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On Properties of Certain Special Zeros of Functions in the Selberg Class

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Abstract In this paper, assuming generalized Riemann hypothesis, we give an upper bound for the multiplicity of eventual zero at central point $1/2$ and location of the first zero with positive imaginary part of function in a certain subclass of the extended Selberg class. We apply our results to automorphic L-functions attached to irreducible unitary automorphic representations of $GL_N(\mathbb{Q})$.

Keywords Selberg class · Explicit formulas · Riemann hypothesis · L-functions

Mathematics Subject Classification 11M41 · 11M36

1 Introduction

In 1989, Selberg [15] defined a general class of Dirichlet series having an Euler product, analytic continuation and a functional equation of Riemann type (plus some side conditions), and formulated some fundamental conjectures concerning them. Especially these conjectures give this class of Dirichlet series a certain structure which applies to central problems in number theory.

The *Selberg class* of functions, denoted by $\mathcal{S}$, is a general class of Dirichlet series F satisfying the following properties:

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(1) (*Dirichlet series*) F posses a Dirichlet series representation

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s},$$

that converges absolutely for $\Re s > 1$.

(2) (*Analytic continuation*) There exists an integer $m \geq 0$ such that the function $(s-1)^m F(s)$ is an entire function of finite order. The smallest such number is denoted by m_F and is called the *polar order* of F.

(3) (*Functional equation*) The function F satisfies the functional equation

$$\Phi_F(s) = w\overline{\Phi_F(1-\bar{s})},$$

where

$$\Phi_F(s) = F(s) Q_F^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j),$$

with $Q_F > 0, r \geq 0, \lambda_j > 0, |w| = 1, \Re(\mu_j) \geq 0, j = 1, \ldots, r$.

(4) (*Ramanujan hypothesis*) For every $\epsilon > 0$ we have $a_F(n) \ll n^{\epsilon}$.

(5) (*Euler product*)

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b_F(n)}{n^s},$$

where $b_F(n) = 0$ for all $n \neq p^m$ with $m \geq 1$ and p prime, and $b_F(n) \ll n^{\theta}$ for some $\theta < 1/2$.

We also recall that degree and conductor, defined by

$$d_F = 2\sum_{j=1}^{r} \lambda_j, \quad q_F = (2\pi)^{d_F} Q_F^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j}, \tag{1}$$

respectively, are invariants of $F \in \mathcal{S}$ (see [8]).

In fact, by the conductor hypothesis it is assumed that for every $F \in \mathcal{S}$ one has $q_F \in \mathbb{N}$. In the special case when $F(s) = \zeta(s)$, where $\zeta(s)$ is the Riemann zeta function then $q_\zeta = 1$. If $F(s) = \zeta_K(s)$, where $\zeta_K(s)$ is the Dedekind zeta function of a number field K, then $q_{\zeta_K} = |d_K|$ (see e.g. [12]). The extended Selberg class $\mathcal{S}^\sharp$, introduced in [7], is the class of functions satisfying axioms (1), (2) and (3). For more information on properties of Selberg class and extended Selberg class see e.g. [1], [6], [12] and [13].

It is conjectured that the Selberg class coincides with the class of all automorphic L-functions.

In order to apply some of our results unconditionally to automorphic L-functions attached to irreducible unitary automorphic representations of $GL_N(\mathbb{Q})$, we also consider class $\mathcal{S}^{\sharp\flat}$, introduced in [10]. It consists of functions satisfying axioms (1), (2) and the two following axioms:

(3') (*Functional equation*) The function F satisfies the functional equation

$$\Phi_F(s) = w\overline{\Phi_F(1-\bar{s})},$$

where

$$\Phi_F(s) = F(s)Q_F^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j),$$

with $Q_F > 0, r \geq 0, \lambda_j > 0, |w| = 1, \Re(\mu_j) > -\frac{1}{4}, \Re(\lambda_j + 2\mu_j) > 0, j = 1, \ldots, r$.

(5') (*Euler sum*) The logarithmic derivative of the function F possesses a Dirichlet series representation

$$\frac{F'}{F}(s) = -\sum_{n=1}^{\infty} \frac{c_F(n)}{n^s},$$

converging absolutely for $\Re s > 1$.

Let us note that (3') implies that $\Re(\lambda_j + \mu_j) > 0$. If $F \in \mathcal{S}$ then

$$c_F(n) = b_F(n) \log n. \tag{2}$$

Assuming *GRH*, we give an upper bound for the multiplicity of eventual zero at central point $1/2$. Moreover, we give a bound for the location of the first zero with positive imaginary part of function F in $\mathcal{S}^{\sharp\flat}$ such that $\Re(c_F(n)) \geq 0$ for all $n \in \mathbb{N}$.

Similar results for Dedekind zeta function were obtained in [11].

The paper is organized as follows. In Sect. 2 we will give the main results of the paper. In Sect. 3 we recall an explicit formula we use in the proof of our main results. In Sect. 4 we prove preliminary lemmas. In Sect. 5 we prove main results of the paper. In Sect. 6 we apply results of Sect. 2 to automorphic L-functions attached to irreducible unitary automorphic representations of $GL_N(\mathbb{Q})$.

2 Main Results

In this section we give two main results of the paper. Namely, we give an upper bound for the multiplicity of eventual zero at central point $1/2$ and provide an upper bound for the height of the first zero with positive imaginary part of function F in $\mathcal{S}^{\sharp\flat}$ such that $\Re(c_F(n)) \geq 0$ for all $n \in \mathbb{N}$.

Throughout this section we assume the *GRH* i.e. we assume that all non-trivial zeros of $F \in \mathcal{S}^{\sharp\flat}$ are on the line $\Re s = 1/2$.

2.1 Multiplicity of Eventual Zero at Central Point

Theorem 1 *Let R be the multiplicity of eventual zero at central point $1/2$ of function $F \in \mathcal{S}^{\sharp\flat}$ such that $\Re(c_F(n)) \geq 0$ and let $B(F) = 2\sum_{j=1}^{r} \lambda_j \Big(\Re\Big(\Psi\Big(\frac{\lambda_j}{2} + \mu_j\Big)\Big) - \log(2\pi\lambda_j)\Big)$.*

(a) If $q_F > e$ then

$$R \leq \frac{(4m_F + 1)\log q_F + B(F)}{2\log\log q_F}.$$

(b) If $0 < q_F \leq e$ then
(i) $R = 0$, for $m_F = 0$,
(ii) $R \leq \frac{4m_F e^{W\left(\frac{B(F)+1}{4em_F}\right)+1} + B(F)+1}{2\left(W\left(\frac{B(F)+1}{4em_F}\right)+1\right)}$, for $4m_F + B(F) + 1 > 0$,

where m_F is the polar order of F, q_F is the conductor of F, λ_j, μ_j are given as in axiom (3') and W denotes the Lambert function.

2.2 Location of the First Zero with Positive Imaginary Part

Theorem 2 *Let h be the height of the first zero with imaginary part different from zero of the function $F \in \mathcal{S}^{\sharp\flat}$. Assume that F satisfies axiom (5) of the Selberg class and $\Re(c_F(n)) \geq 0$. Then, for $q_F > e$ we have the bound*

$$h \leq \max\left\{\frac{16\sqrt{2}\Big[(4m_F + 1)\log q_F + B(F)\Big]}{\pi \log q_F \log\log q_F}, \frac{(2\theta + 1)\pi}{\sqrt{2}\log[\log q_F/16(K_F + \delta)]}\right\}.$$

Here q_F is the conductor of F, m_F is the polar order of F, $B(F)$ is given in Theorem 1, K_F is defined in Lemma 3, $\theta < 1/2$ stemmed from axiom (5) of the Selberg class and $\delta > 0$.

In the case when $F \in \mathcal{S}$ with non-negative coefficients, we can get sharper upper bound for the height of the first zero of F with positive imaginary part, as stated in the following

Theorem 3 *Let h be the height of the first zero with imaginary part different from zero of the function $F \in \mathcal{S}$ and $F(1 + it) \neq 0$ for all $t \in \mathbb{R}$ such that $a_F(n) \geq 0$ for all $n \in \mathbb{N}$. Then, for $q_F > e$ we have the bound*

$$h \leq \max\left\{\frac{16\sqrt{2}\Big[(4m_F + 1)\log q_F + B(F)\Big]}{\pi \log q_F \log\log q_F}, \frac{\pi}{\sqrt{2}\log[\log q_F/16(m_F + \tau)]}\right\},$$

where q_F is conductor of F, m_F is the polar order of F, $B(F)$ is given in Theorem 1 and $\tau > 0$.

3 Preliminaries

3.1 Explicit Formula for Functions in $\mathcal{S}^{\sharp\flat}$

The universal class of test functions in this paper is the class W of regulated functions [3] i.e. functions possessing the one-sided limits at each point. For $f \in W$, we always suppose $2f(x) = f(x+0) + f(x-0)$. If I is an interval with endpoints a and $b(a < b)$, we write $f(I) = f(b) - f(a)$.

Let ϕ be continuous function defined on $[0, \infty)$ and strictly increasing from 0 to ∞. A function f is said to be of ϕ-bounded variation on I if

$$V_\phi(f, I) = \sup \sum_n \phi(|f(I_n)|),$$

where the supremum is taken over all systems $\{I_n\}$ of non-overlapping subintervals of I (cf. [22]).

The crucial tool for deriving our main results is the explicit formula for functions in the Selberg class and its generalizations, applied to suitably constructed test functions.

Theorem 4 [20, Theorem 3.1], [10, Proposition 2.2] *Let a regularized function G satisfy the following conditions:*

1. $G \in \phi BV(\mathbb{R}) \cap L^1(\mathbb{R})$.
2. $G(x)e^{(1/2+\epsilon)|x|} \in \phi BV(\mathbb{R}) \cap L^1(\mathbb{R})$, *for some* $\epsilon > 0$.
3. $G(x) + G(-x) - 2G(0) = O(|\log|x||^{-\alpha})$, *as* $x \to 0$, *for some* $\alpha > 2$.

Let $g(x) = G(-\log x)$, *for* $x > 0$, $G_j(x) = G(x)\exp\left(\frac{ix\Im\mu_j}{\lambda_j}\right)$ *and* $Z(F)$ *the set of all non-trivial zeros of* $F \in \mathcal{S}^{\sharp\flat}$. *Then, the formula*

$$\begin{aligned}
&\lim_{a\to\infty} \sum_{\rho\in Z(F)|\Im\rho|\le a} ord(\rho) M_{\frac{1}{2}} g(\rho) \\
&= m_F M_{\frac{1}{2}} g(0) + m_F M_{\frac{1}{2}} g(1) \\
&- \sum_n \frac{c_F(n)}{n^{\frac{1}{2}}} g(n) - \sum_n \frac{\overline{c_F}(n)}{n^{\frac{1}{2}}} g(1/n) + 2G(0)\log Q_F \\
&+ \sum_{j=1}^{r} \int_0^\infty \left[\frac{2\lambda_j G_j(0)}{x} - \frac{\exp((1-\frac{\lambda_j}{2} - \Re\mu_j)\frac{x}{\lambda_j})}{1-e^{\frac{-x}{\lambda_j}}}(G_j(x) + G_j(-x))\right] e^{\frac{-x}{\lambda_j}}\,\mathrm{d}x
\end{aligned}$$

holds true for an arbitrary function $F \in \mathcal{S}^{\sharp\flat}$, *where*

$$M_{\frac{1}{2}} g(s) = \int_{-\infty}^{\infty} G(x)e^{(s-1/2)x}\,\mathrm{d}x$$

denotes the translate by $1/2$ *of the Mellin transform of the function* g.

Corollary 1 *Let G be an even regularized function satisfying conditions of Theorem 4 then, the formula*

$$\lim_{a\to\infty} \sum_{\rho\in Z(F)|\Im\rho|\le a} ord(\rho) M_{\frac{1}{2}} g(\rho)$$
$$= m_F M_{\frac{1}{2}} g(0) + m_F M_{\frac{1}{2}} g(1) - 2\sum_n \frac{\Re(c_F(n))}{n^{\frac{1}{2}}} g(1/n)$$
$$+ G(0)\Big(\log q_F - d_F \log(2\pi) - 2\sum_{j=1}^{r} (\lambda_j \log \lambda_j)\Big)$$
$$+ 2\sum_{j=1}^{r} \int_0^\infty \left[\frac{\lambda_j G(0)}{x} - \frac{\exp((1-\frac{\lambda_j}{2} - \Re\mu_j)\frac{x}{\lambda_j})}{1-e^{\frac{-x}{\lambda_j}}} G(x)\cosh\left(\frac{ix\Im\mu_j}{\lambda_j}\right)\right] e^{\frac{-x}{\lambda_j}}\,\mathrm{d}x \tag{3}$$

holds true for an arbitrary function $F \in \mathcal{S}^{\sharp\flat}$.

Proof Since G is even function then

$$G(-\log x) = G(\log x), \quad x > 0,$$

hence $g(x) = g(1/x)$, which yields

$$\sum_n \frac{c_F(n)}{n^{\frac{1}{2}}} g(n) + \sum_n \frac{\overline{c_F}(n)}{n^{\frac{1}{2}}} g(1/n) = 2\sum_n \frac{\Re(c_F(n))}{n^{\frac{1}{2}}} g(1/n),$$

and

$$G_j(x) + G_j(-x) = 2G(x)\cosh\left(\frac{ix\Im\mu_j}{\lambda_j}\right).$$

Furthermore, from (1) we get

$$2\log Q_F = \log q_F - d_F \log(2\pi) - 2\sum_{j=1}^{r} \lambda_j \log \lambda_j.$$

This completes the proof. □

3.2 The Prime Number Theorem in the Selberg Class

For $F \in \mathcal{S}$ let us denote by

$$\psi_F(x) = \sum_{n\le x} c_F(n)$$

the analogue of the Chebyshev ψ-function, where $c_F(n)$ is defined by (2).

The Selberg class analogue of the prime number theorem is a theorem that explains the asymptotic behaviour of the function $\psi_F(x)$, as $x \to \infty$.

Kaczorowski and Perelli [9] have proved the equivalence between the prime number theorem for the Selberg class and non-vanishing on the line $\Re s = 1$ for every function in $\mathcal{S}$, without using Tauberian arguments. They proved the following theorem.

Theorem 5 [9, Theorem 1] *Let $F \in \mathcal{S}$. Then $\psi_F(x) = m_F x + p_F(x)$, where $p_F(x) = o(x)$ as $x \to \infty$ if and only if $F(1+it) \neq 0$ for every $t \in \mathbb{R}$.*

4 Preliminary Lemmas

In the proof of our main results, we will need the following lemmas.

Lemma 1 [11, p. 63] *Let G be defined by*

$$G(x) = \begin{cases} 1 - |x|, & \text{if } |x| \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then G satisfies the conditions of Corollary 1 and

$$\hat{G}(u) = \left(\frac{2 \sin \frac{u}{2}}{u} \right)^2,$$

where $\hat{G}$ is the Fourier transform of G.

Lemma 2 [11, Lemma 1] *Let H be the function with compact support on* $[0, \infty]$ *defined by*

$$H(x) = \begin{cases} (1-x)\cos(\pi x) + \frac{3}{\pi}\sin(\pi x), & \text{if } 0 \le x \le 1, \\ (1+x)\cos(\pi x) - \frac{3}{\pi}\sin(\pi x), & \text{if } -1 \le x < 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then H satisfies the condition of Corollary 1 and

$$\hat{H}(u) = 2\left(2 - \frac{u^2}{\pi^2}\right)\left(\frac{2\pi}{\pi^2 - u^2} \cos \frac{u}{2}\right)^2.$$

The proof of Lemmas 1 and 2 is based on partial integration of the Mellin transform.

Let $b_F(n)$ be as in axiom (5) of the Selberg class. Then there exists $C_F \ge 1$ such that

$$|b_F(n)| \le C_F n^{\theta}, \quad \theta < 1/2. \tag{4}$$

The Chebyshev function is defined by $\psi(x) = \sum_{n \le x} \Lambda(n)$, where $\Lambda(n)$ is von Mangoldt function. It satisfies the asymptotic formula

$$\psi(x) = x + r(x), \tag{5}$$

where $r(x) = O(x\exp(-a\sqrt{\log x}))$ for some $a > 0$ and x large enough (see e.g. [2, p.111]).

Lemma 3 *Let $H_T(x) = H(x/T)$, where H is defined in Lemma 2 and $g_T(1/n) = H\left(\frac{\log n}{T}\right)$.*

(a) For $F \in \mathcal{S}^{\sharp\flat}$ satisfying axiom (5) of the Selberg class we have

$$\sum_n \frac{|c_F(n)|}{n^{\frac{1}{2}}} g_T(1/n) \le 4K_F e^{\frac{T}{2}(2\theta+1)} + r_2(T),$$

where

$$K_F = \frac{C_F}{2\theta+1},$$

$$r_2(T) = C_F\left(2e^{\frac{T}{2}(2\theta-1)} r(e^T) + 2r_1(e^T)e^{-\frac{T}{2}} + \int_1^{e^T} x^{\theta-\frac{3}{2}} r(x)dx + \int_1^{e^T} x^{-\frac{3}{2}} r_1(x)\mathrm{d}x\right),$$

$$r_1(x) = \frac{\theta}{\theta+1} - \theta\int_1^x t^{\theta-1} r(t)\mathrm{d}t,$$

and C_F, $r(x)$ are as in (4), (5), respectively.

(b) For $F \in \mathcal{S}$ and $F(1+it) \neq 0$ such that $a_F(n) \geq 0$ for all $n \in \mathbb{N}$ we have

$$\sum_n \frac{c_F(n)}{n^{\frac{1}{2}}} g_T(1/n) \le 4m_F e^{\frac{T}{2}} + P_F(T),$$

where

$$P_F(T) = 2p_F(e^T)e^{-\frac{T}{2}} + \int_1^{e^T} p_F(x)x^{-\frac{3}{2}}\mathrm{d}x,$$

m_F is as in axiom (2) of the Selberg class and p_F is as in Theorem 5.

Proof Definition of H yields

$$H\left(\frac{\log n}{T}\right) = \begin{cases} \left(1-\frac{\log n}{T}\right)\cos\left(\frac{\pi\log n}{T}\right) + \frac{3}{\pi}\sin\left(\frac{\pi\log n}{T}\right), & \text{if } 0 \le \log n \le T, \\ \left(1+\frac{\log n}{T}\right)\cos\left(\frac{\pi\log n}{T}\right) - \frac{3}{\pi}\sin\left(\frac{\pi\log n}{T}\right), & \text{if } -T \le \log n < 0, \\ 0, & \text{otherwise,} \end{cases}$$

hence

$$-2 \le H\left(\frac{\log n}{T}\right) \le 2, \quad \text{for } e^{-T} \le n \le e^{T}.$$

(a) Let $\varphi_F(x) = \sum_{n\le x} |c_F(n)|$. From (2) and (4) we get

$$\varphi_F(x) = \sum_{n\le x} |b_F(n)| \log n \le C_F \sum_{p^k\le x} p^{k\theta} \log p^k = C_F \sum_{n\le x} n^{\theta} \Lambda(n).$$

Therefore

$$\varphi_F(x) \le C_F \sum_{n\le x} n^{\theta} \Lambda(n) = C_F \int_1^x t^{\theta} d\psi(t).$$

With partial integration we have

$$\int_1^x t^{\theta} d\psi(t) = \frac{1}{\theta+1} x^{\theta+1} + x^{\theta} r(x) + r_1(x),$$

hence

$$\varphi_F(x) \le \frac{C_F}{\theta+1} x^{\theta+1} + C_F x^{\theta} r(x) + C_F r_1(x).$$

Now, we have the following estimate of the sum

$$\sum_n \frac{|b_F(n)| \log n}{n^{\frac{1}{2}}} g_T(1/n) \le 2 \sum_{n\le e^T} \frac{|b_F(n)| \log n}{n^{\frac{1}{2}}} = 2 \int_1^{e^T} \frac{1}{x^{1/2}} d\varphi_F(x).$$

An integration by parts of the last integral gives

$$\int_1^{e^T} \frac{1}{x^{1/2}} d\varphi_F(x) \le \frac{\varphi_F(e^T)}{e^{T/2}} + \frac{1}{2} \int_1^{e^T} \frac{\varphi_F(x)}{x^{3/2}} \mathrm{d}x \le 2K_F e^{\frac{T}{2}(2\theta+1)} + \frac{1}{2} r_2(t),$$

it follows

$$\sum_n \frac{|b_F(n)| \log n}{n^{\frac{1}{2}}} g_T(1/n) \le 4K_F e^{\frac{T}{2}(2\theta+1)} + r_2(t).$$

(b) Since $F \in \mathcal{S}$ and $F(1+it) \ne 0$ from (2), Theorem 5 and definition of g_T we have

$$\sum_n \frac{c_F(n)}{n^{\frac{1}{2}}} g_T(1/n) \le 2 \sum_{1\le n\le e^T} \frac{c_F(n)}{n^{\frac{1}{2}}} = 2 \int_1^{e^T} \frac{1}{x^{\frac{1}{2}}} d\psi_F(x).$$

With partial integration we have

$$\int_{1}^{e^T} \frac{1}{x^{\frac{1}{2}}} d\psi_F(x) \leq 2m_F e^{\frac{T}{2}} + \frac{1}{2} P_F(T),$$

hence

$$\sum_{n} \frac{c_F(n)}{n^{\frac{1}{2}}} g_T(1/n) \leq 4m_F e^{\frac{T}{2}} + P_F(T).$$

□

Lemma 4 [11, Lemma 3] *Let A, B, C be three positive real constants and $\alpha > 0$. If $T > 0$ satisfies $AT + Be^{\alpha T} \geq C$, then*

$$T \geq \min\left\{\frac{C}{2A}, \frac{\log(C/2B)}{\alpha}\right\}.$$

Proof By contradiction. □

5 Proof of Main Results

In this section we prove main results of the paper given in Sect. 2.

5.1 Proof of Theorem 1

Let $s = \sigma + it$. The Mellin transform of G is given by

$$M_{\frac{1}{2}} g(s) = \int_{-\infty}^{\infty} G(x) e^{(s-1/2)x} \mathrm{d}x = \int_{-\infty}^{\infty} G(x) e^{(\sigma-1/2)x} e^{itx} \mathrm{d}x = \hat{G}_\sigma(t),$$

where

$$G_\sigma(t) = G(x) e^{(\sigma-1/2)x}.$$

If $\sigma = 1/2$ then

$$M_{\frac{1}{2}} g\left(\frac{1}{2} + it\right) = \int_{-\infty}^{\infty} G(x) e^{itx} \mathrm{d}x = \hat{G}(t)$$

For $t = 0$ we have

$$M_{\frac{1}{2}} g\left(\frac{1}{2}\right) = \hat{G}(0) = 1.$$

Now,

$$M_{\frac{1}{2}}g(0)+M_{\frac{1}{2}}g(1)=\int\limits_{-\infty}^{\infty}G(x)e^{-\frac{x}{2}}\mathrm{d}x+\int\limits_{-\infty}^{\infty}G(x)e^{\frac{x}{2}}\mathrm{d}x=4\int\limits_{0}^{\infty}G(x)\cosh\Big(\frac{x}{2}\Big)\mathrm{d}x.$$

Setting $G_T(x)=G(x/T)$ for $T>0$ we get

$$\hat{G}_T(u)=\int\limits_{-\infty}^{\infty}G_T(x)e^{iux}\mathrm{d}x=\int\limits_{-\infty}^{\infty}G(x/T)e^{iux}\mathrm{d}x.$$

Substituting $x/T=t$ we get

$$\hat{G}_T(u)=T\int\limits_{-\infty}^{\infty}G(t)e^{iuTt}\mathrm{d}t=T\hat{G}(Tu).$$

If R is order of eventual zero of $F(s)\in\mathcal{S}^{\sharp\flat}$ at point $\rho=1/2$ then applying explicit formula (3) for the function $G_T(x)$ we obtain the inequalities

$$\begin{aligned}
&RM_{\frac{1}{2}}g_T(1/2)\\
&\le\lim_{a\to\infty}\sum_{\rho\in Z(F)|\Im\rho|\le a}ord(\rho)M_{\frac{1}{2}}g_T(\rho)\\
&=m_F(M_{\frac{1}{2}}g_T(0)+M_{\frac{1}{2}}g_T(1))-2\sum_n\frac{\Re(c_F(n))}{n^{\frac{1}{2}}}g_T(1/n)\\
&\quad+G_T(0)\Big(\log q_F-d_F\log(2\pi)-2\sum_{j=1}^{r}(\lambda_j\log\lambda_j)\Big)\\
&\quad+2\sum_{j=1}^{r}\int\limits_{0}^{\infty}\Big[\frac{\lambda_jG_{T,j}(0)}{x}-\frac{\exp((1-\frac{\lambda_j}{2}-\Re(\mu_j))\frac{x}{\lambda_j})}{1-e^{\frac{-x}{\lambda_j}}}G_T(x)\cosh\Big(\frac{ix\Im\mu_j}{\lambda_j}\Big)\Big]e^{\frac{-x}{\lambda_j}}\mathrm{d}x\\
&\le 4m_F\int\limits_{0}^{\infty}G_T(x)\cosh\Big(\frac{x}{2}\Big)\mathrm{d}x+\log q_F-2\sum_{j=1}^{r}\lambda_j\log(2\pi\lambda_j))\\
&\quad+\sum_{j=1}^{r}\int\limits_{0}^{\infty}\Big[\frac{2\lambda_jG_{T,j}(0)}{x}-\frac{\exp((1-\frac{\lambda_j}{2}-\Re(\mu_j))\frac{x}{\lambda_j})}{1-e^{\frac{-x}{\lambda_j}}}(G_{T,j}(x)+G_{T,j}(-x))\Big]e^{\frac{-x}{\lambda_j}}\mathrm{d}x.
\end{aligned}\tag{6}$$

We denote by

$$I=\int\limits_{0}^{\infty}\Big[\frac{2\lambda_jG_{T,j}(0)}{x}-\frac{\exp((1-\frac{\lambda_j}{2}-\Re\mu_j)\frac{x}{\lambda_j})}{1-e^{\frac{-x}{\lambda_j}}}(G_{T,j}(x)+G_{T,j}(-x))\Big]e^{\frac{-x}{\lambda_j}}\mathrm{d}x.$$

Substituting in the above integral $\frac{x}{\lambda_j} = t$, employing the equality $G_{T,j}(0) = 1$, we get

$$I = \lambda_j \int_0^\infty \left[\frac{2}{t} - \frac{\exp((1 - \frac{\lambda_j}{2} - \Re\mu_j)t)}{1 - e^{-t}}(G_{T,j}(\lambda_j t) + G_{T,j}(-\lambda_j t))\right] e^{-t} \mathrm{d}t.$$

By the definition of function G_j it follows that

$$G_{T,j}(\lambda_j t) + G_{T,j}(-\lambda_j t) = \begin{cases} \left(1 - \frac{\lambda_j t}{T}\right)\left(e^{it\Im\mu_j} + e^{-it\Im\mu_j}\right), & \text{if } 0 \le t \le \frac{T}{\lambda_j} \\ 0, & \text{otherwise.} \end{cases}$$

For $0 \le t \le \frac{T}{\lambda_j}$

$$\left(1 - \frac{\lambda_j t}{T}\right) \le 1,$$

hence

$$\begin{aligned} I &\le \lambda_j \left[\int_0^\infty \left[\frac{1}{t} - \frac{\exp((1 - \frac{\lambda_j}{2} - \Re\mu_j)t)}{1 - e^{-t}} e^{it\Im\mu_j} \right] e^{-t} \mathrm{d}t \right. \\ &\quad \left. + \int_0^\infty \left[\frac{1}{t} - \frac{\exp((1 - \frac{\lambda_j}{2} - \Re\mu_j)t)}{1 - e^{-t}} e^{-it\Im\mu_j} \right] e^{-t} \mathrm{d}t \right] \\ &= \lambda_j \left[\int_0^\infty \left[\frac{1}{t} - \frac{\exp((1 - \frac{\lambda_j}{2} - \bar{\mu}_j)t)}{1 - e^{-t}} \right] e^{-t} \mathrm{d}t \right. \\ &\quad + \int_0^\infty \left[\frac{1}{t} - \frac{\exp((1 - \frac{\lambda_j}{2} - \mu_j)t)}{1 - e^{-t}} e^{-it\Im\mu_j} \right] e^{-t} \mathrm{d}t \\ &= \lambda_j \left(\Psi\left(\frac{\lambda_j}{2} + \bar{\mu}_j\right) + \Psi\left(\frac{\lambda_j}{2} + \mu_j\right) \right). \end{aligned}$$

Since

$$\begin{aligned} 4 \int_0^\infty G_T(x) \cosh\left(\frac{x}{2}\right) \mathrm{d}x &= 4 \int_0^T (1 - \tfrac{x}{T}) \cosh\left(\tfrac{x}{2}\right) \mathrm{d}x \\ &\le 4 \int_0^T \cosh\left(\tfrac{x}{2}\right) \mathrm{d}x \le 4e^{T/2}, \end{aligned}$$

$$M_{\frac{1}{2}} g_T(1/2) = T\hat{G}(T \cdot 0) = T \cdot 1 = T,$$

we get

$$\begin{aligned} RT &\le 4m_F e^{T/2} + \log q_F - 2 \sum_{j=1}^r \lambda_j \log(2\pi\lambda_j) \\ &\quad + \sum_{j=1}^r \lambda_j \left(\Psi\left(\frac{\lambda_j}{2} + \bar{\mu}_j\right) \right) + \sum_{j=1}^r \lambda_j \left(\Psi\left(\frac{\lambda_j}{2} + \mu_j\right) \right). \end{aligned}$$

It follows that

$$\begin{aligned}RT &\leq 4m_F e^{T/2} + \log q_F - 2\sum_{j=1}^{r} \lambda_j \log(2\pi\lambda_j) + 2\sum_{j=1}^{r} \lambda_j \Re\Big(\Psi\Big(\frac{\lambda_j}{2} + \mu_j\Big)\Big)\\ &= 4m_F e^{T/2} + \log q_F + B(F).\end{aligned}$$

Setting $T = 2\log\log q_F$ for $q_F > e$ we get

$$2R\log\log q_F \leq 4m_F \log q_F + \log q_F + B(F),$$

hence

$$R \leq \frac{4m_F \log q_F + \log q_F + B(F)}{2\log\log q_F}.$$

If $0 < q_F \leq e$, then

$$RT \leq 4m_F e^{T/2} + B(F) + 1,$$

hence

$$R \leq \inf_{T>0}\Big\{\frac{4m_F e^{T/2} + B(F) + 1}{T}\Big\}.$$

If $m_F = 0$ then $\inf_{T>0}\Big\{\frac{4m_F e^{T/2}+B(F)+1}{T}\Big\} = 0$. Otherwise, let

$$f(T) = \frac{4m_F e^{T/2} + B(F) + 1}{T}.$$

Then function f has minimum at point $T > 0$ satisfying equation

$$2m_F e^{T/2}(T-2) = B(F) + 1.$$

We can solve the last equation using the Lambert W-function and get

$$T = 2\Big(W\Big(\frac{B(F)+1}{4em_F}\Big) + 1\Big).$$

This proves our theorem.

As an immediate consequence of the above theorem, in the case when the conductor of function F is small, we get the following

Corollary 2 *Let $F \in \mathcal{S}^{\sharp\flat}$ be such that $\Re(c_F(n)) \geq 0$. Assume also that the conductor, q_F of F is less then or equal to e and that F is holomorphic. Then, $F(1/2) \neq 0$, i.e. F is non-vanishing at the central point.*

Remark 1 From the proof of the Theorem 1 it is easy to see that the statement of theorem holds true under slightly less restrictive assumptions on $\Re(c_F(n))$. Namely, it is sufficient to assume that

$$\sum_n \frac{\Re(c_F(n))}{n^{\frac{1}{2}}} g_T(1/n) \geq 0,$$

see formula (6).

5.2 Proof of Theorem 2

For $T = \sqrt{2}\pi/h$ and $u \geq h$ it is easy to see that

$$M_{\frac{1}{2}} g_T\left(\frac{1}{2} + iu\right) = \hat{H}_T(u) \leq 0,$$

hence from the *GRH* and Lemma 2 we have

$$\sum_{\rho \in Z(F)|\Im\rho|\leq a} ord(\rho) M_{\frac{1}{2}} g_T(\rho) = R M_{\frac{1}{2}} g_T(0) + \sum_{\rho\in Z(F)\rho\neq 1/2|\Im\rho|\leq a} ord(\rho) M_{\frac{1}{2}} g_T(\rho)$$
$$\leq R\hat{H}_T(0) = \frac{16}{\pi^2} RT,$$

for all $a > 1$. Therefore letting $a \to \infty$ and applying explicit formula (3) we obtain the inequality

$$\begin{aligned}\frac{16}{\pi^2} RT &\geq m_F(M_{\frac{1}{2}} g_T(0) + M_{\frac{1}{2}} g_T(1)) - 2\sum_n \frac{\Re(c_F(n))}{n^{\frac{1}{2}}} g_T(1/n)\\ &+ H_T(0)\Big(\log q_F - d_F \log(2\pi) - 2\sum_{j=1}^{r}(\lambda_j \log \lambda_j)\Big)\\ &+ \sum_{j=1}^{r}\int_0^\infty 2\left[\frac{\lambda_j H_{T,j}(0)}{x} - \frac{\exp((1-\frac{\lambda_j}{2} - \Re\mu_j)\frac{x}{\lambda_j})}{1-e^{\frac{-x}{\lambda_j}}} H_T(x)\cosh\left(\frac{ix\Im\mu_j}{\lambda_j}\right)\right] e^{\frac{-x}{\lambda_j}}\,\mathrm{d}x.\end{aligned} \tag{7}$$

Since

$$M_{\frac{1}{2}} g_T(0) + M_{\frac{1}{2}} g_T(1) = 4\int_0^\infty H_T(x)\cosh\left(\frac{x}{2}\right)\mathrm{d}x,$$

by the definition of function $H_T(x)$ we have

$$M_{\frac{1}{2}} g_T(0) + M_{\frac{1}{2}} g_T(1) = 4\int_0^T \left[\left(1 - \frac{x}{T}\right)\cos\left(\frac{\pi x}{T}\right) + \frac{3}{\pi}\sin\left(\frac{\pi x}{T}\right)\right]\cosh\left(\frac{x}{2}\right)\mathrm{d}x.$$

Using partial integration we get

$$M_{\frac{1}{2}}g_T(0)+M_{\frac{1}{2}}g_T(1)\geq\frac{16T^3}{(4\pi^2+T^2)^2}e^{\frac{T}{2}}. \tag{8}$$

Since $H_{T,j}(0)=1$ and from the definition of the function $H_T(x)$ we get

$$\begin{aligned}
&\int_0^\infty 2\left[\frac{\lambda_j H_{T,j}(0)}{x}-\frac{\exp((1-\frac{\lambda_j}{2}-\Re\mu_j)\frac{x}{\lambda_j})}{1-e^{\frac{-x}{\lambda_j}}}H_T(x)\cosh\left(\frac{ix\Im\mu_j}{\lambda_j}\right)\right]e^{\frac{-x}{\lambda_j}}\,\mathrm{d}x\\
&=\int_0^T\left[\frac{\lambda_j e^{\frac{-x}{\lambda_j}}}{x}-\frac{e^{-(\frac{\lambda_j}{2}+\mu_j)\frac{x}{\lambda_j}}}{1-e^{\frac{-x}{\lambda_j}}}\left(\left(1-\frac{x}{T}\right)\cos\left(\frac{\pi x}{T}\right)+\frac{3}{\pi}\sin\left(\frac{\pi x}{T}\right)\right)\right]\mathrm{d}x\\
&\quad+\int_0^T\left[\frac{\lambda_j e^{\frac{-x}{\lambda_j}}}{x}-\frac{e^{-(\frac{\lambda_j}{2}+\bar{\mu}_j)\frac{x}{\lambda_j}}}{1-e^{\frac{-x}{\lambda_j}}}\left(\left(1-\frac{x}{T}\right)\cos\left(\frac{\pi x}{T}\right)+\frac{3}{\pi}\sin\left(\frac{\pi x}{T}\right)\right)\right]\mathrm{d}x\\
&=I_1+I_2.
\end{aligned}$$

For simplicity, we evaluate I_1. Substituting $x=tT$ we have

$$I_1=T\int_0^1\left[\frac{e^{\frac{-Tt}{\lambda_j}}}{\frac{Tt}{\lambda_j}}-\frac{e^{-(\frac{\lambda_j}{2}+\mu_j)\frac{Tt}{\lambda_j}}}{1-e^{\frac{-Tt}{\lambda_j}}}\left((1-t)\cos(\pi t)+\frac{3}{\pi}\sin(\pi t)\right)\right]\mathrm{d}x.$$

Expanding the function under the integral sign in the Taylor series at $t=0$, we see that it is bounded as $t\to 0$ and the bound is independent of T, hence

$$\begin{aligned}
&\sum_{j=1}^r\int_0^\infty 2\left[\frac{\lambda_j H_{T,j}(0)}{x}-\frac{\exp\left(\left(1-\frac{\lambda_j}{2}-\Re\mu_j\right)\frac{x}{\lambda_j}\right)}{1-e^{\frac{-x}{\lambda_j}}}H_T(x)\cosh\left(\frac{ix\Im\mu_j}{\lambda_j}\right)\right]\\
&\quad e^{\frac{-x}{\lambda_j}}\,\mathrm{d}x=cT.
\end{aligned}\tag{9}$$

Now, using Lemma 3a, (8) and (9), inequality (7) gives an inequality

$$\begin{aligned}
\frac{16}{\pi^2}RT&\geq\frac{16m_F T^3}{(4\pi^2+T^2)^2}e^{\frac{T}{2}}-8K_F e^{\frac{T}{2}(2\theta+1)}-2r_2(T)\\
&\quad+\log q_F-2\sum_{j=1}^r\lambda_j\log(2\pi\lambda_j)+cT.
\end{aligned}$$

For T large enough there exists $\delta > 0$ such that

$$\left|\frac{16m_F T^3}{(4\pi^2+T^2)^2}e^{\frac{T}{2}} - 2r_2(T) - 2\sum_{j=1}^{r}\lambda_j \log(2\pi\lambda_j) + cT\right| \leq 8\delta e^{\frac{T}{2}(2\theta+1)},$$

hence

$$\frac{16}{\pi^2}RT \geq -8(K_F+\delta)e^{\frac{T}{2}(2\theta+1)} + \log q_F.$$

Setting

$$A = \frac{8}{\pi^2}\frac{(4m_F+1)\log q_F + B(F)}{\log\log q_F},\ B = 8(K_F+\delta),\ C = \log q_F \text{ and } \alpha = \theta + \frac{1}{2}$$

the result of Theorem 2 easily follows from Theorem 1 and Lemma 4.

5.3 Proof of Theorem 3

For $T = \sqrt{2}\pi/h$, applying the explicit formula (3) as in the proof of Theorem 2 we obtain the inequality (7).

For $a_F(n) \geq 0$, by [21, p. 294] we have $m_F > 0$, hence using (8), (9) and Lemma 3b, inequality (7) yields the inequality

$$\frac{16}{\pi^2}RT \geq \frac{16m_F T^3}{(4\pi^2+T^2)^2}e^{\frac{T}{2}} - 8m_F e^{\frac{T}{2}} - 2P_F(T) + \log q_F - 2\sum_{j=1}^{r}\lambda_j\log(2\pi\lambda_j) + cT.$$

For T large enough there exists $\tau > 0$ such that

$$\left|\frac{16m_F T^3}{(4\pi^2+T^2)^2}e^{\frac{T}{2}} - 2P_F(T) - 2\sum_{j=1}^{r}\lambda_j\log(2\pi\lambda_j) + cT\right| \leq 8\tau e^{\frac{T}{2}},$$

hence

$$\frac{16}{\pi^2}RT \geq -8(m_F+\tau)e^{\frac{T}{2}} + \log q_F.$$

Setting

$$A = \frac{8}{\pi^2}\frac{(4m_F+1)\log q_F + B(F)}{\log\log q_F},\ B = 8(m_F+\tau),\ C = \log q_F \text{ and } \alpha = \frac{1}{2}$$

the result of Theorem 3 easily follows from Theorem 1 and Lemma 4.

6 An Application to Automorphic L-Functions

Let π be an irreducible unitary cuspidal representation of $GL_N(\mathbb{Q})$. Then the (finite) automorphic L-function $L(s,\pi)$ attached to π is given by products of local factors for $\Re s > 1$ (see e.g. [4])

$$L(s,\pi) = \prod_p \prod_{j=1}^{N} \left(1 - \alpha_{p,j}(\pi) p^{-s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{a_n(\pi)}{n^s},$$

where

$$a_n(\pi) = \sum_{j=1}^{N} \alpha_{p,j}(\pi)^k.$$

Therefore

$$\log L(s,\pi) = \sum_{n=1}^{\infty} \frac{b_n(\pi)}{n^s}$$

and

$$\frac{L'}{L}(s,\pi) = -\sum_{n=1}^{\infty} \frac{c_n(\pi)}{n^s},$$

where

$$b_n(\pi) = \frac{\Lambda(n) a_n(\pi)}{\log n},$$

$$c_n(\pi) = b_n(\pi) \log n. \tag{10}$$

In the series of papers [16–19], Shahidi has shown that the complete L-function

$$\Lambda(s,\pi) = Q(\pi)^{s/2} L_\infty(s,\pi_\infty) L(s,\pi),$$

where $Q(\pi) > 0$ is the conductor of π and

$$L_\infty(s,\pi_\infty) = \prod_{j=1}^{N} \Gamma_{\mathbb{R}}(s + \kappa_j(\pi)).$$

is the archimedean factor, satisfies the functional equation

$$\Lambda(s,\pi) = \epsilon(\pi) \overline{\Lambda(1-\bar{s},\pi)}$$

with a constant $\epsilon(\pi)$ of absolute value 1. Here, $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and the parameters κ_j satisfy the inequality $\Re \kappa_j > -1/2$, as proved by Rudnick and Sarnak in [14]. Jacquet and Shalika in [5] proved that

$$L(s,\pi) \neq 0 \quad \text{for} \quad \Re s = 1.$$

It is easy to see that $L(s,\pi) \in \mathcal{S}^{\sharp\flat}$ [10, pp. 533–534] with $r = N, Q_F = Q(\pi)^{1/2}\pi^{-N/2}, \lambda_j = \frac{1}{2}, \mu_j = \frac{1}{2}\kappa_j(\pi), j = 1, \ldots, N, d_F = N$ and the parameters κ_j satisfy the inequality $\Re\kappa_j > -1/2$.

Assuming *GRH* for automorphic L-functions and applying results of Theorems 1 and 2 to $L(s,\pi) \in \mathcal{S}^{\sharp\flat}$ we get the following corollaries.

Corollary 3 *Let R be the multiplicity of the eventual zero at the central point $1/2$ of $L(s,\pi)$ such that $\Re(c_n(\pi)) \geq 0$ and let*

$$B(L) = \sum_{j=1}^{N} \Re\Big(\Psi\Big(\frac{1}{4} + \frac{1}{2}\kappa_j(\pi)\Big)\Big) - N\log\pi.$$

(a) If $Q(\pi) > e$ then

$$R \leq \frac{(4m_L + 1)\log Q(\pi) + B(L)}{2\log\log Q(\pi)}.$$

(b) If $0 < Q(\pi) \leq e$ then
  (i) $R = 0$, when $N > 1$ or $N = 1$ and $\pi \neq Id$,
  (ii) $R \leq \dfrac{4m_L e^{W\left(\frac{1-\gamma-\pi/2-\log 8\pi}{4e}\right)+1} + 1 - \gamma - \pi/2 - \log 8\pi}{2\left(W\left(\frac{1-\gamma-\pi/2-\log 8\pi}{4e}\right)+1\right)}.$

where W denotes the Lambert function. Specially, if $L(s,\pi) \neq \zeta(s)$ is automorphic L-function with analytic conductor $Q(\pi)$ less than or equal to e, then $L(s,\pi)$ is non-vanishing at central point $s = 1/2$.

Proof Part a) of the statement follows immediately from Theorem 1a with $r = N$ and $\lambda_j = \frac{1}{2}, \mu_j = \frac{1}{2}\kappa_j(\pi), j = 1, \ldots, N, Q_F = Q(\pi)^{1/2}\pi^{-N/2}$.

When $N > 1$ or $N = 1$ and $\pi \neq Id$ L-function is entire, hence $m_L = 0$, thus Theorem 1b yields $R = 0$.

When $N = 1$ and $\pi = Id, L(s,\pi) = \zeta(s)$, hence $B(L) = \Psi(1/4) - \log\pi$. Moreover, $\Psi(1/4) = -\frac{\pi}{2} - 3\log 2 - \gamma$, thus $B(L) = -\frac{\pi}{2} - \log 8\pi - \gamma$. The proof is complete. □

Corollary 4 *Let h be the height of the first zero with imaginary part different from zero of the function $L(s,\pi)$. Assume that $L(s,\pi)$ satisfies axiom (5) of the Selberg class and $\Re(c_n(\pi)) \geq 0$, where $c_n(\pi)$ are given by (10). Then, for $Q(\pi) > e$ we have the bound*

$$h \leq \max\left\{\frac{16\sqrt{2}\Big[\log Q(\pi) + B(L)\Big]}{\pi\log Q(\pi)\log\log Q(\pi)}, \frac{(2\theta+1)\pi}{\sqrt{2}\log[\log Q(\pi)/16(K_L + \delta)]}\right\}.$$

Here m_L is defined in axiom (2) of the Selberg class, $B(L)$ is given in Corollary *3, K_L is defined in* Lemma *3, $\theta < 1/2$ and $\delta > 0$.*

Proof We proceed analogously as in the proof of Corollary 3, by putting $r = N$, $Q_F = Q(\pi)^{\frac{1}{2}}\pi^{-\frac{N}{2}}$, $\lambda_j = \frac{1}{2}$, $\mu_j = \frac{1}{2}\kappa_j(\pi)$, $j = 1, \ldots, N$. Then, we observe that the conductor q_ζ of the Riemann zeta function is equal to one, hence applying the relation (1) with $r = 1, \quad \lambda_1 = \frac{1}{2}$, we see that $q_\zeta = 1 = \pi Q_\zeta^2$, thus the analytic conductor of the Riemann zeta function is also equal to 1.

Therefore, assumption $Q(\pi) > e$ yields that $L(s, \pi) \neq \zeta(s)$, hence $L(s, \pi)$ is holomorphic and $m_L = 0$.

Once we observe that,

$$q_L = (2\pi)^N Q_L^2 \left(\frac{1}{2}\right)^N = \pi^N Q(\pi)\pi^{-N} = Q(\pi),$$

the corollary follows immediately from Theorem 2. □

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