ or }\ Q\le 0,\ P\ge-\frac{Q}{4};\\ \frac{4PR-Q^2}{4P}, & Q>0,\ P\le-\frac{Q}{8}.\end{cases}$$

where P, Q, $areR$ are given by (2.52), (2.53), and (2.54), respectively. □

Remark 2.18 When $\alpha=0$, Theorem 2.17 reduces to [26, Theorem 2]. Then Corollary 2.3 comes as a particular case.

Our next theorem gives the Fekete–Szegö inequality for functions in the class $\mathcal{K}_\alpha(\varphi)$.

Theorem 2.19 *Let φ be defined as in (2.1) and let the function $f(z)=z+a_2z^2+a_3z^3+\cdots\in\mathcal{K}_\alpha(\varphi)$. Then we have the following:*

(1) *If B_1, B_2 and μ satisfy the condition*

$$(2-\alpha)B_1^2\mu\le(1-\alpha)B_1^2+(B_2-B_1),$$

then

$$|a_3-\mu a_2^2|\le\frac{1}{2-\alpha}\Big(B_2+(1-\alpha)B_1^2-(2-\alpha)\mu B_1^2\Big).$$

(2) *If B_1, B_2 and μ satisfy the condition*

$$(1-\alpha)B_1^2+(B_2-B_1)\le(2-\alpha)B_1^2\mu\le(1-\alpha)B_1^2+(B_2+B_1),$$

then

$$|a_3-\mu a_2^2|\le\frac{B_1}{2-\alpha}.$$

(3) *If B_1, B_2 and μ satisfy the condition*

$$(1-\alpha)B_1^2 + (B_2 + B_1) \leq (2-\alpha)B_1^2\mu,$$

then

$$|a_3 - \mu a_2^2| \leq \frac{1}{2-\alpha}\Big(-B_2 - (1-\alpha)B_1^2 + (2-\alpha)\mu B_1^2\Big).$$

Proof Using (2.48) and (2.49) we get,

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{4(2-\alpha)}\Big(2c_2B_1 - c_1^2B_1 + c_1^2B_2 + (1-\alpha)c_1^2B_1^2 - \mu(2-\alpha)B_1^2c_1^2\Big) \\ &= \frac{B_1}{2(2-\alpha)}\left(c_2 - \frac{c_1^2}{2}\left(\left(1 - \frac{B_2}{B_1}\right) - (1-\alpha)B_1 + (2-\alpha)\mu B_1\right)\right) \\ &= \frac{B_1}{2(2-\alpha)}(c_2 - \nu c_1^2), \end{aligned}$$

where $\nu = \dfrac{1}{2}\left(\dfrac{-B_2}{B_1} + 1 - (1-\alpha)B_1 + \mu(2-\alpha)B_1\right)$. Using Lemma 2.6, we get the desired result. □

Remark 2.20 Bounds for the second and the third coefficients for f can be directly obtained from Theorem 2.15 as follows:

$$|a_2| \leq B_1,$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_2 + (1-\alpha)B_1^2}{2-\alpha}, & (1-\alpha)B_1^2 + (B_2 - B_1) \geq 0; \\ \dfrac{B_1}{2-\alpha}, & (1-\alpha)B_1^2 + (B_2 - B_1) \leq 0 \quad \text{or} \\ & (1-\alpha)B_1^2 + (B_2 + B_1) \geq 0; \\ \dfrac{-B_2 - (1-\alpha)B_1^2}{2-\alpha}, & (1-\alpha)B_1^2 + (B_2 + B_1) \leq 0. \end{cases}$$

Let $\varphi : \mathbb{D} \to \mathbb{C}$ be an analytic function given by (2.1). For $0 \leq \alpha \leq 1$, the class $\mathcal{T}_\alpha(\varphi)$ consists of functions $f \in \mathcal{A}$ satisfying the following subordination:

$$\left(\frac{f(z)}{z}\right)^\alpha \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z).$$

We see that

$$\mathcal{T}_0(\varphi) = \mathcal{K}(\varphi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}$$

is the class of Ma–Minda convex functions.

Theorem 2.21 *Let the function* $f \in \mathcal{T}_\alpha(\varphi)$ *be given by* $f(z) = z + a_2z^2 + a_3z^3 + \cdots$. *Then,*

(1) *If* B_1, B_2 *and* B_3 *satisfy the conditions*

$$\begin{aligned}4(1-\alpha)(2-\alpha)&|6-7\alpha||B_2| + 2(1-\alpha)|7\alpha^2-8\alpha-6|B_1^2\\ &\leq ((2-\alpha)^2(12-11\alpha) - 2(1-\alpha)\,(2-\alpha)\,|6-7\alpha|)B_1,\end{aligned}$$

and

$$\begin{aligned}&\Big|(1-\alpha)(67\alpha^4-329\alpha^3+516\alpha^2+168\alpha-576)B_1^4 - 12(2-\alpha)^4(12-11\alpha)B_2^2\\ &\quad + 12(6-5\alpha)^2\,(2-\alpha)^3\,B_1B_3 - 24(1-\alpha)(2-\alpha)^2(7\alpha^2-8\alpha-6)B_1^2B_2\Big|\\ &\quad - 12(2-\alpha)^4(12-11\alpha)B_1^2 \leq 0,\end{aligned}$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{(6-5\alpha)^2}.$$

(2) *If* B_1, B_2 *and* B_3 *satisfy the conditions*

$$\begin{aligned}4(1-\alpha)(2-\alpha)&|6-7\alpha||B_2| + 2(1-\alpha)|7\alpha^2-8\alpha-6|B_1^2\\ &\geq ((2-\alpha)^2(12-11\alpha) - 2(1-\alpha)\,(2-\alpha)\,|6-7\alpha|)B_1,\end{aligned}$$

and

$$\begin{aligned}&\Big|(1-\alpha)(67\alpha^4-329\alpha^3+516\alpha^2+168\alpha-576)B_1^4 - 12(2-\alpha)^4(12-11\alpha)B_2^2\\ &\quad + 12(6-5\alpha)^2\,(2-\alpha)^3\,B_1B_3 - 24(1-\alpha)(2-\alpha)^2(7\alpha^2-8\alpha-6)B_1^2B_2\Big|\\ &\quad - 24(1-\alpha)(2-\alpha)^3|6-7\alpha||B_2|B_1 - 12(1-\alpha)(2-\alpha)^2|7\alpha^2-8\alpha-6|B_1^3\\ &\quad - 6(2-\alpha)^3(2(1-\alpha)|6-7\alpha| + (2-\alpha)(12-11\alpha))B_1^2 \geq 0,\end{aligned}$$

or the conditions

$$\begin{aligned}4(1-\alpha)(2-\alpha)&|6-7\alpha||B_2| + 2(1-\alpha)|7\alpha^2-8\alpha-6|B_1^2\\ &\leq \Big((2-\alpha)^2(12-11\alpha) - 2(1-\alpha)\,(2-\alpha)\,|6-7\alpha|\Big)B_1,\end{aligned}$$

and

$$\begin{aligned}&\Big|(1-\alpha)(67\alpha^4-329\alpha^3+516\alpha^2+168\alpha-576)B_1^4 - 12(2-\alpha)^4(12-11\alpha)B_2^2\\ &\quad + 12(6-5\alpha)^2\,(2-\alpha)^3\,B_1B_3 - 24(1-\alpha)(2-\alpha)^2(7\alpha^2-8\alpha-6)B_1^2B_2\Big|\\ &\quad - 12(2-\alpha)^4(12-11\alpha)B_1^2 \geq 0,\end{aligned}$$

then the second Hankel determinant satisfies

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{12(2-\alpha)^4(6-5\alpha)^2(12-11\alpha)}\Big|-12(2-\alpha)^4(12-11\alpha)B_2^2$$
$$+ (1-\alpha)(67\alpha^4 - 329\alpha^3 + 516\alpha^2 + 168\alpha - 576)B_1^4$$
$$+ 12(2-\alpha)^3(6-5\alpha)^2B_1B_3 - 24(1-\alpha)(2-\alpha)^2(7\alpha^2 - 8\alpha - 6)B_1^2B_2\Big|.$$

(3) *If B_1, B_2 and B_3 satisfy the conditions*

$$4(1-\alpha)(2-\alpha)|6-7\alpha||B_2| + 2(1-\alpha)|7\alpha^2 - 8\alpha - 6|B_1^2$$
$$> \Big((2-\alpha)^2(12-11\alpha) - 2(1-\alpha)\,(2-\alpha)\,|6-7\alpha|\Big)B_1,$$

and

$$|(1-\alpha)(67\alpha^4-329\alpha^3+516\alpha^2+168\alpha - 576)B_1^4 - 12(2-\alpha)^4(12-11\alpha)B_2^2$$
$$+ 12(6-5\alpha)^2\,(2-\alpha)^3\,B_1B_3 - 24(1-\alpha)(2-\alpha)^2(7\alpha^2 - 8\alpha - 6)B_1^2B_2|$$
$$-\ 24(1-\alpha)(2-\alpha)^3|6-7\alpha||B_2|B_1 - 12(1-\alpha)(2-\alpha)^2|7\alpha^2 - 8\alpha - 6|B_1^3$$
$$-\ 6(2-\alpha)^3\Big(2(1-\alpha)|6-7\alpha| + (2-\alpha)(12-11\alpha)\Big)B_1^2 \leq 0,$$

then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2M}{(2(6-5\alpha)^2(12-11\alpha))N},$$

where

$$M = \Big|2(1-\alpha)(12-11\alpha)(67\alpha^4 - 329\alpha^3 + 516\alpha^2 + 168\alpha - 576)B_1^4$$
$$-\ 24(2-\alpha)^4(12-11\alpha)^2B_2^2 + 24(2-\alpha)^3(12-11\alpha)(6-5\alpha)^2B_1B_3$$
$$- 48(1-\alpha)(2-\alpha)^2(12-11\alpha)(7\alpha^2 -\ 8\alpha - 6)B_1^2B_2\Big|$$
$$-\ 24(1-\alpha)(2-\alpha)|7\alpha^2-8\alpha-6|\Big((2-\alpha)(12-11\alpha)+2(1-\alpha)|6-7\alpha|\Big)B_1^3$$
$$-\ 48(1-\alpha)(2-\alpha)^2|6-7\alpha|\Big(2(1-\alpha)|6-7\alpha|+(2-\alpha)(12-11\alpha)\Big)B_1|B_2|$$
$$-\ 6(2-\alpha)^2\Big(4(1-\alpha)|6-7\alpha|(2-\alpha)(12-11\alpha) + (2-\alpha)^2(12-11\alpha)^2$$
$$+\ 4(1-\alpha)^2(6-7\alpha)^2\Big)B_1^2 - 24(1-\alpha)^2(7\alpha^2 - 8\alpha - 6)^2B_1^4$$
$$- 96(1-\alpha)^2(6-7\alpha)^2(2-\alpha)^2B_2^2 -\ 96(1-\alpha)^2(2-\alpha)|6-7\alpha||7\alpha^2$$
$$- 8\alpha - 6|B_1^2|B_2|$$

and

$$N = \Big|(1-\alpha)(67\alpha^4 - 329\alpha^3 + 516\alpha^2 + 168\alpha - 576)B_1^4$$
$$-\ 12(2-\alpha)^4(12-11\alpha)B_2^2 + 12(2-\alpha)^3(6-5\alpha)^2B_1B_3$$

$$-24(1-\alpha)(2-\alpha)^2(7\alpha^2-8\alpha-6)B_1^2B_2\Big|$$
$$-24(1-\alpha)(2-\alpha)^2|7\alpha^2-8\alpha-6|B_1^3-48(1-\alpha)(2-\alpha)^3|6$$
$$-7\alpha|B_1|B_2|-24(1-\alpha)(2-\alpha)^3|6-7\alpha|B_1^2.$$

Proof Since $f\in\mathcal{T}_\alpha(\varphi)$, there is an analytic function $w(z)=w_1z+w_2z^2+\cdots\in\Omega$, such that

$$\left(\frac{f(z)}{z}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}=\varphi(w(z)). \tag{2.55}$$

Define $p_1(z)$ by

$$p_1(z)=\frac{1+w(z)}{1-w(z)}=1+c_1z+c_2z^2+\cdots,$$

then this implies

$$w(z)=\frac{p_1(z)-1}{p_1(z)+1}=\frac{1}{2}\left(c_1z+\left(c_2-\frac{c_1^2}{2}\right)z^2+\cdots\right).$$

Clearly p_1 is analytic in $\mathbb{D}$ with $p_1(0)=1$ and $p_1\in\mathcal{P}$. Then, since $\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots$, we get

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right)=1+\frac{1}{2}B_1c_1z+\left(\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2\right)z^2+\cdots. \tag{2.56}$$

Also, the Taylor series expansion of f gives

$$\begin{aligned}\left(\frac{f(z)}{z}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}&=1+(2-\alpha)a_2z+\left((6-5\alpha)a_3-\frac{1}{2}(1-\alpha)(8+\alpha)a_2^2\right)z^2\\&\quad+\Big((12-11\alpha)a_4-(1-\alpha)(18+5\alpha)a_2a_3\\&\quad+\frac{1}{6}(1-\alpha)(\alpha^2+28\alpha+48)a_2^3\Big)z^3+\cdots.\end{aligned} \tag{2.57}$$

Then from (2.55), (2.56) and (2.57), we get

$$a_2=\frac{B_1c_1}{2(2-\alpha)}. \tag{2.58}$$

$$a_3=\frac{1}{8(6-5\alpha)}\left[4B_1c_2-\frac{2(B_1-B_2)(2-\alpha)^2-(1-\alpha)(8+\alpha)B_1^2}{(2-\alpha)^2}c_1^2\right], \tag{2.59}$$

$$a_4=\frac{1}{8(12-11\alpha)}\Bigg[4B_1c_3+\Bigg(B_1-\frac{(1-\alpha)(18+5\alpha)}{(2-\alpha)(6-5\alpha)}B_1^2-2B_2+B_3$$

$$- \frac{(1-\alpha)(10\alpha^3+25\alpha^2+186\alpha-144)}{6(2-\alpha)^3(6-5\alpha)} B_1^3 + \frac{(1-\alpha)(18+5\alpha)}{(2-\alpha)(6-5\alpha)} B_1 B_2 \Bigg) c_1^3$$
$$+ \left(-4B_1 + \frac{2(1-\alpha)(18+5\alpha)}{(2-\alpha)(6-5\alpha)} B_1^2 + 4B_2\right) c_1 c_2 \Bigg]. \tag{2.60}$$

Thus,

$$\begin{aligned} a_2a_4 - a_3^2 = \frac{1}{8} B_1 \Bigg[\Bigg\{ & \frac{(1-\alpha)(67\alpha^4-329\alpha^3+516\alpha^2+168\alpha-576)}{24(2-\alpha)^4(6-5\alpha)^2(12-11\alpha)} B_1^3 \\ & - \frac{1}{2(6-5\alpha)^2} \frac{B_2^2}{B_1} + \frac{1}{2(2-\alpha)(12-11\alpha)} B_3 \\ & + \frac{(1-\alpha)(6-7\alpha)}{(2-\alpha)(6-5\alpha)^2(12-11\alpha)} (B_1 - 4B_2) \\ & + \frac{(1-\alpha)(7\alpha^2-8\alpha-6)}{(2-\alpha)^2(6-5\alpha)^2(12-11\alpha)} B_1(B_1 - B_2) \Bigg\} c_1^4 \\ & + \Bigg\{ \frac{4(1-\alpha)(6-7\alpha)}{(2-\alpha)(6-5\alpha)^2(12-11\alpha)} (B_2 - B_1) \\ & - \frac{2(1-\alpha)(7\alpha^2-8\alpha-6)}{(2-\alpha)^2(6-5\alpha)^2(12-11\alpha)} B_1^2 \Bigg\} c_2 c_1^2 - \frac{2}{(6-5\alpha)^2} B_1 c_2^2 \\ & + \frac{2}{(2-\alpha)(12-11\alpha)} B_1 c_1 c_3 \Bigg]. \end{aligned}$$

Again, proceeding as in the proof of Theorem 2.1, we see that $|a_2a_4 - a_3^2|$ is bounded by

$$\begin{aligned} G(c) = \frac{B_1}{8} \Bigg[c^4 \Bigg\{ \Bigg| & - \frac{1}{2(6-5\alpha)^2} \frac{B_2^2}{B_1} \\ & + \frac{(1-\alpha)(67\alpha^4-329\alpha^3+516\alpha^2+168\alpha-576)}{24(2-\alpha)^4(6-5\alpha)^2(12-11\alpha)} B_1^3 \\ & + \frac{1}{2(2-\alpha)(12-11\alpha)} B_3 - \frac{(1-\alpha)(7\alpha^2-8\alpha-6)}{(2-\alpha)^2(6-5\alpha)^2(12-11\alpha)} B_1 B_2 \Bigg| \\ & - \frac{(1-\alpha)|7\alpha^2-8\alpha-6|}{(2-\alpha)^2(6-5\alpha)^2(12-11\alpha)} B_1^2 \\ & - \frac{(1-\alpha)|6-7\alpha|}{(2-\alpha)(6-5\alpha)^2(12-11\alpha)} (B_1 + 2|B_2|) \Bigg\} \\ & + 4c^2 \Bigg\{ \frac{(1-\alpha)|6-7\alpha|}{(2-\alpha)(6-5\alpha)^2(12-11\alpha)} (B_1 + 2|B_2|) \\ & + \frac{(1-\alpha)|7\alpha^2-8\alpha-6|}{(2-\alpha)^2(6-5\alpha)^2(12-11\alpha)} B_1^2 \end{aligned}$$

$$- \frac{1}{2(6-5\alpha)^2} B_1 \Big\} + \frac{8B_1}{(6-5\alpha)^2} \Big]$$

$$= \frac{B_1}{8} \left(Pc^4 + Qc^2 + R \right), \tag{2.61}$$

where

$$\begin{aligned} P = {} & \frac{1}{24}(2-\alpha)^{-4}(6-5\alpha)^{-2}(12-11\alpha)^{-1}\Big(\Big| -12(2-\alpha)^4(12-11\alpha)\frac{B_2^2}{B_1} \\ & + (1-\alpha)(67\alpha^4 - 329\alpha^3 + 516\alpha^2 + 168\alpha - 576)B_1^3 \\ & + 12(2-\alpha)^3(6-5\alpha)^2 B_3 - 24(1-\alpha)(2-a)^2(7\alpha^2 - 8\alpha - 6)B_1 B_2 \Big| \\ & - 48(1-\alpha)(2-\alpha)^3|6-7\alpha||B_2| - 24(1-\alpha)(2-\alpha)^2|7\alpha^2 - 8\alpha - 6|B_1^2 \\ & - 24(1-\alpha)(2-\alpha)^3|6-7\alpha|B_1 \Big), \end{aligned} \tag{2.62}$$

$$\begin{aligned} Q = {} & 2(2-\alpha)^{-2}(6-5\alpha)^{-2}(12-11\alpha)^{-1}\Big(2(1-\alpha)(2-\alpha)|6-7\alpha|(2|B_2| + B_1) \\ & + 2(1-\alpha)|7\alpha^2 - 8\alpha - 6|B_1^2 - (2-\alpha)^2(12-11\alpha)B_1\Big), \end{aligned} \tag{2.63}$$

$$R = 8(6-5\alpha)^{-2}B_1. \tag{2.64}$$

Thus using (2.18) and (2.61) we get,

$$|a_2a_4 - a_3^2| \le \frac{B_1}{8} \begin{cases} R, & Q \le 0,\ P \le -\frac{Q}{4}; \\ 16P + 4Q + R, & Q \ge 0,\ P \ge -\frac{Q}{8} or Q \le 0,\ P \ge -\frac{Q}{4}; \\ \frac{4PR - Q^2}{4P}, & Q > 0,\ P \le -\frac{Q}{8}. \end{cases}$$

where P, Q, R are given by (2.62), (2.63) and (2.64), respectively. □

Remark 2.22 When $\alpha = 0$, Theorem 2.21 reduces to [26, Theorem 2].

Our next theorem gives the Fekete–Szegö inequality for functions in the class $\mathcal{T}_\alpha(\varphi)$.

Theorem 2.23 *Let φ be defined as in (2.1) and let the function $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{T}_\alpha(\varphi)$. Then we have the following:*

(1) *If B_1, B_2, and μ satisfy the condition*

$$2(6-5\alpha)B_1^2\mu \le (1-\alpha)(8+\alpha)B_1^2 + 2(2-\alpha)^2(B_2 - B_1),$$

then

$$|a_3 - \mu a_2^2| \le \frac{1}{2(6-5\alpha)}\left(2B_2 + \frac{(1-\alpha)(8+\alpha)}{(2-\alpha)^2}B_1^2 - \frac{2(6-5\alpha)}{(2-\alpha)^2}\mu B_1^2\right).$$

(2) *If* B_1, B_2 *and* μ *satisfy the condition*

$$\begin{aligned}&(1-\alpha)(8+\alpha)B_1^2+2(2-\alpha)^2(B_2-B_1)\\&\quad\leq 2(6-5\alpha)B_1^2\mu\leq(1-\alpha)(8+\alpha)B_1^2+2(2-\alpha)^2(B_2+B_1),\end{aligned}$$

then

$$|a_3-\mu a_2^2|\leq\frac{B_1}{6-5\alpha}.$$

(3) *If* B_1, B_2 *and* μ *satisfy the condition*

$$(1-\alpha)(8+\alpha)B_1^2+2(2-\alpha)^2(B_2+B_1)\leq 2(6-5\alpha)B_1^2\mu,$$

then

$$|a_3-\mu a_2^2|\leq\frac{1}{2(6-5\alpha)}\left(-2B_2-\frac{(1-\alpha)(8+\alpha)}{(2-\alpha)^2}B_1^2+\frac{2(6-5\alpha)}{(2-\alpha)^2}\mu B_1^2\right).$$

Proof Using (2.58) and (2.59) we get,

$$\begin{aligned}a_3-\mu a_2^2&=\frac{1}{8(6-5\alpha)}\Big(4c_2B_1-2c_1^2B_1+2c_1^2B_2+\frac{(1-\alpha)(8+\alpha)}{(2-\alpha)^2}c_1^2B_1^2\\&\quad-\mu\frac{2(6-5\alpha)}{(2-\alpha)^2}B_1^2c_1^2\Big)\\&=\frac{B_1}{2(6-5\alpha)}\left(c_2-\frac{c_1^2}{4}\left(2\left(1-\frac{B_2}{B_1}\right)-\frac{(1-\alpha)(8+\alpha)}{(2-\alpha)^2}B_1\right.\right.\\&\quad\left.\left.+\frac{2(6-5\alpha)}{(2-\alpha)^2}\mu B_1\right)\right)\\&=\frac{B_1}{2(6-5\alpha)}\left(c_2-\nu c_1^2\right),\end{aligned}$$

where $\nu=\frac{1}{4}\left[2\left(1-\frac{B_2}{B_1}\right)-\frac{(1-\alpha)(8+\alpha)}{(2-\alpha)^2}B_1+\frac{2(6-5\alpha)}{(2-\alpha)^2}\mu B_1\right]$. Using Lemma 2.6 we get the desired result. □

Remark 2.24 Bounds for the second and the third coefficients for f can be directly obtained from Theorem 2.17 as follows:

$$|a_2|\leq\frac{B_1}{2-\alpha},$$

and

$$|a_3| \leq \begin{cases} \dfrac{B_2 + \frac{(1-\alpha)(8+\alpha)}{2(2-\alpha)^2} B_1^2}{6-5\alpha}, & (1-\alpha)(8+\alpha)B_1^2 + 2(2-\alpha)^2(B_2 - B_1) \geq 0; \\ \dfrac{B_1}{6-5\alpha}, & (1-\alpha)(8+\alpha)B_1^2 + 2(2-\alpha)^2(B_2 - B_1) \leq 0 \quad \text{or} \\ & (1-\alpha)(8+\alpha)B_1^2 + 2(2-\alpha)^2(B_2 + B_1) \geq 0; \\ \dfrac{-B_2 - \frac{(1-\alpha)(8+\alpha)}{2(2-\alpha)^2} B_1^2}{6-5\alpha}, & (1-\alpha)(8+\alpha)B_1^2 + 2(2-\alpha)^2(B_2 + B_1) \leq 0. \end{cases}$$

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