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On Choosability with Separation of Planar Graphs Without Adjacent Short Cycles

Min Chen[1] · **Ko-Wei Lih**[2] · **Weifan Wang**[1]

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Abstract A (k, d)-list assignment L of a graph is a function that assigns to each vertex v a list $L(v)$ of at least k colors satisfying $|L(x) \cap L(y)| \leq d$ for each edge xy. An L-coloring is a vertex coloring π such that $\pi(v) \in L(v)$ for each vertex v and $\pi(x) \neq \pi(y)$ for each edge xy. A graph G is (k, d)-choosable if there exists an L-coloring of G for every (k, d)-list assignment L. This concept is known as choosability with separation. In this paper, we prove that planar graphs without 4-cycles adjacent to 4^--cycles are (3, 1)-choosable. This is a strengthening of a result which says that planar graphs without 4-cycles are (3, 1)-choosable.

Keywords Planar graphs · Choosability with separation · List coloring · Cycles

1 Introduction

A graph G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set $E(G)$, disjoint from $V(G)$, of edges, together with an incidence function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. All graphs considered in this paper are finite, loopless, and without multiple edges, unless otherwise stated. A graph G is *planar* if it can be drawn on the

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✉ Min Chen
chenmin@zjnu.cn

Weifan Wang
wwf@zjnu.cn

[1] Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China

[2] Institute of Mathematics, Academia Sinica, Nankang, Taipei 115, Taiwan

plane so that its edges meet only at the vertices of the graph. Given a graph G, a *list assignment* L is a mapping that assigns to each vertex $v \in V(G)$ a list $L(v)$ of colors. An *L-coloring* is a vertex coloring π such that $\pi(v) \in L(v)$ for each vertex v and $\pi(x) \neq \pi(y)$ for each edge xy. If there is an L-coloring for each list assignment L with $|L(v)| \geq k$ for each vertex v, then we say G is *k-choosable* and the minimum such integer k is the *list chromatic number* of G, denoted by $\chi_l(G)$.

Motivated by forcing the lists of the adjacent vertices to be somewhat separated, the concept known as *choosability with separation* was raised. A graph G is said to be *(k, d)-choosable* if there is an L-coloring for each list assignment L with $|L(v)| \geq k$ for each vertex v such that $|L(x) \cap L(y)| \leq d$ for each edge xy. Obviously, G is (k, k)-choosable if and only if it is k-choosable. Moreover, if G is (k, d)-choosable, then it is (k', d')-choosable for all $k' \geq k$ and $d' \leq d$. This concept was introduced by Kratochvíl, Tuza, and Voigt [4]. They investigated this concept for complete graphs and sparse graphs. Recently, Füredi, Kostochka, and Kumbhat [2,3] have extended the study of dense graphs to complete bipartite graphs and multipartite graphs.

One significant theorem established in a paper by Thomassen [7] is that every planar graph is 5-choosable. The upper bound 5 is best possible since it cannot be lowered to 4 by an example given in [8]. It follows that every planar graph is $(5, d)$-choosable for all non-negative integers d and there exists a non-$(4, 4)$-choosable planar graph. Moreover, there exist non-$(4, 3)$-choosable planar graphs given by Mirzakhani [5]. On the other hand, Kratochvíl, Tuza, and Voigt [4] positively confirmed the $(4, 1)$-choosability of planar graphs. The question of whether every planar graph is $(4, 2)$-choosable seems to be a difficult open problem.

Now we turn our attention to $(3, d)$-choosability of planar graphs. It is proved in [1] that every triangle-free planar graph is $(3, 1)$-choosable. This result is sharp since non-$(3, d)$-choosable triangle-free planar graphs with $d = 2$ and $d = 3$ were constructed by Škrekovski [6] and Voigt [9], respectively. In addition, Choi, Lidický, and Stolee [1] proved that planar graphs without 4-cycles are $(3, 1)$-choosable and planar graphs without 5- and 6-cycles are $(3, 1)$-choosable.

In this paper, we aim to study $(3, 1)$-choosability of planar graphs in which certain 4-cycles can be allowed. More precisely, we will prove the following theorem that is a strengthening of a result in [1].

Theorem 1 *Every planar graph without 4-cycles adjacent to 4^--cycles is $(3, 1)$-choosable.*

Before proving our main result, we need to introduce some notation and terminology. Suppose that G is a planar graph embedded on the plane. We denote its vertex set, edge set, order, and size by $V(G)$, $E(G)$, $|G|$, and $|E|$, respectively. Suppose that C is a cycle in G. C is called a k^--cycle if the length of C is at most k. A *triangle* is the same as a 3-cycle. Two cycles are called *adjacent* if they share at least one edge. A *walk* of G is a non-empty alternating sequence of vertices and edges denoted by $W = v_0e_1v_1e_2 \ldots e_kv_k$, where $e_i = v_{i-1}v_i$ for each $i \in \{1, 2, \ldots, k\}$. If all the vertices of a walk $v_0e_1v_1e_2 \ldots e_kv_k$ are mutually distinct, then we call such a walk a *path*, simply denoted by $P = v_0v_1 \ldots v_{k-1}v_k$. If $S \subset V(G)$, then $G - S$ represents the subgraph obtained from G by deleting the vertices in S and all the edges incident with some vertices in S.

Let $\mathcal{G}$ denote the class of planar graphs without 4-cycles adjacent to 4^--cycles. If $G \in \mathcal{G}$, then the following three configurations will be excluded from G:

(A1) two 3-cycles sharing an edge;
(A2) two 4-cycles sharing an edge;
(A3) a 3-cycle and a 4-cycle sharing an edge.

2 Proof of Theorem 1

In what follows, let L be a list assignment on $V(G)$. A $(*, 1)$-*list assignment* is a list assignment L such that $|L(v)| \geq 1$ for each vertex v and $|L(x) \cap L(y)| \leq 1$ for each edge xy. For $i \in \{1, 2, 3\}$, a vertex v is called an L_i-*vertex* if $|L(v)| = i$, denoted by $v \in L_i$.

Instead of showing Theorem 1, we prove Theorem 2 which is a stronger result inspired by the proof with clever ideas in [1]. Since any (3, 1)-list assignment for $G \in \mathcal{G}$ satisfies all conditions of Theorem 2, we may easily derive Theorem 1 from Theorem 2.

Theorem 2 *Let $G \in \mathcal{G}$ with an outer face F and let P be a subpath of F with order at most* 3. *If L is a* $(*, 1)$-*list assignment satisfying the following six conditions, then G is L-colorable.*

(C1) $|L(v)| \geq 3$ *for* $v \in V(G) \setminus V(F)$;
(C2) $|L(v)| \geq 2$ *for* $v \in V(F) \setminus V(P)$;
(C3) $|L(v)| = 1$ *for* $v \in V(P)$;
(C4) *the subgraph induced by $V(P)$ in G is L-colorable;*
(C5) *no L_2-vertices are adjacent in G;*
(C6) *no L_2-vertex is adjacent to at least two vertices in P.*

Proof Suppose the theorem is not true. Among counterexamples with minimum $|G|+|E|$, we choose G to possess the smallest $\sum_{v \in V(G)} |L(v)|$, that is, the sum of list sizes is the smallest. Note that G is connected.

In the following, for $v \in V(G)$, we use $N_G(v)$ to denote the neighborhood of v in G. For simplicity, we use $N(v)$ instead of $N_G(v)$ when G is clear. If $v \in V(P)$, then v is called a P-*vertex*. A *middle P-vertex* is a P-vertex which has exactly two neighbors in P. The following claim will be used often. □

Claim 1 *For each edge $xy \in E(G) \setminus E(P)$, $|L(x) \cap L(y)| = 1$.*

Proof By definition, $|L(x) \cap L(y)| \leq 1$. If $|L(x) \cap L(y)| = 0$, then it suffices to L-color $G - xy$ and extend this coloring to G , which is a contradiction. □

For our convenience, in what follows, we will use the notation $(G; P, L)$ to denote by a graph G with respect to the path P under the list assignment L.

Claim 2 *G is* 2-*connected. In particular, F is a cycle.*

Proof Suppose to the contrary that there exists a cut vertex $v \in V(G)$, and let G_1 and G_2 be two connected subgraphs such that $V(G_1) \cap V(G_2) = \{v\}$ and $G_1 \cup G_2 = G$.

Clearly, both G_1 and G_2 have at least two vertices. If P is fully contained in some G_i, say G_1, then G_1 admits an L-coloring π_1 by the minimality of G. Let L' be a list assignment on $V(G_2)$ such that $L'(u) = \{\pi_1(u)\}$ if $u = v$, and $L'(u) = L(u)$ otherwise. Consider $P' = v$ as a subpath of the outer face of G_2. Since $(G_2; P', L')$ satisfies all conditions (C1)-(C6) of Theorem 2, there is an L'-coloring π_2 of G_2. Combining π_1 and π_2, we get an L-coloring of G, a contradiction. Now suppose that $P = xvy$ with $xv \in E(G_1)$ and $vy \in E(G_2)$. Let $P_1 = xv$ and $P_2 = yv$. Let L^{i*} be the list assignment L restricted to G_i. Then $(G_i; P_i, L^{i*})$ satisfies (C1)-(C6). By the minimality of G, there exists an L^{i*}-coloring π_i of G_i for each $i \in \{1, 2\}$. Taking $\pi_1(v) = \pi_2(v)$, we get an L-coloring $\pi_1 \cup \pi_2$ of G, a contradiction. □

An edge v_0v_1 is called a *chord* of C if $v_0, v_1 \in V(C)$ but $v_0v_1 \notin E(C)$. A 2-*chord* of C is defined to be a path $Q = v_0v_1v_2$ such that $v_0, v_2 \in V(C)$ and $v_1 \notin V(C)$. Further, if $v_0v_2 \notin E(C)$, then we call Q a *nice* 2-*chord*. Let $V_{\text{int}}(C)$ and $V_{\text{ext}}(C)$ denote the sets of vertices located inside and outside C, respectively.

Claim 3 *G contains no triangle T with $V_{\text{int}}(T) \neq \emptyset$.*

Proof Assume to the contrary that G contains a triangle $T = xyzx$ such that $\text{int}(T) \neq \emptyset$. Let $G_1 = G[V_{\text{ext}}(T) \cup T]$ and $G_2 = G[V_{\text{int}}(T) \cup T]$. Clearly, $P \subseteq G_1$. Since $|G_1| < |G|$, G_1 has an L-coloring π. So x, y, z are colored. Let $G' = G_2 - xy$, and $P' = xzy$. Let L' be a list assignment on $V(G')$ such that $L'(u) = \{\pi(u)\}$ if $u \in \{x, y, z\}$, and $L'(u) = L(u)$ otherwise. It is easy to check that (C1)-(C6) hold for $(G'; P', L')$. Thus, by the minimality of G, G' has an L'-coloring π'. Consequently, the coloring $\pi \cup \pi'$ is an L-coloring of G, a contradiction. □

Claim 4 $|F| \neq 3$.

Proof Suppose to the contrary that $|F| = 3$. Then $V_{\text{int}}(F) = \emptyset$ by Claim 3, and thus $G = F$. Let $F = v_1v_2v_3v_1$. If $|P| = 3$, then G is L-colorable by (C4). If $|P| = 2$, say $P = v_1v_2$, then $v_3 \in L_3$ by (C6). It suffices to color v_3 with a color in $L(v_3) \setminus (L(v_1) \cup L(v_2))$ after v_1 and v_2 have been colored. If $|P| = 1$, say $P = v_1$, then by (C5), we may assume that $v_2 \in L_2 \cup L_3$ and $v_3 \in L_3$. We color v_1 with $a \in L(v_1)$, v_2 with $b \in L(v_2) \setminus \{a\}$, and v_3 with $c \in L(v_3) \setminus \{a, b\}$. In all cases, we reach a contradiction. □

A chord xy is called *bad* if there exists an L_2-vertex $z \in V(F)$ such that $zx, zy \in E(F)$. Note that $xyzx$ is a 3-cycle. Otherwise, xy is called *good*.

For our convenience, in the proofs of Claims 5 and 6, we always use F_1 and F_2 to denote the two cycles in $F \cup \{xy\}$ that contain the chord xy. Let $G_i = G[V_{\text{int}}(F_i) \cup V(F_i)]$ for $i \in \{1, 2\}$. For two vertices $u, v \in V(F_i)$, let $F_i(u, v)$ denote the path in F_i from u to v along the boundary of F (except u and v).

Claim 5 *If xy is a good chord of F, then either x or y is a middle P-vertex.*

Proof Assume that neither x nor y is a middle P-vertex. Without loss of generality, assume that P is fully contained in $F_1 - xy$. Moreover, xy is chosen as a good chord such that $|V(G_2)|$ is as small as possible. It means that xy is the unique good chord located in G_2.

Clearly, G_1 admits an L-coloring π by the minimality of G. Let $P' = xy$. Let L' be a list assignment on $V(G_2)$ such that $L'(u) = \{\pi(u)\}$ if $u \in \{x, y\}$, and $L'(u) = L(u)$ otherwise. One may inspect that (C1)-(C4) are valid for $(G_2; P', L')$. Since every L'_2-vertex in G_2 is just an L_2-vertex in G, (C5) holds for $(G_2; P', L')$. If (C6) is also true, then G_2 has an L'-coloring π', and therefore $\pi \cup \pi'$ is an L-coloring of G, a contradiction. Otherwise, suppose that there exists an L'_2-vertex $z \in F_2(x, y)$ such that $zx, zy \in E(G_2)$. Note that $z \in L_2$ and $xyzx$ is a triangle. Since xy is good, we see that at least one of xz and yz is a chord, say zx. Then $|F_2(x, z)| \geq 3$ due to $G \in \mathcal{G}$, implying that zx is a good chord of F_2 (also a good chord of F), contradicting the choice of xy. □

Claim 6 *Each chord of F is bad.*

Proof Suppose to the contrary that F has a good chord xy. By Claim 5, we may assume that $P = wxv$ such that $w \in V(F_1)$ and $v \in V(F_2)$. Without loss of generality, assume that $|F_1(w, y)| \leq |F_2(v, y)|$. Let $P_1 = wx$. By the minimality of G, G_1 admits an L-coloring π with respect to P_1. Let L' be a list assignment on $V(G_2)$ such that $L'(u) = \{\pi(u)\}$ if $u \in \{x, y\}$, and $L'(u) = L(u)$ otherwise. Let $P_2 = vxy$. One may easily check that (C1), (C2), (C3), and (C5) are valid for $(G_2; P_2, L')$ since L_2-vertices remain L'_2-vertices. Next, we will show that (C4) and (C6) are also satisfied for $(G_2; P_2, L')$, implying that G_2 has an L'-coloring π' and therefore $\pi \cup \pi'$ is an L-coloring of G, a contradiction.

The proof splits into the following two cases.

Case 1. $0 \leq |F_1(w, y)| \leq 1$.

Note that F_1 is a 3-cycle or a 4-cycle. Since $G \in \mathcal{G}$, we derive that $vy \notin E(G)$ and therefore $|F_2(v, y)| \geq 2$. This implies immediately that (C4) holds for $(G_2; P_2, L')$. If (C6) is not true for $(G_2; P_2, L')$, then there exists an L'_2-vertex $z \in F_2(v, y)$ adjacent to at least two vertices of P_2. Since $z \in L_2$, $x, v \in V(P)$, and $(G; P, L)$ satisfies (C6), we see that $zy \in E(G)$ and exactly one of x and v is adjacent to z. If $zx \in E(G)$, then $xyzx$ is a 3-cycle. If $zv \in E(G)$, then $xyzvx$ is a 4-cycle. Both cases contradict the assumption that $G \in \mathcal{G}$.

Case 2. $|F_1(w, y)| \geq 2$.

Then $|F_2(v, y)| \geq 2$. Moreover, xy is chose as a chord that minimizes $|V(G_2)|$ under the assumption $|F_2(v, y)| \geq 2$. It means that if there is $t \in F_2(v, y)$ such that xt is a chord of F and $|F_2(v, t)| \geq 2$, then we select xt instead of xy.

If $yv \in E(G)$, then $yv \notin E(F_2)$ and thus yv is a chord of F_2 (also a good chord of F). Further, yv is a good chord of F since $G \in \mathcal{G}$. However, neither v nor y is a middle P-vertex, which contradicts Claim 5. Thus, $yv \notin E(G)$, implying that (C4) holds for $(G_2; P_2, L')$. Also if (C6) is false for $(G_2; P_2, L')$, then there exists an L'_2-vertex $z \in F_2(v, y)$ such that $zy \in E(G)$ and exactly one of x and v is adjacent to z. If $zx \in E(G)$, then $|F_2(v, z)| \geq 2$, hence zx is a good chord of F_2 as $G \in \mathcal{G}$. This contradicts the choice of xy. So $zv \in E(G)$, and this implies that $xyzvx$ is a 4-cycle. Since $|F_2(v, y)| \geq 2$, at least one of zv and zy is a chord of F_2. If zy is a chord of F_2, then $|F_2(y, z)| \geq 3$. Thus, either zv or zy is a good chord of F since no L_2-vertex can be adjacent to both endpoints. However, none of v, y, and z is a middle P-vertex, contradicting Claim 5. □

Claim 7 *If xy is a chord of F, then x and y are either L_2-vertices or L_3-vertices.*

Proof By Claim 6, xy is bad, so there is an L_2-vertex $z \in V(F)$ such that $zx, zy \in E(F)$ and $xyzx$ forms a 3-cycle. By Claim 3, $V_{\text{int}}(xyzx) = \emptyset$. By symmetry, it suffices to show that $x \in L_3$. By (C5), $x \notin L_2$. Let $x \in L_1$, that is, $x \in V(P)$. Since $|P| \leq 3$ and there are no adjacent triangles in G, we deduce that $y \notin L_1$, and thus $y \in L_3$ by (C5). Without loss of generality, assume that $L(x) = \{a\}$, $L(z) = \{a, b\}$, and $L(y) = \{a, c, d\}$ such that $b \notin \{c, d\}$ by Claim 1. By the minimality of G, $G - yz$ admits an L-coloring π. In fact, π is also an L-coloring of G since no conflict is caused by adding the edge yz back to $G - yz$, a contradiction. □

Recall that a *nice* 2-*chord* of a cycle C is a path $Q = v_0v_1v_2$ such that $v_0, v_2 \in V(C)$, $v_1 \notin V(C)$, and $v_0v_2 \notin E(C)$. In the proofs of Claims 8–10, we shall define F_1 and F_2 to be the two cycles in $F \cup Q$ that contain Q. For $i \in \{1, 2\}$, let $G_i = G[V_{\text{int}}(F_i) \cup V(F_i)]$, and for $u, v \in V(F_i)$, let $F_i(u, v)$ denote the path in F_i from u to v along the boundary of F (except u and v). A nice 2-chord $Q = v_0v_1v_2$ of F is called *worse* if $v_0 \in V(P)$, $v_2 \in L_3$ and there is an L_2-vertex $v^* \in V(F)$ such that $v^*v_0, v^*v_2 \in E(F)$. Note that $v^*v_0v_1v_2v^*$ is a 4-cycle.

Claim 8 *Let $Q = v_0v_1v_2$ be a nice 2-chord of F. If $v_2 \in L_2$, then v_0 is a middle P-vertex.*

Proof Suppose to the contrary that v_0 is not a middle P-vertex. Without loss of generality, assume that $P \subseteq F_1 \cap F$. Furthermore, choose Q so that $|V(G_2)|$ is as small as possible. By the minimality of G, G_1 has an L-coloring π. Let L' be a list assignment on $V(G_2)$ such that $L'(u) = \{\pi(u)\}$ if $u \in \{v_0, v_1, v_2\}$, and $L'(u) = L(u)$ otherwise. Let $P' = v_0v_1v_2$. It is not difficult to see that (C1)–(C3) and (C5) hold for $(G_2; P', L')$. If (C4) fails, implying $v_0v_2 \in E(G_2)$, then the definition of a nice 2-chord asserts that $|F_2(v_0, v_2)| \geq 1$ and hence v_0v_2 is a chord of F. By Claim 6, v_0v_2 is bad. However, v_2 is an L_2-vertex, which contradicts Claim 7.

If (C6) is true, then an L'-coloring π' of G_2 can be established. Consequently, combining π and π' constructs an L-coloring of G, a contradiction. Now assume that (C6) is not true for $(G_2; P', L')$. Then there exists an L'_2-vertex $z \in F_2(v_0, v_2)$ adjacent to at least two of v_0, v_1, v_2. Note that $z \in L_2$ and $v_2 \in L_2$. Since (C5) holds for $(G; P, L)$, we know that $zv_2 \notin E(G)$ and thus $|F_2(z, v_2)| \geq 1$. So $zv_0, zv_1 \in E(G)$. It follows that zv_1v_2 is a nice 2-chord where $|V(G_2)|$ is smaller, which contradicts the choice of $v_0v_1v_2$. □

Claim 9 *Let $Q = v_0v_1v_2$ be a nice 2-chord of F. If $v_2 \in L_3$ and v_0 is a non-middle P-vertex, then Q is worse.*

Proof First we note that $|F_i(v_0, v_2)| \geq 1$ for $i \in \{1, 2\}$. Since v_0 is not a middle vertex, without loss of generality, we may assume that $P \subseteq F_1 \cap F$. Then G_1 has an L-coloring π by the minimality of G. Let L' be a list assignment on $V(G_2)$ such that $L'(u) = \{\pi(u)\}$ if $u \in \{v_0, v_1, v_2\}$, and $L'(u) = L(u)$ otherwise. Let $P' = v_0v_1v_2$.

Using an argument similar to Claim 8, we can show that all conditions (C1)–(C5) are satisfied for $(G_2; P', L')$. Thus, it remains to check that (C6) holds for $(G_2; P', L')$. Let $z \in F_2(v_0, v_2)$ be an L'_2-vertex adjacent to at least two of v_0, v_1, v_2. If $v_1z \notin E(G_2)$, then z is adjacent to both v_0 and v_2. Moreover, by Claim 7, we see that zv_0 and zv_2 are both edges of F_2, and thus Q is worse. So now assume that $v_1z \in E(G_2)$. That is, v_0v_1z is a 2-chord. If $|F_2(z, v_0)| \geq 1$, then v_0 is a middle P-vertex by Claim 8, a contradiction. So $|F_2(z, v_0)| = 0$. Similarly, $|F_2(z, v_2)| = 0$ since v_2 cannot be a middle P-vertex due to $v_2 \in L_{3+}$. So $v_0z \in E(F_2)$ and $v_2z \in E(F_2)$. However, adjacent triangles zv_0v_1z and zv_1v_2z are established, which is a contradiction to the assumption that $G \in \mathcal{G}$. □

Claim 10 *Let $Q = v_0v_1v_2$ be a nice 2-chord of F with $v_0 \in V(P)$ and $v_2 \in L_2 \cup L_3$. Then v_1 is not adjacent to any vertex in $V(P) \setminus \{v_0\}$.*

Proof Assume to the contrary that there is $u \in V(P) \setminus \{v_0\}$ such that $v_1u \in E(G)$. Without loss of generality, assume that $u \in V(F_1 \cap F)$. Then $v_0v_1u \ldots v_0$ is a 4^--cycle since $|P| \leq 3$. Since $G \in \mathcal{G}$, we have that $|F_1(u, v_2)| \geq 2$ and $|F_2(v_0, v_2)| \geq 2$. This implies that uv_1v_2 and $v_0v_1v_2$ are both nice 2-chords that are not worse. If $v_2 \in L_2$, then both v_0 and u are middle P-vertices by Claim 8, which is impossible since there is at most one middle P-vertex. So assume that $v_2 \in L_3$. Then at least one of v_0 and u, say u, is not a middle P-vertex. By Claim 9, uv_1v_2 is worse, a contradiction. □

In the rest of the paper, we let $N^*(v) = N(v) \cap V_{\text{int}}(F)$ for any $v \in V(F)$.

Claim 11 *If xyz is a subpath of F with $|L(y)| = 2$, then $L(x) \cap L(y) \neq L(y) \cap L(z)$.*

Proof Suppose to the contrary that $L(y) = \{a, b\}$ and $a \in L(x) \cap L(y)$ and $a \in L(y) \cap L(z)$. By (C5), $x, z \in L_1 \cup L_3$. By Claim 6, there does not exist $t \in V(F) \setminus \{x, z\}$ adjacent to y. Namely, $N^*(y) = N(y) \setminus \{x, z\}$.

Let L' be a list assignment on the vertices of $G - y$ such that $L'(u) = L(u) \setminus \{b\}$ for $u \in N^*(y)$, and $L'(u) = L(u)$ otherwise. Let G' be the graph obtained from $G - y$ by removing the edges between L'_2-vertices with disjoint lists. Let $P' = P$. Now we are going to verify that (C1)–(C6) are all valid for $(G'; P', L')$.

Let $u \in V(G')$. Note that $|L'(u)| \geq 2$ if $u \in N^*(y)$, and $L'(u) = L(u)$ if $u \in V(G') \setminus N^*(y)$. So (C1)–(C4) hold automatically for $(G'; P', L')$. By the definition of G', there are no adjacent L'_2-vertices in $N^*(y)$. So if (C5) is false for $(G'; P', L')$, then the only possibility is that there is an edge $ty^* \in E(G)$ such that $y^* \in N^*(y)$, $t \in L_2 \cap (V(F) \setminus \{x, y, z\})$. However, yy^*t forms a nice 2-chord such that neither of its ends is a middle P-vertex, contradicting Claim 8. Hence (C5) holds for $(G'; P', L')$.

If (C6) is false for $(G'; P', L')$, then there must exist an L'_2-vertex $y^* \in N^*(y)$ adjacent to at least two vertices in P', say w_1 and w_2. If $w_i \in V(F) \setminus \{x, y, z\}$ for some $i \in \{1, 2\}$, then yy^*w_i is a nice 2-chord. By Claim 10, y^* is not adjacent to any P-vertex except w_i, a contradiction. So assume that $\{w_1, w_2\} = \{x, z\}$. That is, $y^*x, y^*z \in E(G)$, so xyy^*x and zyy^*z are two adjacent triangles, also a contradiction. Therefore (C6) holds for $(G'; P', L')$.

Now, by the minimality of G, G' admits an L'-coloring π. Extend π to G by coloring y with b to get an L-coloring of G, a contradiction. □

Claim 12 *If $F = xyzwx$ is a 4-cycle with $x \in V(P)$ and $y, w \in L_1 \cup L_2$, then $z \in L_1 \cup L_2$.*

Proof Suppose to the contrary that $z \in L_3$. Obviously, there is a color $c \in L(z) \setminus (L(y) \cup L(w))$. Since $xz, yw \notin E(G)$ by $G \in \mathcal{G}$, we see that $N^*(z) = N(z) \setminus \{y, w\}$. Let L' be a list assignment on $V(G) \setminus \{z\}$ such that $L'(u) = L(u) \setminus \{c\}$ for $u \in N^*(z)$, and $L'(u) = L(u)$ otherwise. Let G' be the graph obtained from $G - z$ by removing edges between L'_2-vertices with disjoint lists. Let $P' = P$. It is easy to check that (C1)–(C4) hold for $(G'; P', L')$. By the definition of G', no two L'_2-vertices in $N^*(z)$ are adjacent in G'. Since $G \in \mathcal{G}$, none of $z^* \in N^*(z)$ is adjacent to a vertex in $\{x, y, w\}$. So (C5) and (C6) both hold for $(G'; P', L')$. Hence, by the minimality of G, G' is L'-colorable and we may easily obtain an L-coloring of G by further coloring z with c, a contradiction. □

Claim 13 *If* $|F| = 4$, *then* $|P| \leq 2$.

Proof Let $F = xyzwx$. Suppose to the contrary that $|P| = 3$, say $P = wxy$. Then $x, y, w \in L_1$. By (C6), we know that $z \in L_3$, contradicting Claim 12. □

Claim 14 *If* xyz *is a subpath of* F *with* $y \in L_3$, *then* $x, z \in L_2$.

Proof Assume that the claim is not true. By Claim 4, $|F| \geq 4$. So the proof can be divided into the following two cases by symmetry.

Case 1. $x, z \in L_1 \cup L_3$.

Choose $a \in L(y)$. Let L' be the list assignment on $V(G)$ such that $L'(y) = L(y) \setminus \{a\}$, and $L'(u) = L(u)$ for $u \in V(G) \setminus \{y\}$. This will change y into an L'_2-vertex, whereas all other vertices keep their color lists unaltered. Then (C1)–(C4) hold obviously for $(G; P, L')$. If there is an L_2-vertex $t \in V(F) \setminus \{x, y, z\}$ adjacent to y, that is, ty is a chord, then we see that t should be an L_3-vertex by Claim 7. This shows that (C5) holds for $(G; P, L')$. Moreover, if (C6) fails for $(G; P, L')$, then x and z are both P-vertices. This leads to $|F| = 4$ and $|P| = 3$, contradicting Claim 13. Thus, (C6) holds for $(G; P, L')$. By the choice of L, G has an L'-coloring, which is also an L-coloring of G.

Case 2. $x \in L_2$ and $z \in L_1 \cup L_3$.

In this case, we select $c \in L(x) \cap L(y)$. Let $G' = G - xy$. Let L' be the list assignment on the vertices of G' such that $L'(y) = L(y) \setminus \{c\}$, and $L'(u) = L(u)$ for $u \in V(G') \setminus \{y\}$. Let $P' = P$. Similarly, as only the vertex y has changed its color list, (C1)–(C4) hold for $(G'; P', L')$. F has at most one bad chord with one end y, and it must be bad by Claim 6. Let yy' be the chord, if any, so that $yy'xy$ forms a 3-cycle. This implies that $N(y) = N^*(y) \cup \{x, z\}$ or $N(y) = N^*(y) \cup \{x, z, y'\}$. This fact, together with Claim 7, shows that (C6) holds for $(G'; P', L')$. If (C5) fails for $(G'; P', L')$, then the only possible case is that $yy' \in E(G')$ with $y, y' \in L'_2$. Since $y' \in L_2$, we get a contradiction to Claim 7. Thus, (C5) holds for $(G'; P', L')$. By the minimality of G, G' is L'-colorable, and hence G is L-colorable which is also an L-coloring for G, a contradiction. □

Claim 15 *If* $u \in V(F) \cap L_3$, *then* u *is adjacent to exactly two* L_2*-vertices in* $V(F)$.

Proof We know $|F| \geq 4$ and let xuy be a subpath of F. By Claim 14, x and y are L_2-vertices. If $u' \in V(F) \setminus \{x, u, y\}$, then $u' \in L_3$ by Claim 7. □

Claim 16 $|F| \geq 5$.

Proof Suppose to the contrary that $|F| \leq 4$. By Claim 4, $|F| = 4$, say $F = xyzwx$. Then $|P| \leq 2$ by Claim 13. If $|P| = 2$, say $P = xy$, then at least one of z and w, say w, belongs to L_3 by (C5). However, this contradicts Claim 14. Now assume $|P| = 1$, say $P = x$. Again, by Claim 14, we deduce that $y, w \in L_2$ and thus $z \in L_3$ by (C5). However, this is impossible by Claim 12. □

In what follows, we assume that $F = v_1 v_2 \dots v_m v_{m+1} \dots v_t v_1$ such that $v_i \in V(F) \setminus V(P)$ for $i = 1, \dots, m$ and $v_j \in V(P)$ for $j = m+1, \dots, t$. By Claim 16, $t \geq 5$. Since $|P| \leq 3$, we have that $m \geq 2$ and $t \leq m + 3$. Moreover, by Claim 14, $v_i \in L_2$ for odd i and $v_j \in L_3$ for even j, where $i, j \in \{1, \dots, m\}$. It follows from $v_m \in L_2$ that m is odd and thus at least 3.

The proof of Theorem 2 splits into Cases I and II. In each case, we first pick a special subset X of vertices in G, and then color the vertices in X with distinct colors from their corresponding color lists. Let π denote the L-coloring of X. Second, we define L' to be a list assignment on the vertices of $G - X$ such that $L'(u) = L(u) \setminus \{\pi(x) \mid x \in X$ and $xu \in E(G)\}$, and $L'(u) = L(u)$ otherwise. Let G' be the graph obtained from $G - X$ by removing all edges among the vertices with disjoint L'-lists that are not in $E(P)$. Let $P' = P$. If (C1)–(C6) hold for $(G'; P', L')$, then G' admits an L'-coloring π'. Combining π and π', we get an L-coloring of G, a contradiction.

Since $v_1, v_3 \in L_2$, and $v_2 \in L_3$, any chord with v_2 as an end is bad and both ends are L_3-vertices by Claim 6. This consequence gives us the following important observation.

Observation 1 $N(v_2) = N^*(v_2) \cup \{v_1, v_3\}$ *or* $N(v_2) = N^*(v_2) \cup \{v_1, v_3, v_4\}$.

From now on, there are no more claims and the proof of our main theorem is starting.

Case I $v_2 v_4 \notin E(G)$.

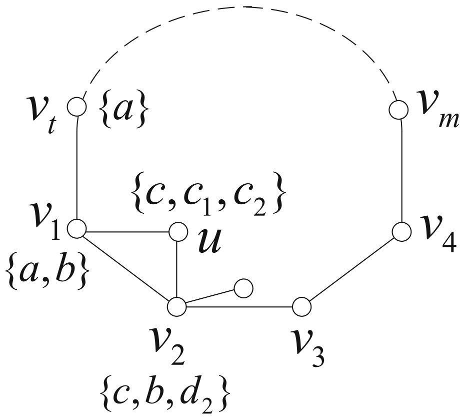
Then $N(v_2) = N^*(v_2) \cup \{v_1, v_3\}$ by Observation 1. We select a color $c \in L(v_2) \setminus (L(v_1) \cup L(v_3))$. Since $G \in \mathcal{G}$, we know that $|N^*(v_2) \cap N^*(v_3)| \leq 1$. So we have two subcases to consider.

Subcase I$_1$ There is no vertex $w \in N^*(v_2) \cap N^*(v_3)$ such that $c \in L(w)$.

Let $X = \{v_2\}$. Set $\pi(v_2) = c$. Observe that only vertices in $N^*(v_2)$ have changed their color lists to L' lists, while other vertices in $V(G') \setminus N^*(v_2)$ have kept their color lists unaltered. Since $|L'(u)| \geq 2$ for each $u \in N^*(v_2)$, (C1)–(C4) hold automatically for $(G'; P', L')$. If (C6) fails for $(G'; P', L')$, then there must exist $x \in N^*(v_2)$ adjacent to at least two P'-vertices. This contradicts Claim 10 since $P' = P$. Now we are going to show that (C5) holds for $(G'; P', L')$.

Let Y_1 and Y_2 denote the set of L'_2-vertices in $N^*(v_2)$ and in $\{v_5, \dots, v_m\}$, respectively. Let $Y_3 = \{v_1, v_3\}$. So $Y_1 \cup Y_2 \cup Y_3$ forms the set of L'_2-vertices in $V(G')$. The definition of G' implies that $E(G'[Y_1]) = \emptyset$. Since each vertex in $Y_2 \cup Y_3$ is also an L_2-vertex in G and (C5) is valid for $(G; P, L)$, we conclude that $E(G'[Y_2 \cup Y_3]) = \emptyset$. So, if (C5) is not true for $(G'; P', L')$, then there must exist a vertex $u \in Y_1$ adjacent to some vertex $u' \in Y_2 \cup Y_3$. By Claim 8, $u' \notin Y_2$ since v_2 is not the middle P-vertex. Thus, $u' \in Y_3$, that is, $u' = v_1$ or $u' = v_3$. If $u' = v_3$, then $u \in N^*(v_2) \cap N^*(v_3)$. This contradicts the assumption on the current subcase.

Fig. 1 One possible case that $uv_1 \in E(G')$ of Subcase I_1



So now we assume that $uv_1 \in E(G)$, see Fig. 1. Note that $c \in L(u)$. Without loss of generality, assume that $L(v_1) = \{a, b\}$, $L(v_2) = \{c, d_1, d_2\}$, and $L(u) = \{c, c_1, c_2\}$. By Claim 1, we may assume that $L(v_t) = \{a\}$. By Claim 11, $a \notin L(v_2)$, and thus $b \in L(v_2)$ by Claim 1. Since $c \notin L(v_1)$, we may assume that $b = d_1$. Since $b \neq c$, Claim 1 implies that $b \notin L(u)$. By the minimality of G, $G' - uv_1$ admits an L-coloring π' and thus $G - uv_1$ is L-colorable by combining π and π'. Therefore, G is L-colorable after replacing the edge uv_1 back since the color b is not in $L(u)$, a contradiction.

Subcase I_2 There exists a vertex $w \in N^*(v_2) \cap N^*(v_3)$ such that $c \in L(w)$.

Let $X = \{v_2, v_3\}$. Set $\pi(v_2) = c$ and $\pi(v_3) = d \in L(v_3) \setminus L(v_4)$. Note that $d \neq c$. Moreover, we claim that $d \notin L(w)$. Otherwise, we assume that $L(v_3) = \{d, d_1\}$, $L(w) = \{c, d, f\}$ and $L(v_2) = \{c, c_1, c_2\}$. Observe that $L(v_2) \cap L(w) = \{c\}$ and thus $d \notin \{c_1, c_2\}$ by Claim 1. It follows that $d_1 \in \{c_1, c_2\}$ by $c \neq d$. By Claim 11, $d_1 \notin L(v_4)$, hence $L(v_3) \cap L(v_4) = \emptyset$, contradicting Claim 1.

Since $v_3 \in L_2$, we have $N(v_3) = N^*(v_3) \cup \{v_2, v_4\}$. Note that only vertices in $N^*(v_2) \cup N^*(v_3)$ have changed their color lists, whereas other vertices in $V(G') \setminus (N^*(v_2) \cup N^*(v_3))$ have kept their color lists to L' lists unchanged. The fact that $d \notin L(w)$ guarantees that $|L'(u)| \geq 2$ for each $u \in N^*(v_2) \cup N^*(v_3)$. So (C1)–(C4) hold for $(G'; P', L')$. By Claim 10, it is easy to check that (C6) holds for $(G'; P', L')$. Now we are about to verify (C5) for $(G'; P', L')$.

Denote by Y_1, Y_2, and Y_3 the sets of L'_2-vertices in $N^*(v_2)$, $N^*(v_3)$, and $\{v_5, \ldots, v_m\}$, respectively. Let $Y_4 = \{v_1\}$. Then $Y_1 \cup Y_2 \cup Y_3 \cup Y_4$ forms the set of L'_2-vertices in $V(G')$. By the definition of G' and that G' is in $\mathcal{G}$, $E(G'[Y_1 \cup Y_2]) = \emptyset$. Since each vertex in $Y_3 \cup Y_4$ is also an L_2-vertex in G and (C5) is true for $(G; P, L)$, we deduce that $E(G'[Y_3 \cup Y_4]) = \emptyset$. Moreover, $E(G'[Y_2 \cup Y_4]) = \emptyset$ by the absence of adjacent triangles, and $E(G'[Y_3 \cup Y_i]) = \emptyset$ for $i \in \{1, 2\}$ since there does not exist $u \in Y_3$ adjacent to $u' \in Y_1 \cup Y_2$ by Claim 8. So if (C5) fails for $(G'; P', L')$, then $E(G'[Y_1 \cup Y_4]) \neq \emptyset$. Similarly, we may let $L(v_1) = \{a\}$, $L(v_1) = \{a, b\}$, $L(v_2) = \{c, d_1, d_2\}$, and $L(u) = \{c, c_1, c_2\}$. By a similar discussion as the previous Case I_1, we deduce that $b \notin L(u)$ and thus we may obtain an L-coloring of G after replacing the edge uv_1 back, a contradiction.

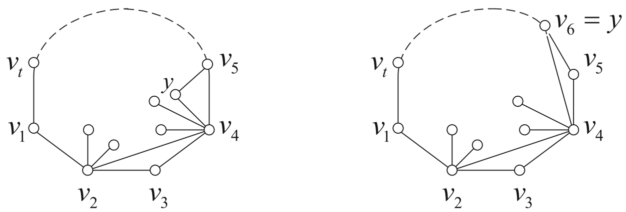


Fig. 2 Two possible cases in Subcase II_1

Case II $v_2v_4 \in E(G)$.

In this case, $N(v_2) = N^*(v_2) \cup \{v_1, v_3, v_4\}$. Note that v_2v_4 is a bad chord of F. By Claim 7, $v_4 \in L_3$, and hence $v_5 \in L_2$ and $v_5 \neq v_t$. It is easy to see that $N(v_4) = N^*(v_4) \cup \{v_2, v_3, v_5\}$ or $N(v_4) = N^*(v_4) \cup \{v_2, v_3, v_5, v_6\}$, as shown in Fig. 2.

Subcase II_1 There is a vertex $y \in N^*(v_5) \cup \{v_6\}$ such that $yv_4 \in E(G)$.

Since G contains no adjacent triangles, the existence of y is unique. Let $X = \{v_2, v_3, v_4\}$. Set $\pi(v_4) = a \in L(v_4) \setminus (L(y) \cup L(v_5))$ and $\pi(v_2) = b \in L(v_2) \setminus (L(v_1) \cup \{a\})$. If $L(v_3) \neq \{a, b\}$, we set $\pi(v_3) = c \in L(v_3) \setminus \{a, b\}$. Otherwise, $L(v_3) = \{a, b\}$ and thus $a \notin L(v_2)$. We color v_3 with b and recolor v_2 with a color $b' \in L(v_2) \setminus (L(v_1) \cup \{b\})$. Thus, v_2, v_3, v_4 can always be precolored with different colors.

It is easy to see that $y \in L_3$ by Claim 7 and (C1), and $yv_2, yv_3 \notin E(G)$ by $G \in \mathcal{G}$. Further, since $a \notin L(y)$, we have $y \in L_3'$. Since $N(v_2) = N^*(v_2) \cup \{v_1, v_3, v_4\}$, there is no other chord starting from v_2 except v_2v_4. Hence, (C1)–(C4) all hold for $(G'; P', L')$. By Claim 10, one may easily verify that (C6) is true for $(G'; P', L')$. It suffices to check that (C5) holds for $(G'; P', L')$.

Let Y_1, Y_2, and Y_3 denote the set of L_2'-vertices in $N^*(v_2)$, $N^*(v_4)$, and $\{v_7, \ldots, v_m\}$, respectively. Let $Y_4 = \{v_1, v_5\}$. Similar to the discussion for Subcase I_2, we know that $E(G'[Y_1 \cup Y_2]) = \emptyset$ and $E(G'[Y_3 \cup Y_4]) = \emptyset$. If $E(G'[Y_2 \cup Y_4]) \neq \emptyset$, in other words, there exists $t \in Y_2$ such that $tv_1 \in E(G)$ or $tv_5 \in E(G)$, then since $t \neq y$, we can find a 3-cycle adjacent to a 4-cycle, a contradiction. Thus, $E(G'[Y_2 \cup Y_4]) = \emptyset$. Moreover, by Claim 8, we confirm that there does not exist a vertex $u \in Y_3$ adjacent to some vertex $u' \in Y_1 \cup Y_2$, and thus $E(G'[Y_3 \cup Y_i]) = \emptyset$ for $i \in \{1, 2\}$. So if (C5) fails for $(G'; P', L')$, then $E(G'[Y_1 \cup Y_4]) \neq \emptyset$. It implies that there exists $u \in Y_1$ such that $uv_1 \in E(G)$ or $uv_5 \in E(G)$. Obviously, it must be the case that $uv_1 \in E(G)$ since $G \in \mathcal{G}$. The following discussion is the same as previous Case I_1.

Subcase II_2 No vertex $y \in N^*(v_5) \cup \{v_6\}$ is adjacent to v_4.

It follows that $N(v_4) = N^*(v_4) \cup \{v_2, v_3, v_5\}$. Let $X = \{v_2, v_3, v_4\}$. Define $\pi(v_4) = a \in L(v_4) \setminus L(v_5)$, $\pi(v_2) = b \in L(v_2) \setminus (L(v_1) \cup \{a\})$, and $\pi(v_3) = c \in L(v_3) \setminus \{b\}$ such that $c \neq a$. Reasoning as for Subcase II_1, one can verify that a, b, c exist. It remains us to check that all (C1)–(C6) hold for $(G'; P', L')$. Though its proof is very similar to that of above Case II_1, we like to write, for completeness, its details.

Clearly, (C1)–(C4) all hold for $(G'; P', L')$. Again, by Claim 10, we see that (C6) is valid for $(G'; P', L')$. We only need to check that (C5) holds for $(G'; P', L')$.

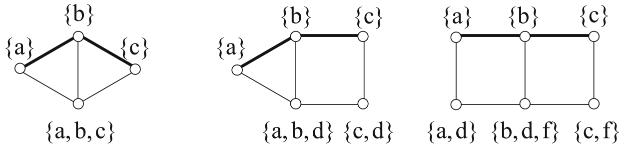


Fig. 3 Three structures with given $(*, 1)$-list assignment

Let Y_1, Y_2, and Y_3 denote the set of L'_2-vertices in $N^*(v_2)$, $N^*(v_4)$, and $\{v_7, \ldots, v_m\}$, respectively. Let $Y_4 = \{v_1, v_5\}$. Similarly, we know that $E(G'[Y_1 \cup Y_2]) = \emptyset$ and $E(G'[Y_3 \cup Y_4]) = \emptyset$. If $E(G'[Y_2 \cup Y_4]) \neq \emptyset$, then there exists $t \in Y_2$ such that $tv_1 \in E(G)$ or $tv_5 \in E(G)$. By the assumption on the current subcase, $tv_5 \notin E(G)$ and so $tv_1 \in E(G)$. However, we can find a 3-cycle $v_2v_3v_4v_2$ adjacent to a 4-cycle $v_1v_2v_4tv_1$, a contradiction. Thus, $E(G'[Y_2 \cup Y_4]) = \emptyset$. Moreover, by Claim 8, we confirm that there does not exist a vertex $u \in Y_3$ adjacent to some vertex $u' \in Y_1 \cup Y_2$, and thus $E(G'[Y_3 \cup Y_i]) = \emptyset$ for $i \in \{1, 2\}$. So if (C5) fails for $(G'; P', L')$, then $E(G'[Y_1 \cup Y_4]) \neq \emptyset$. It follows that there is $u \in Y_1$ such that $uv_1 \in E(G)$ or $uv_5 \in E(G)$. If $uv_5 \in E(G)$, then a 3-cycle $v_2v_3v_4v_2$ is adjacent to a 4-cycle $v_2v_4v_5uv_2$, a contradiction. So $uv_1 \in E(G)$. By applying a similar argument as Case I_1, we may obtain an L-coloring of G. □

In Fig. 3, let thick edges denote the selected subpath P. Clearly, all conditions (C1)–(C6) are satisfied for each of these three configurations with respect to the given $(*, 1)$-list assignment L. However, none of them are L-colorable. These examples show that Theorem 2 is best possible in the sense that none of the forbidden configurations (A1), (A2), and (A3), stated in Sect. 2, can be allowed.

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