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A Variation of Coretractable Modules

Nil Orhan Ertaş[1] · Derya Keskin Tütüncü[2] · Rachid Tribak[3]

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Abstract A right R-module M is called *coretractable* (*s-coretractable*) if $\text{Hom}(M/K, M) \neq 0$ for any proper submodule (supplement submodule) K of M. In this article, we continue the study of coretractable modules. Then we study s-coretractable modules. It is shown that this property is not inherited by direct summands and a direct sum of s-coretractable modules may not be s-coretractable. Examples are provided to illustrate and delineate the results.

Keywords Coretractable modules · s-Coretractable modules · Supplement (complement) submodules

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✉ Rachid Tribak
tribak12@yahoo.com

Nil Orhan Ertaş
orhannil@yahoo.com

Derya Keskin Tütüncü
keskin@hacettepe.edu.tr

1 Department of Mathematics, Karabuk University, 78050 Karabuk, Turkey

2 Department of Mathematics, Hacettepe University, Beytepe, 06800 Ankara, Turkey

3 Centre Régional des Métiers de L'Education et de la Formation (CRMEF)-Tanger, Avenue My Abdelaziz, Souani, B.P. 3117, 90000 Tangier, Morocco

1 Introduction

In this paper, R will be a ring with identity and all modules will be unitary right R-modules. We will use the symbol $N \leq M$ to denote that N is a submodule of a module M. The notation $N \ll M$ will mean that N is a small submodule of a module M, namely $M \neq N + X$ for any proper submodule X of M. For a module M, we write $\mathrm{Rad}(M)$, $E(M)$ and $\mathrm{End}_R(M)$ for the Jacobson radical, the injective hull and the endomorphism ring of M, respectively. The Jacobson radical of the ring R will be denoted by $Jac(R)$. By $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$, we denote the ring of rational, integer and natural numbers, respectively. $\mathbb{Z}(p^\infty)$ denotes the Prüfer p-group and $\mathbb{Z}_n$ denotes $\mathbb{Z}/n\mathbb{Z}$.

We also denote $l_S(K) = \{s \in S \mid s(K) = 0\}$ for $K \leq M$ and $r_M(I) = \cap\{\mathrm{Ker} g \mid g \in I\}$ for a left ideal I of $S = \mathrm{End}_R(M)$.

A right R-module M is said to be *retractable* if $\mathrm{Hom}_R(M, U) \neq 0$ for any nonzero submodule U of M. For a background on retractable modules, we refer the reader to [17]. Dually, the notion of coretractable modules was introduced and studied by Amini, Ershad and Sharif in [1]. A right R-module M is said to be *coretractable* if $\mathrm{Hom}_R(M/K, M) \neq 0$ for any proper submodule K of M. In Sect. 2, we explore further properties of coretractable modules. For example, we investigate the factor modules of these modules.

Section 3 is devoted to the connections between a coretractable module M and its endomorphism ring $S = \mathrm{End}_R(M)$. It is shown that for some properties (P) of modules, if ${}_S S$ satisfies (P), then M_R satisfies the dual property of (P). Among other results, we prove that if M is a coretractable module such that ${}_S S$ is Rickart, then M is a dual Rickart module (see Proposition 3.15).

The investigations in Sect. 4 focus on another variation of coretractability which will be called *s-coretractable modules*. This notion is introduced as the dual of the notion of *e-retractable modules* studied in [17]. We provide many examples of these modules. Moreover, some examples are presented to show that an s-coretractable module need not be coretractable. Also, we study rings R for which the module R_R is s-coretractable. Then we deal with direct summands and direct sums of s-coretractable modules.

For undefined terms and terminology, the reader is referred to [2,6,7,10].

2 Some Properties of Coretractable Modules

It is of natural interest to investigate whether or not an algebraic notion is inherited by factor modules. It is shown that a factor module of a coretractable module is not coretractable, in general (Proposition 2.9 and Example 2.10). We present some cases when some factor modules of a coretractable module are coretractable. In [27], Rizvi and Roman introduced a weaker form of nonsingularity. A module M is said to be $\mathcal{K}$*-nonsingular* if, for every nonzero endomorphism φ of M, $\mathrm{Ker}\varphi$ is not essential in M. Note that every nonsingular module is $\mathcal{K}$-nonsingular (see [28, Corollary 2.4]). In [1, Proposition 2.3(b)], it is proved that for a coretractable module M, if M is nonsingular, then M is semisimple. We begin with a slight generalization of this result.

Proposition 2.1 *Let M be a coretractable module. Then M is $\mathcal{K}$-nonsingular if and only if M is semisimple.*

Proof Suppose that M is $\mathcal{K}$-nonsingular but not semisimple. Hence M has a proper essential submodule N by [32, 20.2]. Since M is coretractable, there exists a nonzero homomorphism $\varphi : M \longrightarrow M$ such that $\varphi(N) = 0$. Then $N \subseteq \mathrm{Ker}\varphi$. Since N is essential in M, $\mathrm{Ker}\varphi$ is essential in M. As M is $\mathcal{K}$-nonsingular, we have $\varphi = 0$. This is a contradiction. The converse is immediate. □

Let M be a right R-module and $S = \mathrm{End}_R(M)$. The module M is called a *Rickart module* if for every $\varphi \in S$, $Ker\varphi = \{x \in M \mid \varphi(x) = 0\} = e_\varphi M$ for some idempotent $e_\varphi \in S$ (see [21]).

Recall that the module M is called a *Baer module* if for any submodule N of M, $l_S(N)$ is a direct summand of ${}_SS$, equivalently, for every $I \leq {}_SS$, $r_M(I)$ is a direct summand of M (see [27]).

It is clear that the following implications hold:

$$\text{Baer} \Rightarrow \text{Rickart} \Rightarrow \mathcal{K}\text{-nonsingular}.$$

A ring R (not necessarily commutative) is called a *domain* if for all $x, r \in R$, $xr = 0 \Rightarrow x = 0$ or $r = 0$. The next result is a direct consequence of Proposition 2.1.

Corollary 2.2 *A coretractable module M is semisimple if at least one of the following conditions holds:*

(i) for any $(x, r) \in M \times R$, $xr = 0 \Rightarrow x = 0$ or $r = 0$ (e.g. M is a torsion-free module over a commutative domain);
(ii) M is a Rickart module;
(iii) M is a Baer module.

Next, we provide another application of Proposition 2.1.

Corollary 2.3 *Let R be a commutative domain which is not a field. Let P be a nonzero projective R-module. Then P is not coretractable.*

Proof Note that R has no simple projective modules, since otherwise R has a maximal ideal which is a direct summand. Therefore P is not semisimple. Moreover, it is well known that P is isomorphic to a direct summand of a free R-module. Thus P is torsion-free. The result follows by Corollary 2.2. □

The next example shows that the assumption of coretractability is not superfluous in Proposition 2.1.

Example 2.4 (1) Let R be a commutative domain which is not a field. By Corollary 2.3, the R-module R_R is not coretractable. On the other hand, it is easily seen that R_R is a $\mathcal{K}$-nonsingular R-module which is not semisimple.

(2) Let R be a von Neumann regular ring which is not semisimple. By [1, Proposition 4.4], the R-module R_R is not coretractable. In addition, it is clear that R_R is a $\mathcal{K}$-nonsingular R-module (see [18, Corollary 7.7]). Now, we present two explicit examples of rings which satisfy these conditions.

(a) We can take $R = \prod_{n\geq 1} F_i$, where each $F_i = F$ is a field.
(b) (See [3, Example 1.6] and [21, Example 2.19]) For a field F, let $F_n = F$ for each $n \in \mathbb{N}$. Consider the ring

$$R = \begin{bmatrix} \prod_{n=1}^{\infty} F_n & \oplus_{n=1}^{\infty} F_n \\ \oplus_{n=1}^{\infty} F_n & < \oplus_{n=1}^{\infty} F_n, 1 > \end{bmatrix}$$

which is a subring of the 2×2 matrix ring over the ring $\prod_{n=1}^{\infty} F_n$, where $< \oplus_{n=1}^{\infty} F_n, 1 >$ is the F-algebra generated by $\oplus_{n=1}^{\infty} F_n$ and 1. Then R is a von Neumann regular ring which is not semisimple.

As an application of Corollary 2.2, we can give the following example which will be needed in the last section.

Example 2.5 Let R be the ring as in Example 2.4(2)(b). Consider the right R-module $M = eR$, where e is the idempotent $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. By [21, Example 2.19], M is a Rickart module which is not semisimple. Hence the module M_R is not coretractable by Corollary 2.2.

Recall that a module M is called *quasi-injective* if it is M-injective, that is for every submodule X of M, any homomorphism $\varphi : X \to M$ can be extended to a homomorphism $\psi : M \to M$.

A submodule N of an R-module M is called *fully invariant* if $f(N)$ is contained in N for every R-endomorphism f of M.

Let M be a module with $S = \mathrm{End}_R(M)$. It is clear that $N \subseteq r_M l_S(N)$ for every submodule N of M. In [1, Corollary 4.2], the authors provided some conditions under which $N = r_M l_S(N)$. In a similar vein, we show the next result.

Proposition 2.6 *Let M_R be a quasi-injective coretractable module with $S = \mathrm{End}_R(M)$. Let $N = xR$ be a cyclic fully invariant submodule of M. Then $r_M l_S(N) = N$.*

Proof We only need to show that $r_M l_S(N) \subseteq N$. Let $u \in r_M l_S(N)$. Without loss of generality, we can assume that $N \neq M$. Then $l_S(N) \neq 0$ as M is coretractable. Define the R-homomorphism $\varphi : S/l_S(N) \longrightarrow M$ by $s + l_S(N) \mapsto s(u)$ and the R-monomorphism $\eta : S/l_S(N) \longrightarrow M$ by $s + l_S(N) \mapsto s(x)$. Since M is quasi-injective, there exists a homomorphism $h \in S$ such that $h\eta = \varphi$. Then, $h(x) = u$. Hence $u \in N$ as N is fully invariant. □

Example 2.7 Consider the $\mathbb{Z}$-module $M = \oplus_{i=1}^{n} \mathbb{Z}(p_i^{\infty})$, where p_i $(1 \leq i \leq n)$ are distinct prime integers. By [24, Theorem 3.10], every submodule of M is fully invariant. Moreover, M is an injective coretractable module by [1, Example 2.2 and Proposition 2.6]. Applying Proposition 2.6, it follows that $r_M l_S(N) = N$ for every cyclic submodule of M.

Next, we study the question: does any factor module of a coretractable module inherit the property? We begin with an example which shows that the coretractability property does not transfer from a module to each of its direct summands (see also [1, p 291] for other examples).

Example 2.8 In [1, Example 3.15], it is given a ring R over which every free R-module is coretractable, but R has a projective module P which is not coretractable. Note that P is isomorphic to a direct summand of a free R-module.

Recall that a ring R is said to be *right Kasch* if every simple right R-module can be embedded in R_R (see [18, p. 280]). It is shown in [1, Theorem 2.14] that a ring R is right Kasch if and only if the module R_R is coretractable.

Proposition 2.9 *Let R be a right Kasch ring which is not left perfect. Then R has a coretractable module M which contains a submodule N such that M/N is not coretractable.*

Proof By [1, Theorem 2.14], the R-module R_R is coretractable. On the other hand, since the ring R is not left perfect, R has a cyclic R-module M which is not coretractable by [1, Proposition 3.8]. Therefore, there exists a right ideal I of R such that the R-module R/I is not coretractable. □

The next example guarantees the existence of a ring with the required conditions of Proposition 2.9. This ring is taken from [18, Proposition 8.30] and [19, Exercise 8.18].

Example 2.10 Let $\{p_i : i \in I\}$ be the set of prime integers. Consider the $(\mathbb{Z}, \mathbb{Z})$-bimodule $M = \oplus_{i \in I} \mathbb{Z}/p_i\mathbb{Z}$ (in the natural way with identical left, right $\mathbb{Z}$-actions). Let $R = M \oplus \mathbb{Z}$ be the trivial extension of $\mathbb{Z}$ by M. By [18, Proof of Proposition 8.30], R is a commutative Kasch ring which has infinitely many simple modules. So R is not semilocal. Hence, R is not perfect.

The next proposition deals with a special case of factor modules of coretractable modules.

Proposition 2.11 *Let M be a coretractable module. If N is a submodule of M such that $\varphi(M)$ is not contained in N for every nonzero $\varphi \in End_R(M)$, then M/N is coretractable.*

Proof Let T/N be a proper submodule of M/N. Since M is coretractable, there exists a nonzero homomorphism $\alpha : M \to M$ such that $\alpha(T) = 0$. Define $\psi : M/N \longrightarrow M/N$ by $m + N \mapsto \alpha(m) + N$. Clearly, ψ is well defined and $\psi(M/N) = (\alpha(M) + N)/N$. Since $\alpha(M) \nsubseteq N$, $\psi \neq 0$. Also, we have $\psi(T/N) = 0$. Thus, M/N is coretractable. □

Let M be a module. In [29], it was introduced the submodule $\overline{Z}(M) = \cap\{N \leq M \mid M/N \ll E(M/N)\}$. The module M is called *noncosingular* if $\overline{Z}(M) = M$. Note that noncosingularity is inherited by homomorphic images (see [29, Proposition 2.4]).

Corollary 2.12 *Let M be a noncosingular coretractable module and $N \ll M$. Then M/N is coretractable.*

Proof Let $0 \neq \varphi \in \mathrm{End}_R(M)$. Since M is noncosingular, we have $\overline{Z}(\varphi(M)) = \varphi(M)$ by [29, Proposition 2.4]. Therefore $\varphi(M)$ is not small in M, since otherwise $\overline{Z}(\varphi(M)) = 0$. Thus, $\varphi(M) \nsubseteq N$. The result follows by Proposition 2.11. □

Corollary 2.13 *Let R be a right Noetherian ring. Let M be a coretractable R-module with $Rad(M) = M$. If N is a finitely generated submodule of M, then M/N is coretractable.*

Proof By Proposition 2.11. □

3 Some Notions Versus Their Duals

Let M be a coretractable R-module with $S = \mathrm{End}_R(M)$. In the literature, we can find several examples of properties (P) of modules for which the fact that the left S-module ${}_SS$ satisfies (P) implies that M_R satisfies the dual property of (P). Among others, we can cite the following examples:

1. If ${}_SS$ is uniform, then M_R is hollow (see [1, Corollary 4.6]).
2. If ${}_SS$ has finite uniform dimension, then M_R has finite hollow dimension (see [1, Proposition 4.10]).
3. If S_S has the summand intersection property, then M_R has the summand sum property (see [9, Proposition 2.5]).
4. If S is a Baer ring, then M_R is a dual Baer module (see [16, Theorem 3.6]).

These links between some concepts and their duals are the motivations for the investigations in this section. We prove that if ${}_SS$ satisfies (C_{11}), then M is $\oplus$-supplemented (Theorem 3.6). We show that if M is quasi-injective and $\oplus$-supplemented, then every finitely generated left ideal of S has a complement which is a direct summand of ${}_SS$ (Theorem 3.11). It is also shown that if ${}_SS$ is Rickart, then M is a d-Rickart module (Proposition 3.15).

Proposition 3.1 *Let M be a nonzero coretractable module with $S = \mathrm{End}_R(M)$. If every descending chain $V_1 \supseteq V_2 \supseteq \cdots$ of left ideals of S with $\bigcap_{i\geq 1} V_i = 0$ becomes stationary after finitely many steps, then every ascending chain $U_1 \subseteq U_2 \subseteq \cdots$ of right submodules of M with $\bigcup_{i\geq 1} U_i = M$ becomes stationary after finitely many steps.*

Proof Let $U_1 \subseteq U_2 \subseteq \cdots$ be an ascending chain of right submodules of M with $\bigcup_{i\geq 1} U_i = M$. Therefore $l_S(U_1) \supseteq l_S(U_2) \supseteq \cdots$. In addition, we have $l_S(\bigcup_{i\geq 1} U_i) = l_S(\sum_{i\geq 1} U_i) = \bigcap_{i\geq 1} l_S(U_i) = l_S(M) = 0$ (see [2, Proposition 2.16]). By hypothesis, there exists $k \geq 1$ such that $l_S(U_i) = l_S(U_k)$ for every $i \geq k$. We claim that $U_k = M$, for if not, there exists $0 \neq \varphi \in S$ such that $\varphi(U_k) = 0$ since M is a coretractable module. It follows that $\varphi \in l_S(U_k)$. Hence $\varphi \in l_S(U_i)$ for all $i \geq 1$. Therefore $\varphi(M) = \varphi(\sum_{i\geq 1} U_i) = 0$, a contradiction. □

Let N and K be submodules of a module M. Recall that K is said to be a *complement* of N in M if K is maximal with respect to the property $K \cap N = 0$. Dually, we say that K is a *supplement* of N in M if $M = N + K$ and K is minimal with respect to this property, equivalently, $M = N + K$ and $N \cap K \ll K$. Note that every direct summand of M can be considered as a supplement and a complement of some submodule of M. Recall that a submodule K of M is called *coclosed* in M if whenever $K/L \ll M/L$ for a submodule L of K, then $L = K$. Note that every direct summand of M is coclosed in M. A right R-module M is called *semi-injective* if for any $f \in S$, $Sf = l_S(\mathrm{Ker} f) = l_S r_M(Sf)$, where $S = \mathrm{End}_R(M)$. Note that every quasi-injective module is semi-injective (see [6]).

Proposition 3.2 *Let K and N be submodules of a coretractable module M_R such that K is coclosed in M. Let $S = \mathrm{End}_R(M)$. If $l_S(K)$ is a complement of $l_S(N)$ in ${}_SS$, then K is a supplement of N in M. The converse holds when M_R is semi-injective and K is a direct summand of M.*

Proof Assume that $l_S(K)$ is a complement of $l_S(N)$ in ${}_SS$. Suppose that $M \neq N + K$. Since M is coretractable, there exists $0 \neq f \in S$ such that $f(N + K) = 0$. Thus $f \in l_S(N) \cap l_S(K)$. Hence $f = 0$, a contradiction. Thus, $M = N + K$. Moreover, $l_S(N) \oplus l_S(K)$ is essential in ${}_SS$ by [10, Proposition 1.3]. Therefore, $l_S(N \cap K)$ is essential in ${}_SS$ as $l_S(N) \oplus l_S(K) \subseteq l_S(N \cap K)$. By [1, Proposition 4.5(a)], we have $N \cap K \ll M$. Since K is coclosed, $N \cap K \ll K$ (see [6, 3.7(3)]). It follows that K is a supplement of N in M.

Conversely, assume that M is semi-injective and K is a direct summand of M which is a supplement of N in M. So, there exists a submodule L of M such that $M = N + K = L \oplus K$. Note that $l_S(N) \cap l_S(K) = l_S(L) \cap l_S(K) = 0$. Then $l_S(N \cap K) = l_S(N) \oplus l_S(K)$ and $S = l_S(L \cap K) = l_S(L) \oplus l_S(K)$ by [1, Lemma 4.9]. Since $N \cap K \ll K$ and M is semi-injective, $l_S(N \cap K)$ is essential in ${}_SS$ by [1, Proposition 4.5(c)]. Thus, $l_S(N) \oplus l_S(K)$ is essential in ${}_SS$. Let Y be a complement of $l_S(N)$ in ${}_SS$ containing $l_S(K)$. Therefore, $l_S(N) \oplus l_S(K)$ is essential in $l_S(N) \oplus Y$. From [2, Proposition 5.20], it follows that $l_S(K)$ is essential in Y. Since $S = l_S(L) \oplus l_S(K)$, $l_S(K)$ is a direct summand of Y by modularity. So $Y = l_S(K)$. This completes the proof. □

Recall that a module M is called *cosemisimple* if every simple module is M-injective. Note that a module M is cosemisimple if and only if every submodule of M is coclosed in M if and only if the radical of every factor module of M is zero (see [6, 3.8] and [32, 23.1]). The following two corollaries are now immediate since 0 is the only small submodule of a cosemisimple module.

Corollary 3.3 *Let K be a submodule of a coretractable cosemisimple module M_R with $S = \mathrm{End}_R(M)$. If $l_S(K)$ is a complement of $l_S(N)$ in ${}_SS$, then $M = K \oplus N$.*

Corollary 3.4 *Let M_R be a coretractable cosemisimple module with $S = \mathrm{End}_R(M)$. Assume that for every left ideal I of S, there exists a submodule K of M_R such that $l_S(K) = I$. Then M is semisimple.*

Example 3.5 Consider the $\mathbb{Z}$-module $M = \mathbb{Q}$. It is well known that M is not a coretractable module and every nonzero endomorphism of M is bijective. Let N be a nonzero proper submodule of M. Then we have $l_S(N) = 0$. Thus $l_S(0) = S$ is a complement of $l_S(N)$ in ${}_SS$. But 0 is not a supplement of N in M. Note that 0 is a coclosed submodule of M. This shows that coretractability is needed in Proposition 3.2.

Following [4], we say that a module M *satisfies* (C_{11}) (or M is a (C_{11})-*module*) if every submodule of M has a complement which is a direct summand of M. Dually, a module M is called $\oplus$-*supplemented* if every submodule of M has a supplement which is a direct summand of M (see [12]).

Theorem 3.6 *Let M_R be a coretractable module with $S = \mathrm{End}_R(M)$. If the module ${}_SS$ satisfies (C_{11}), then M is $\oplus$-supplemented.*

Proof Let N be a proper submodule of M. Then $l_S(N) \neq 0$ as M is coretractable. Since ${}_SS$ satisfies (C_{11}), there exists a direct summand I of ${}_SS$ such that I is a complement of $l_S(N)$ in ${}_SS$. Assume that $I = Se$ for some idempotent element e in S. Clearly, $(1-e)(M)$ is a direct summand of M and $l_S((1-e)(M)) = Se$. Applying Proposition 3.2, $(1-e)(M)$ is a supplement of N in M. Therefore, M is $\oplus$-supplemented. □

The coretractability assumption is not superfluous in Theorem 3.6 as shown below.

Example 3.7 Consider the $\mathbb{Z}$-module $\mathbb{Q}_{\mathbb{Z}}$ again. Since $\mathrm{End}_{\mathbb{Z}}(\mathbb{Q}_{\mathbb{Z}}) \cong \mathbb{Q}$, $\mathrm{End}_{\mathbb{Z}}(\mathbb{Q}_{\mathbb{Z}})$ satisfies (C_{11}). However, $\mathbb{Q}_{\mathbb{Z}}$ is not $\oplus$-supplemented (see [6, Example 20.12]). Note that the module $\mathbb{Q}_{\mathbb{Z}}$ is not coretractable.

Combining Theorem 3.6, [1, Theorem 2.14] and [32, 42.6], we obtain the following result.

Corollary 3.8 *If R is a right Kasch ring such that the R-module ${}_RR$ satisfies (C_{11}), then R is a semiperfect ring.*

Lemma 3.9 *Let M_R be a quasi-injective module with $S = \mathrm{End}_R(M)$. Let I be a left ideal of S such that $l_S r_M(I) = I$. Then $l_S r_M(I + Sf) = I + Sf$ for any $f \in S$.*

Proof The proof is analogous to that of [11, Lemma 1]. Let $f \in S$. Clearly, $I + Sf \subseteq l_S r_M(I + Sf)$. Let $g \in l_S r_M(I + Sf) = l_S(r_M(I) \cap r_M(Sf)) = l_S(r_M(I) \cap \mathrm{Ker} f)$. Since $g(r_M(I) \cap \mathrm{Ker} f) = 0$, $r_M(I) \cap \mathrm{Ker} f \subseteq \mathrm{Ker} g = r_M(Sg)$. Define the map $\theta : f(r_M(I)) \longrightarrow g(M)$ by $f(a) \mapsto g(a)$ for all $a \in r_M(I)$. Then θ is well defined and it is an R-homomorphism. Since M is quasi-injective, there exists a homomorphism $s \in S$ such that $si = i'\theta$, where $i : f(r_M(I)) \longrightarrow M$ and $i' : g(M) \longrightarrow M$ are the inclusion maps. It follows that for every $a \in r_M(I)$, $(g - sf)(a) = (i'\theta - si)(f(a)) = 0$. Therefore, $g - sf \in l_S r_M(I) = I$. Hence, $g \in I + Sf$. □

Corollary 3.10 *Let M_R be a quasi-injective module with $S = \mathrm{End}_R(M)$. Then $l_S r_M(I) = I$ for any finitely generated left ideal I of S.*

Proof This follows from Lemma 3.9 and the fact that M_R is a semi-injective module. □

Theorem 3.11 *Let M be a coretractable quasi-injective module with $S = \mathrm{End}_R(M)$. Assume that M is $\oplus$-supplemented. Then every finitely generated left ideal of S has a complement which is a direct summand of ${}_SS$.*

Proof Let I be a nonzero finitely generated left ideal of S. Then, $r_M(I) \neq M$. Since M is $\oplus$-supplemented, there exist submodules K and K' of M such that $M = K \oplus K'$ and K is a supplement of $r_M(I)$ in M. By Proposition 3.2 and Corollary 3.10, it follows that $l_S(K)$ is a complement of $I = l_S r_M(I)$ in ${}_SS$. Moreover, since $l_S(K) \cap l_S(K') = 0$, we have $S = l_S(K) \oplus l_S(K')$ by [1, Lemma 4.9]. Thus, $l_S(K)$ is a direct summand of ${}_SS$. □

The following result is a direct consequence of Theorem 3.11.

Corollary 3.12 *Let M be a coretractable quasi-injective module such that $S = \mathrm{End}_R(M)$ is a left Noetherian ring. If the R-module M is $\oplus$-supplemented, then the S-module ${}_SS$ satisfies (C_{11}).*

Example 3.13 Consider the $\mathbb{Z}$-module $M = \oplus_{i=1}^{n} \mathbb{Z}(p_i^{\infty})$, where n is a natural number and p_i $(1 \leq i \leq n)$ are prime numbers. By [23, Propositions A.7 and A.8], M is a $\oplus$-supplemented module. Moreover, M is a coretractable module by [1, Proposition 2.8]. Since $S = \mathrm{End}_{\mathbb{Z}}(M)$ is a Noetherian ring (see [8, Proposition 111.4]) and M is a quasi-injective module, it follows that the module ${}_SS$ satisfies (C_{11}) by Corollary 3.12.

The next example shows that the condition "M is a quasi-injective module" in Corollary 3.12 is not superfluous.

Example 3.14 Let A be the polynomial algebra $k[x, y]$ over a field k, and let $R = A/U$, where

$$U = (x, y)^{n+1} = \sum_{i+j=n+1} x^i y^j A (n \geq 1).$$

From [18, Example 3.69], it follows that R is a commutative local ring with maximal ideal $m = \bar{x}R + \bar{y}R$. It is clear that $ann(m) \neq 0$. Hence R is a Kasch ring by [18, Corollary 8.23]. Therefore the module R_R is coretractable by [1, Theorem 2.14]. Also, it is shown in [18, Example 3.69] that the R-module R_R is not quasi-injective and R is an Artinian ring (hence R is a Noetherian ring) such that the socle of R_R is essential in R_R and $\mathrm{Soc}(R_R) = \bigoplus_{i+j=n} \overline{x}^i \overline{y}^j k$ is a direct sum of $n + 1$ simple ideals. So, R_R is not a uniform R-module. Hence, the module R_R does not satisfy (C_{11}) since R_R is indecomposable. On the other hand, the R-module R_R is $\oplus$-supplemented as R is a local ring.

Recall that a ring R is called a *right Rickart ring* if the right annihilator of any element in R is of the form eR for some idempotent $e \in R$ (see [18, p. 260] and [21]).

According to [22], a module M is called a *d-Rickart* (or dual Rickart) module if for all $\varphi \in S = \mathrm{End}_R(M)$, $\mathrm{Im}\varphi$ is a direct summand of M. In [22, Proposition 3.1], it is shown that if M is a d-Rickart module, then $\mathrm{End}_R(M)$ is a left Rickart ring. The next result is a partial converse of [22, Proposition 3.1].

Proposition 3.15 *Let M_R be a coretractable module and $S = \mathrm{End}_R(M)$. If ${}_SS$ is Rickart, then M is a d-Rickart module.*

Proof Assume that ${}_SS$ is Rickart. Let $0 \neq \varphi \in S$ and $K = \mathrm{Im}\varphi$. Then $l_S(K) = \{f \in S \mid f\varphi = 0\}$ is a direct summand of ${}_SS$. So there exists an idempotent $h \in S$ such that $l_S(K) \oplus Sh = S$. It follows that $0 = r_M(S) = r_M l_S(K) \cap r_M(Sh) = r_M l_S(K) \cap \mathrm{Ker}h$. Since $K \subseteq r_M l_S(K)$, we have $K \cap \mathrm{Ker}h = 0$. Suppose that $N = K + \mathrm{Ker}h \neq M$. As M is coretractable, there exists $0 \neq g \in S$ such that $g(N) = 0$. Thus $g(K) = g(\mathrm{Ker}h) = 0$. Note that $\mathrm{Ker}h = (1 - h)(M)$. Therefore, $g(K) = 0$ and $g(1 - h) = 0$, that is, $g \in l_S(K)$ and $g = gh \in Sh$. As $l_S(K) \cap Sh = 0$, it follows that $g = 0$. This contradicts the fact that $g \neq 0$. So $K + \mathrm{Ker}h = M$. Since $K \cap \mathrm{Ker}h = 0$, we have $K \oplus \mathrm{Ker}h = M$. This proves the proposition. □

The next example shows that the condition "M_R is coretractable" cannot be omitted from Proposition 3.15.

Example 3.16 Let R be a commutative domain which is not a field. Then $\mathrm{End}_R(R) \cong R$ is a left Rickart ring. On the other hand, the R-module R_R is not d-Rickart by [22, Remark 2.2]. Note that R_R is not a coretractable module (see [1, Theorem 2.14] or Corollary 2.3).

4 s-Coretractable Modules

A module M is said to be *e-retractable* if $\mathrm{Hom}(M, C) \neq 0$ for every nonzero complement submodule C of M (see [17]). Dually, we introduce the notion of s-coretractable modules. A module M will be called *s-coretractable* if $\mathrm{Hom}(M/K, M) \neq 0$ for every proper supplement submodule K of M. Examples are provided to show that this concept is a proper generalization of coretractability (Example 4.9). We show that for every domain R, the R-module R_R is s-coretractable (Proposition 4.8). We obtain a characterization for a direct sum of two cyclic modules over a commutative local ring to be s-coretractable (Proposition 4.15). It is shown in Example 4.17 that a direct sum of s-coretractable modules is not s-coretractable, in general. Then we provide some conditions under which a direct sum of s-coretractable modules is s-coretractable (Propositions 4.18 and 4.19).

Recall that a module M is called *lifting* if for every submodule N of M, there exists a direct summand K of M such that $K \subseteq N$ and $N/K \ll M/K$.

Example 4.1 It is clear that every module whose supplement submodules are direct summands is s-coretractable. So, every lifting module is s-coretractable by [6, 22.3]. Also, every module which has no nonzero proper supplement submodules is s-coretractable (e.g. the $\mathbb{Z}$-module $\mathbb{Z}$). Note that $\mathbb{Z}$ is not a coretractable $\mathbb{Z}$-module.

Example 4.2 Let M be an injective module over a Dedekind domain R. Let L and K be submodules of M such that K is a supplement of L in M. Let $0 \neq r \in R$. Then $Lr + Kr = (L + K)r = Mr = M$ as M is divisible. So $L + Kr = M$. Thus $K = Kr$ by the minimality of K. Therefore K is divisible, hence injective. It follows that K is a direct summand of M. So M is an s-coretractable module.

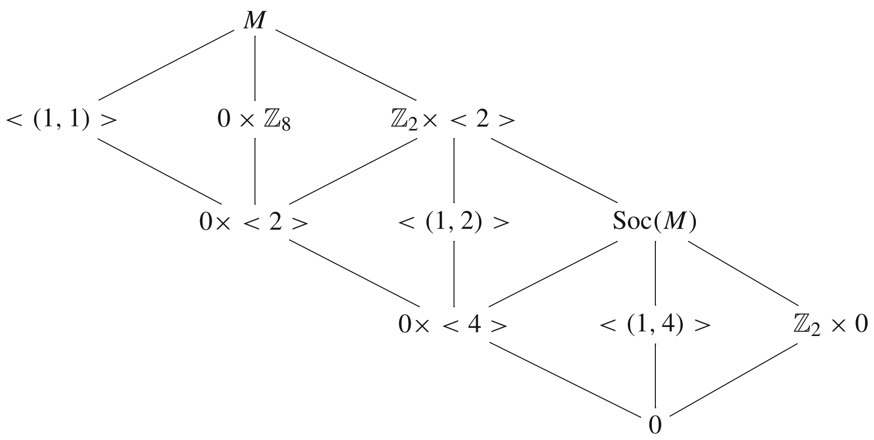
Remark 4.3 (i) In [33], Zöschinger examined the rings R such that all finitely generated projective R-modules satisfy the property that all supplement submodules are direct summands. He called these rings *right L-rings*. He showed that a ring R is a left L-ring if and only if R is a right L-ring if and only if every projective right R-module P with $P/\mathrm{Rad}(P)$ finitely generated is finitely generated. Clearly, any finitely generated projective module over an L-ring is s-coretractable. Next, we exhibit some classes of rings which are L-rings.

(a) In [30], Valette showed that if R is a ring such that the factor ring R/P is a right Goldie ring for every prime ideal P (in particular, if R is a right or a left Noetherian ring), then R is an L-ring.
(b) In [14, Theorem 2.5], it is shown that any ring with polynomial identity is an L-ring.
(c) [25, Corollary 3.3] shows that if a ring R has either left or right Krull dimension, then R is an L-ring.
(d) From [20, Proposition 5] or [31, Theorem 2.1], we see that every commutative ring is an L-ring.
(e) If R is a semihereditary ring or an exchange ring (in particular, if R is a von Neumann regular ring or a semiregular ring), then R is an L-ring (see, for example, [13, Example 3.10](i) and (iii)).

(ii) Zöschinger showed also that if R is a commutative integral domain or a right Noetherian ring, then in every projective R-module supplement submodules are direct summands (see [33, Remark page 202 and Theorem 3.3]). Hence, every projective R-module is s-coretractable.
(iii) If R is a hereditary ring, then every supplement submodule in a projective R-module is a direct summand. Therefore, every projective R-module is s-coretractable (see, for example, [13, Example 3.10](i)).

Next, we give an s-coretractable module which contains a supplement submodule that is not a direct summand.

Example 4.4 Let M be the $\mathbb{Z}$-module $\mathbb{Z}_2 \times \mathbb{Z}_8$. Note that M is coretractable by [1, Example 2.2 and Proposition 2.6], and hence it is s-coretractable. We investigate below the proper supplement submodules of M which are not direct summands. The module M has the following lattice of submodules:



Let $N =< (1, 2) >$. It is easy to see that N is the only proper supplement submodule of M which is not a direct summand of M and it is a supplement of the submodules $< (1, 1) >$ and $0 \times \mathbb{Z}_8$ in M. Note that $M/N =< (0, 1) + N >$. On the other hand, we can define the nonzero homomorphism $f : M/N \longrightarrow M$ with the definition that $(0, x) + N \mapsto (x, 4x)$, where $x \in \mathbb{Z}_8$.

In contrast to Remark 4.3, in the next example we present a ring R for which R_R is not an s-coretractable R-module. Recall that a uniserial domain R is called *nearly simple* if R is not Artinian, and $Jac(R)$ is the unique nontrivial two-sided ideal of R. For an example of a nearly simple uniserial domain see [26, p. 325].

Example 4.5 Let R be a nearly simple uniserial domain, $0 \neq r \in Jac(R)$, $M = R/rR$, and $S = \mathrm{End}_R(M)$. Then S is a semilocal ring by [7, Corollary 4.16]. By [25, Proposition 6.7], S has exactly three nonzero proper two-sided ideals, namely two maximal ideals I and K and the Jacobson radical $J = Jac(S) = I \cap K$ which is idempotent. Note that K is a cyclic left ideal of S by [25, p. 239]. Since $S = I + K$ and $I \cap K = J = J^2 \subseteq JK = \mathrm{Rad}(K) \ll K$, K is a supplement of I in ${}_SS$. Suppose that the S-module ${}_SS$ is s-coretractable. Then there exists a nonzero S-homomorphism $\varphi : S/K \to S$. Since S/K is simple, $\mathrm{Ker}\varphi = 0$. Then $S/K \cong X$ for some left simple ideal X of S. It follows that $KX = 0$. Note that S is left uniform and the left singular ideal of S is $Z_l(S) = K$ by [25, Proposition 6.4]. Therefore $X \subseteq K$ as K is essential in ${}_SS$. Thus $K^2 = 0$. But S is a prime ring by [25, Proposition 6.7]. Then $K = 0$, a contradiction. Consequently, the S-module ${}_SS$ is not s-coretractable.

Next, we characterize rings R for which the R-module R_R is s-coretractable.

Lemma 4.6 *For any right ideal K of a ring R, $Hom(R/K, R) \neq 0$ if and only if $l_R(K) \neq 0$.*

Proof ($\Rightarrow$) Let K be a right ideal of R such that $Hom(R/K, R) \neq 0$. So there exists a nonzero homomorphism $f : R_R \longrightarrow R_R$ such that $f(K) = 0$. Let $f(1) = r$. Then $r \neq 0$ and $r \in l_R(K)$.

($\Leftarrow$) Let K be a right ideal of R such that $l_R(K) \neq 0$. Then there is a nonzero element $a \in R$ such that $aK = 0$. Consider the nonzero homomorphism $f : R_R \to R_R$ defined by $f(x) = ax$ for every $x \in R$. Then $f(K) = aK = 0$. This implies that $Hom(R/K, R) \neq 0$. □

Proposition 4.7 *The following conditions are equivalent for a ring R:*

(i) the R-module R_R is s-coretractable;
(ii) for every proper right ideal K of R which is a supplement in R_R, $l_R(K) \neq 0$.

Proof This follows from Lemma 4.6. □

The next result provides other examples of s-coretractable modules.

Proposition 4.8 *Let R be a domain. Then R_R is an s-coretractable R-module.*

Proof Let K be a nonzero supplement submodule in R_R. By [13, Theorem 1.6], there exist $0 \neq x \in R$ and $r_0 \in R$ such that $K = xR$ and $x^2 r_0 = x$. Then $x(xr_0 - 1) = 0$. Since R is a domain, we get $xr_0 = 1 \in K$. So, $K = R$. Therefore, the zero submodule of R_R is the only proper supplement in R_R. It follows that R_R is an s-coretractable module. □

Next, some examples are presented to show that the class of coretractable modules is a proper subclass of the class of s-coretractable modules.

Example 4.9 (1) By [1, Theorem 2.14], for a ring R, the R-module R_R is coretractable if and only if $l_R(I) \neq 0$ for any right ideal I of R, where $l_R(I) = \{r \in R \mid rx = 0$ for all $x \in I\}$. Hence for any domain R which is not a division ring, R_R is an s-coretractable R-module which is not coretractable by Proposition 4.8.

(2) Let R and M be as in Example 2.5. Then M is not coretractable. On the other hand, R is an L-ring as R is von Neumann regular (see Remark 4.3(i)(e)). Since M is a cyclic projective R-module, M is s-coretractable by Remark 4.3(i).

(3) Let P be a nonzero projective $\mathbb{Z}$-module. Then the module P is not coretractable by Corollary 2.3. However, P is an s-coretractable module by Remark 4.3(ii).

Next, we investigate when a direct summand of an s-coretractable module is s-coretractable. First, we present an example which shows that the s-coretractability property is not inherited by direct summands.

Example 4.10 Let R be a ring and let $C = \oplus_{T \in \mathcal{S}} E(T)$, where $\mathcal{S}$ is an irredundant set of representatives of the simple R-modules. It is well known that C is a cogenerator (see [2, Corollary 18.16]). Then for any module M, it is clear that $C \oplus M$ is again a cogenerator and so it is coretractable (see also [1, p. 291]). So, $C \oplus M$ is s-coretractable. But M need not be s-coretractable (see Example 4.5).

The last example shows also that the class of s-coretractable modules is not closed under factor modules.

As in [1, Proposition 2.5], the next result deals with some special cases.

Proposition 4.11 *Let $M = K \oplus L$ be an s-coretractable module such that either K is a fully invariant submodule of M or K cogenerates M. Then K is s-coretractable.*

Proof Let X be a proper supplement submodule of K. Then $X \oplus L$ is a proper supplement submodule of M by [15, Lemma 2.2]. Since M is s-coretractable, there is a nonzero homomorphism $f : M \longrightarrow M$ such that $f(X \oplus L) = 0$. If K is fully invariant in M, then $f(K) \subseteq K$. As f is nonzero, $f(K)$ is nonzero. Thus $f\,|_K : K \longrightarrow K$ is nonzero and $(f\,|_K)(X) = 0$. So K is s-coretractable. Now assume that K cogenerates M. In the same manner as in the proof of [1, Proposition 2.5], we can see that K is s-coretractable. □

Lemma 4.12 *Let M be a module and let L and K be submodules of M such that $L \ll M$ and $L \subseteq K$. Then K is a supplement submodule in M if and only if K/L is a supplement submodule of M/L.*

Proof (⇒) By [6, 20.5(1)]. (⇐) Let N be a submodule of M such that $L \subseteq N$, $(K + N)/L = M/L$ and $(K \cap N)/L \ll K/L$. Let X be a submodule of M such that $(K \cap N) + X = K$. Then $X + L = K$. Moreover, we have $L \ll K$ by [6, 20.2]. Therefore $X = K$. Hence $K \cap N \ll K$. It follows that K is a supplement of N in M. □

Combining Lemma 4.12 and the proof of Proposition 2.11, we obtain the following result.

Proposition 4.13 *Let M be a noncosingular s-coretractable module and $N \ll M$. Then M/N is s-coretractable.*

Next, we study the question of when a direct sum of s-coretractable modules is also s-coretractable.

Recall that a nonzero module M is called *hollow* if every proper submodule is small in M. The module M is said to be *local* if M has a proper submodule which contains all other proper submodules.

Let $n \in \mathbb{N}$. Recall that a module M is said to have *hollow dimension* n and this is denoted by $h.dim(M) = n$, if there exists an epimorphism from M to a direct sum of n nonzero modules but no epimorphism from M to a direct sum of more than n nonzero modules (see [6, 5.2]). Note that if $M = 0$, then $h.dim(M) = 0$ and a module M is hollow if and only if $h.dim(M) = 1$. If M is a module over a commutative ring R, we denote the *annihilator* of M by $ann(M)$, i.e. $ann(M) = \{r \in R \mid Mr = 0\}$. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in a commutative ring R, their *ideal quotient* is $\mathfrak{a} : \mathfrak{b} = \{x \in R \mid x\mathfrak{b} \subseteq \mathfrak{a}\}$ which is an ideal of R.

It is clear that every local module is s-coretractable. The next proposition characterizes when a direct sum of two local modules over a commutative local ring is s-coretractable. First, we prove the following lemma.

Lemma 4.14 *Let R be a commutative local ring with maximal ideal $\mathfrak{m}$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two proper ideals of R and consider the R-module $M = R/\mathfrak{a} \oplus R/\mathfrak{b}$. Then for any nonzero proper supplement K in M, M/K is cyclic and $ann(M/K) = s\mathfrak{a} + \mathfrak{b}$ or $ann(M/K) = \mathfrak{a} + r\mathfrak{b}$ for some $r, s \in R$.*

Proof First, we point out that $\text{Rad}(M) = \mathfrak{m}/\mathfrak{a} \oplus \mathfrak{m}/\mathfrak{b} \ll M$. Let $H_1 = R/\mathfrak{a} \oplus 0 = (\bar{1}, \tilde{0})R$ and $H_2 = 0 \oplus R/\mathfrak{b} = (\bar{0}, \tilde{1})R$. Thus $M = H_1 + H_2$.

Let K be a nonzero proper supplement of M. Then $h.dim(M) = h.dim(M/K) + h.dim(K)$ by [6, 20.10(2)]. This implies that $h.dim(K) = 1$. Hence K is a hollow module. Since K is finitely generated (see [6, 20.4(2)]), it follows that K is a local submodule of M. Therefore, there exist $r, s \in R$ such that $r \notin \mathfrak{m}$ or $s \notin \mathfrak{m}$ and $K = (\bar{r}, \tilde{s})R$.

Assume that $r \notin \mathfrak{m}$. Then $M = K + H_2$ and $M/K \cong H_2/K \cap H_2$. Thus, $ann(M/K) = ann(H_2/(K \cap H_2)) = \{\alpha \in R \mid (\bar{0}, \tilde{1})\alpha = (\bar{r}, \tilde{s})x \text{ for some } x \in R\}$ $= \{\alpha \in R \mid rx \in \mathfrak{a} \text{ and } \alpha - sx \in \mathfrak{b} \text{ for some } x \in R\}$. Note that r is invertible as $r \notin \mathfrak{m}$. Then $ann(M/K) = \{\alpha \in R \mid \exists x \in \mathfrak{a} \text{ such that } \alpha - sx \in \mathfrak{b}\} = s\mathfrak{a} + \mathfrak{b}$. The same reasoning applies to the case $s \notin \mathfrak{m}$ gives $ann(M/K) = \mathfrak{a} + r\mathfrak{b}$. □

Proposition 4.15 *Let R be a commutative local ring with maximal ideal $\mathfrak{m}$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two proper ideals of R and consider the R-module $M = R/\mathfrak{a} \oplus R/\mathfrak{b}$. Then the following conditions are equivalent:*

(i) *M is an s-coretractable module;*
(ii) $\mathfrak{a} : \mathfrak{b} \neq \mathfrak{a}$ *or* $\mathfrak{b} : \mathfrak{a} \neq \mathfrak{b}$.

Proof (i) ⇒ (ii) Let $K = (\bar{r}, \tilde{s})R$ with $r, s \notin \mathfrak{m}$. It is easily seen that K is a supplement of both H_1 and H_2 in M. By the proof of Lemma 4.14, it follows that M/K is cyclic and $ann(M/K) = \mathfrak{a} + \mathfrak{b}$. Since M is s-coretractable, we have $\text{Hom}(M/K, R/\mathfrak{a}) \neq 0$

or $\mathrm{Hom}(M/K, R/\mathfrak{b}) \neq 0$. If $\mathrm{Hom}(M/K, R/\mathfrak{a}) \neq 0$, then there exists $x \in R$ such that $x \notin \mathfrak{a}$ and $x(\mathfrak{a} + \mathfrak{b}) \subseteq \mathfrak{a}$. So there exists $x \in R$ such that $x \notin \mathfrak{a}$ and $x\mathfrak{b} \subseteq \mathfrak{a}$. That is, $\mathfrak{a} : \mathfrak{b} \neq \mathfrak{a}$. In the same manner, we can see that $\mathrm{Hom}(M/K, R/\mathfrak{b}) \neq 0$ implies $\mathfrak{b} : \mathfrak{a} \neq \mathfrak{b}$.

(ii) ⇒ (i) Assume that $\mathfrak{a} : \mathfrak{b} \neq \mathfrak{a}$. So there exists $x \in R$ such that $x \notin \mathfrak{a}$ and $x\mathfrak{b} \subseteq \mathfrak{a}$. Let K be a nonzero proper supplement in M and let $\mathfrak{c} = ann(M/K)$. By Lemma 4.14, we have $M/K \cong R/\mathfrak{c}$ and $\mathfrak{c} \subseteq \mathfrak{a} + \mathfrak{b}$. Since $x(\mathfrak{a} + \mathfrak{b}) \subseteq \mathfrak{a}$, we have $x\mathfrak{c} \subseteq \mathfrak{a}$. Consider the map $\varphi : R/\mathfrak{c} \to R/\mathfrak{a}$ defined by $\varphi(r + \mathfrak{c}) = xr + \mathfrak{a}$ for all $r \in R$. It is easily seen that φ is well defined and it is a nonzero R-homomorphism. Therefore $\mathrm{Hom}(M/K, R/\mathfrak{a}) \neq 0$. Hence $\mathrm{Hom}(M/K, M) \neq 0$. The same conclusion can be drawn for the case $\mathfrak{b} : \mathfrak{a} \neq \mathfrak{b}$. This completes the proof. □

The next result is a direct consequence of Proposition 4.15.

Corollary 4.16 *Let R be a commutative local ring. If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of R such that $\mathfrak{a} \subseteq \mathfrak{b}$, then the R-module $M = R/\mathfrak{a} \oplus R/\mathfrak{b}$ is s-coretractable.*

Proposition 4.15 provides a source of examples of s-coretractable modules whose direct sum is not s-coretractable.

Example 4.17 Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}$. By [5, Theorem 2], the power series ring $T = R[[X]]$ is a local ring with maximal ideal $\mathfrak{m} + (X)$. By [5, Theorem 4], $\mathfrak{p}_1 = (X)$ and $\mathfrak{p}_2 = \mathfrak{m}[[X]]$ are prime ideals of T. It is clear that $\mathfrak{p}_1 \nsubseteq \mathfrak{p}_2$ and $\mathfrak{p}_2 \nsubseteq \mathfrak{p}_1$. Therefore $\mathfrak{p}_1 : \mathfrak{p}_2 = \mathfrak{p}_1$ and $\mathfrak{p}_2 : \mathfrak{p}_1 = \mathfrak{p}_2$. By Proposition 4.15, it follows that the T-module $T/\mathfrak{p}_1 \oplus T/\mathfrak{p}_2$ is not s-coretractable. However, the T-modules $T/\mathfrak{p}_1$ and $T/\mathfrak{p}_2$ are s-coretractable.

Next, we provide some sufficient conditions for a direct sum of s-coretractable modules to be s-coretractable. Note that the next two results are partial analogues to the relevant results for coretractable modules (see [1, Propositions 2.6 and 2.8]).

Recall that a module M is called a *duo module* provided every submodule of M is fully invariant. A module M is called *distributive* if $A \cap (B + C) = (A \cap B) + (A \cap C)$ for all submodules A, B and C of M. It is well known that if M is duo or distributive, then for every submodule $X \leq M$ and for every decomposition $M = N \oplus K$, we have $X = (X \cap N) \oplus (X \cap K)$ (see, for example, [24, Lemma 2.1]).

Proposition 4.18 *Let $M = \oplus_{i=1}^{n} M_i$ be a module. Assume that M is duo or distributive. If $M_1, M_2, \ldots, M_n$ are s-coretractable, then so is M.*

Proof Without loss of generality, we can assume that $n = 2$. Let K be a proper supplement submodule of M. So $K = A \oplus B$, where $A = K \cap M_1$ and $B = K \cap M_2$. By [6, 20.6], A and B are supplement submodules in M_1 and M_2, respectively. Since $K \neq M$, we have $A \neq M_1$ or $B \neq M_2$. Assume $B \neq M_2$. Since M_2 is s-coretractable, there exists a nonzero homomorphism $f : M_2/B \longrightarrow M_2$. Let $\pi : M/K \to M/(K + M_1)$ be the natural epimorphism and let $\mu : M_2 \to M$ be the natural inclusion. Note that $M/(K + M_1) = [(K + M_1) + M_2]/(K + M_1) \cong M_2/B$ since $M_2 \cap (K + M_1) = B$. Let $g : M/(K + M_1) \to M_2/B$ be an isomorphism. Then $\mu f g \pi : M/K \to M$ is a nonzero homomorphism. The case $A \neq M_1$ can be handled in much the same way. It follows that M is s-coretractable. □

Proposition 4.19 *Let $M = \oplus_{\alpha \in I} M_\alpha$ be the direct sum of s-coretractable submodules $\{M_\alpha \mid \alpha \in I\}$. If for all $\alpha, \beta \in I$, M_α is M_β-injective and M is duo or distributive, then M is s-coretractable.*

Proof Let K be a proper supplement submodule of M. Then there exists $\beta \in I$ such that $M_\beta \nsubseteq K$. Moreover, we have $K = X \oplus Y$ with $X = K \cap M_\beta$ and $Y = K \cap (\oplus_{\alpha \neq \beta} M_\alpha)$. Note that X is a proper supplement submodule of M_β by [6, 20.6]. Since M_α is M_β-injective for all $\alpha, \beta \in I$, by the same method as in the proof of [1, Proposition 2.8], we can see that M is s-coretractable. □

The next example illustrates the last proposition.

Example 4.20 Let $\{p_i \mid i \in I\}$ be a family of distinct prime integers and let $M = \oplus_{i \in I} M_i$ such that for each $i \in I$, either $M_i \cong \mathbb{Z}(p_i^\infty)$ or $M_i \cong \mathbb{Z}/p_i^{k_i}\mathbb{Z}$ for some positive integer k_i. Clearly, each M_i is s-coretractable. Moreover, it is immediate that for all $i, j \in I$, M_i is M_j-injective. By [24, Theorem 3.10], the $\mathbb{Z}$-module M is duo. Therefore by Proposition 4.19, M is s-coretractable. Note that by [1, Proposition 2.8], M is also a coretractable $\mathbb{Z}$-module.

Recall that a module M is called *amply supplemented* if for any two submodules A and B of M with $M = A + B$, B contains a supplement of A. The following characterization is dual to [17, Theorem 2.4] in some sense.

Proposition 4.21 *The following are equivalent for an amply supplemented module M with $S = \mathrm{End}_R(M)$.*

(i) M is s-coretractable;
(ii) for any proper supplement submodule K of M, $r_M(l_S(K))/K \ll M/K$;
(iii) for any supplement submodule K of M, if $r_M(l_S(K))$ is a direct summand of M, then $K = r_M(l_S(K))$.

Proof (i) ⇒ (ii) Let K be a proper supplement submodule of M and $L = r_M(l_S(K))$. To prove that $L/K \ll M/K$, let $(L/K) + (X/K) = M/K$ with $K \leq X \leq M$. Since M/K is amply supplemented (see [32, 41.7(2)]), there exists a submodule $V \leq M$ such that $K \subseteq V \subseteq X$ and V/K is a supplement of L/K in M/K. By [6, 20.5(2)], V is a supplement submodule of M. If $V \neq M$, then by hypothesis there exists $0 \neq f \in \mathrm{End}_R(M)$ with $f(V) = 0$. Therefore, $f \in l_S(K)$ and so $f(L) = 0$. This implies that $f(M) = f(L + V) = 0$, a contradiction. It follows that $V = X = M$. Hence $L/K \ll M/K$.

(ii) ⇒ (iii) and (iii) ⇒ (i) These are similar to the proofs of (b) ⇒ (c) and (c) ⇒ (a) in [1, Lemma 4.1], respectively. □

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